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CANONICAL FORMS OF SELF–ADJOINT BOUNDARY CONDITIONS FOR REGULAR DIFFERENTIAL OPERATORS OF ORDER THREE

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Abstract. In this paper, we find all canonical forms for third order self-adjoint boundary conditions. These canonical forms play an important role in the study of the dependence of the eigenvalues on the problem and for their numerical calculation. In order to obtain those canonical forms, we give a classification of self-adjoint boundary conditions. Those self-adjoint boundary conditions can be categorized into three mutually exclusive classes: coupled, strictly separated and mixed. Unlike the even order case, for the third order case, the strictly separated self-adjoint boundary conditions can not be realized. For coupled and mixed cases, there are some different types for the canonical forms: 2 for coupled and 4 for mixed boundary conditions.

1. Introduction

A regular self-adjoint Sturm-Liouville problem consists of the symmetric differential equation

$$-(py')' + qy = \lambda wy \quad on\ J = (a,b), -\infty < a < b < \infty, \tag{1.1}$$

with coefficients satisfying:

$$\frac{1}{p}, q, w \in L(J,\mathbb{R}),\ \omega > 0 \tag{1.2}$$

and boundary conditions

$$AY(a) + BY(b) = 0, \qquad Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \tag{1.3}$$

where

$$A, B \in M_2(\mathbb{C}),\ AEA^* = BEB^*,\ rank(A:B) = 2,\ E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

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The boundary conditions (1.3) and (1.4) can be categorized into two mutually exclusive classes: separated and coupled, and these have the canonical forms:
(i) Separated self-adjoint boundary condition: these can be formulated as follows:

$$\begin{aligned} \cos(\alpha)y(a)-\sin(\alpha)(py')(a)=0, \alpha\in[0,\pi), \\ \cos(\beta)y(a)-\sin(\beta)(py')(a)=0, \beta\in(0,\pi]. \end{aligned} \tag{1.5}$$

(ii) Coupled self-adjoint boundary condition: these are

$$Y(b)=e^{i\gamma}KY(a), \tag{1.6}$$

where $-\pi<\gamma\leqslant\pi$ and K satisfies

$$K=\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},\ k_{ij}\in\mathbb{R},\ detK=1. \tag{1.7}$$

(see[1, 2]). Given a boundary condition (1.3) with matrices A,B satisfying (1.4), it is equivalent to exactly one of the separated or coupled boundary conditions defined above (see [1]).

These canonical forms play an important role in the study of the dependence of the eigenvalues on the problem and for their numerical calculation (see [3]). Similar to the second order case, Hao, Sun and Zettl have obtained canonical forms for the fourth order self-adjoint boundary conditions (see [4]). By using these forms, lots of researchers have obtained the dependence of the eigenvalues on the fourth order boundary value problems (see[5]- [7]). These papers give us a better understanding of the dependence of eigenvalues on the problem. These canonical forms also have many other potential applications. For more details, one can see book [1].

In this paper, the set of all $n\times n$ matrices over the field $\mathbb{F}$ is denoted by $M_n(\mathbb{F})$. A^* denotes the complex conjugate of the matrix A. The set of all real-valued Lebesgue integrable functions on J is denoted by $L(J,\mathbb{R})$. Given $A,B\in M_n(\mathbb{F})$, the form $(A:B)$ denotes the $n\times 2n$ matrix whose last n columns are the columns of the matrix B and whose first n columns are those of the matrix A.

In this paper, we consider third order boundary value problems, the equation is given by

$$My=[-i(p(py')'-b_0y)-a_1y']'+ib_0y'+a_0y=\lambda wy\ \ on\ \ J=(a,b), \tag{1.8}$$

where

$$p^{-1},b_0p^{-1},b_1p^{-1},a_1p^{-2},a_0\in L(J,\mathbb{R}),\ w>0 \text{ a.e.on } J. \tag{1.9}$$

Note that the coefficients of (1.8) have no smoothness assumptions. For this reason, we give the following quasi-derivatives:

$$y^{[0]}=y, \qquad y^{[1]}=py', \qquad y^{[2]}=ip(y^{[1]})'+a_1y'-ib_0y. \tag{1.10}$$

The existence of the quasi-derivatives at a and b can be guaranteed by the condition (1.9), so this problem is regular (see [1, 2]).

The boundary conditions

$$A\begin{pmatrix} y(a) \\ y^{[1]}(a) \\ y^{[2]}(a) \end{pmatrix} + B\begin{pmatrix} y(b) \\ y^{[1]}(b) \\ y^{[2]}(b) \end{pmatrix} = 0, \quad A, B \in M_3(\mathbb{C}), \tag{1.11}$$

of Eq. (1.8) are self-adjoint if and only if

$$rank(A:B) = 3 \quad and \quad AE_3A^* = BE_3B^*, \; E_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & i & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.12}$$

The proof in [8] and [9] can readily be adapted to this generality.

When multiplied by a nonsingular matrix $G \in M_3(\mathbb{C})$, the boundary conditions (1.11) are clearly invariant. If $AE_3A^* = BE_3B^*$, then

$$(GA)E_3(GA)^* = (GB)E_3(GB)^*.$$

Hence, by using linear transformations of the rows of $(A : B)$, the boundary form is invariant.

REMARK 1. Since $AE_3A^* = BE_3B^*$, if A is nonsingular, then B is nonsingular. Hence, for a self-adjoint boundary conditions, $rank(A)$ and $rank(B)$ satisfy one of the following four cases:

(i) rank(A)=rank(B)=3;
(ii) rank(A)=rank(B)=2;
(iii) rank(A)=1,rank(B)=2;
(iv) rank(A)=2,rank(B)=1.

Similar to the even order case, we give a definition of strictly separated, coupled and mixed boundary conditions. (see [10])

DEFINITION 1. Assume the matrices $A, B \in M_3(\mathbb{C})$ satisfy (1.12). Then the self-adjoint boundary condition (1.11) is

(1) coupled if $rank(A) = rank(B) = 3$,
(2) mixed if $rank(A) = rank(B) = 2$,
(3) strictly separated if $rank(A) = 2$ and $rank(B) = 1$ or $rank(A) = 1$ and $rank(B) = 2$.

In this paper, we obtain canonical forms for the third order self-adjoint boundary conditions (1.11), (1.12). Unlike the even order case, for the third order case, the strictly separated self-adjoint boundary conditions cannot be realized. For coupled and mixed cases, there are some different types for the canonical forms: 2 for coupled and 4 for mixed boundary conditions.

The paper is composed as follows: In Section 2, we prove the strictly separated self-adjoint boundary conditions cannot be realized. In Sections 3 and 4, we prove the coupled and mixed self-adjoint boundary conditions can be realized and obtain their canonical forms.

2. Strictly separated self-adjoint boundary conditions

In this section, we prove that the strictly separated self-adjoint boundary conditions cannot be realized. By Definition 1, if the strictly separated self-adjoint boundary conditions can be realized, then $rank(A)$, $rank(B)$ satisfy not only the requirements of $rank(A)$ and $rank(B)$ but also the self-adjointness conditions (1.12). In what follows, we prove that the strictly separated types cannot be realized.

LEMMA 1. *Assume the matrices $A,B \in M_3(\mathbb{C})$ satisfy (1.12). Then $rank(A) = 1, rank(B) = 2$ cannot be realized.*

Proof. Assume A and B satisfy $rank(A:B) = 3, rank(A) = 1, rank(B) = 2$. By using the elementary row transformation, $(A:B)$ is equivalent to

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix}. \tag{2.1}$$

(i) If $b_{21} \neq 0$, then (2.1) has the following form by a transformation of rows

$$(2.1) \to \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \widetilde{b}_{22} & \widetilde{b}_{23} \\ 0 & 0 & 0 & 0 & \widetilde{b}_{32} & \widetilde{b}_{33} \end{pmatrix} \overset{rewrite}{\longrightarrow} \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & b_{22} & b_{23} \\ 0 & 0 & 0 & 0 & b_{32} & b_{33} \end{pmatrix}.$$

By a computation on the reduced form s of A and B i.e.,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix},$$

we have

$$AE_3A^* = \begin{pmatrix} ia_{12}\bar{a}_{12} + a_{13}\bar{a}_{11} - a_{11}\bar{a}_{13} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{2.2}$$

$$BE_3B^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & ib_{22}\bar{b}_{22} + b_{23} - \bar{b}_{23} & ib_{22}\bar{b}_{32} - \bar{b}_{33} \\ 0 & ib_{32}\bar{b}_{22} + b_{33} & ib_{32}\bar{b}_{32} \end{pmatrix}. \tag{2.3}$$

Assume the self-adjointness conditions (1.12) can be satisfied, then

$$\begin{pmatrix} ia_{12}\bar{a}_{12} + a_{13}\bar{a}_{11} - a_{11}\bar{a}_{13} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & ib_{22}\bar{b}_{22} + b_{23} - \bar{b}_{23} & ib_{22}\bar{b}_{32} - \bar{b}_{33} \\ 0 & ib_{32}\bar{b}_{22} + b_{33} & ib_{32}\bar{b}_{32} \end{pmatrix},$$

i.e., $b_{32} = b_{33} = 0$. It contradicts $rank(B) = 2$.
(ii) If $b_{21} = 0$, then (2.1) has the following form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix}. \tag{2.4}$$

By a computation with

$$B=\begin{pmatrix}0&0&0\\0&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix},$$

we have

$$BE_3B^*=\begin{pmatrix}0&0&0\\0&ib_{22}\overline{b}_{22}&ib_{22}\overline{b}_{32}+b_{23}\overline{b}_{31}\\0&ib_{32}\overline{b}_{22}-b_{31}\overline{b}_{23}&ib_{32}\overline{b}_{32}-b_{31}\overline{b}_{33}+b_{33}\overline{b}_{31}\end{pmatrix}. \tag{2.5}$$

If the self-adjointness conditions (1.12) can be satisfied, then

$$\begin{pmatrix}ia_{12}\overline{a}_{12}+a_{13}\overline{a}_{11}-a_{11}\overline{a}_{13}&0&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&ib_{22}\overline{b}_{22}&ib_{22}\overline{b}_{32}+b_{23}\overline{b}_{31}\\0&ib_{32}\overline{b}_{22}-b_{31}\overline{b}_{23}&ib_{32}\overline{b}_{32}-b_{31}\overline{b}_{33}+b_{33}\overline{b}_{31}\end{pmatrix}.$$

It means $b_{22}=b_{23}=0$ or $b_{22}=b_{31}=b_{32}=0$. They all contradict $rank(B)=2$.

LEMMA 2. *Assume the matrices $A,B\in M_3(\mathbb{C})$ satisfy (1.12). Then $rank(A)=2, rank(B)=1$ cannot be realized.*

Proof. This proof is completely similar to Lemma 1.

Combining Lemmas 1 and 2 we obtain:

THEOREM 1. *For the third order regular differential operator, the strictly separated self-adjoint boundary condition cannot be realized.*

3. Coupled self-adjoint boundary conditions

In this section, we prove that the coupled self-adjoint boundary conditions can be realized, furthermore, every coupled self-adjoint boundary condition is equivalent to one of the two canonical forms given below.

By Definition 1, if the coupled self-adjoint boundary conditions can be realized, then $(A:B)$ can be transformed to

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}&1&0&0\\a_{21}&a_{22}&a_{23}&0&1&0\\a_{31}&a_{32}&a_{33}&0&0&1\end{pmatrix}, \tag{3.1}$$

and the form $(A:B)$ given by (3.1) satisfies the self-adjointness condition (1.12). Below, we verify whether $(A:B)$ satisfies the self-adjointness condition (1.12). In order to make our calculations more simple, we give a kind of classification of $(A:B)$ by discussing the first column of A.

LEMMA 3. *Let $A,B\in M_3(\mathbb{C})$ satisfy $rank(A)=rank(B)=3$, then $(A:B)$ is equivalent to one of the following three cases:*

Case (i):

$$(A:B)=\begin{pmatrix}1 & a_{12} & a_{13} & b_{11} & 0 & 0\\ 0 & a_{22} & a_{23} & b_{21} & 1 & 0\\ 0 & a_{32} & a_{33} & b_{31} & 0 & 1\end{pmatrix}; \tag{3.2}$$

Case (ii):

$$(A:B)=\begin{pmatrix}0 & a_{12} & a_{13} & 1 & 0 & 0\\ 1 & a_{22} & a_{23} & 0 & b_{22} & 0\\ 0 & a_{32} & a_{33} & 0 & b_{32} & 1\end{pmatrix}; \tag{3.3}$$

Case (iii):

$$(A:B)=\begin{pmatrix}0 & a_{12} & a_{13} & 1 & 0 & 0\\ 0 & a_{22} & a_{23} & 0 & 1 & 0\\ 1 & a_{32} & a_{33} & 0 & 0 & b_{33}\end{pmatrix}. \tag{3.4}$$

Proof. (i). If $a_{11}\neq 0$, then (3.1) has the following form by a transformation of rows:

$$(3.1)\rightarrow\begin{pmatrix}1 & \widetilde{a}_{12} & \widetilde{a}_{13} & b_{11} & 0 & 0\\ 0 & \widetilde{a}_{22} & \widetilde{a}_{23} & b_{21} & 1 & 0\\ 0 & \widetilde{a}_{32} & \widetilde{a}_{33} & b_{31} & 0 & 1\end{pmatrix}\overset{rewrite}{\rightarrow}\begin{pmatrix}1 & a_{12} & a_{13} & b_{11} & 0 & 0\\ 0 & a_{22} & a_{23} & b_{21} & 1 & 0\\ 0 & a_{32} & a_{33} & b_{31} & 0 & 1\end{pmatrix}.$$

This is the case (i).
(ii) If $a_{11}=0, a_{21}\neq 0$, then (3.1) has the following form by a transformation of rows:

$$(3.1)\rightarrow\begin{pmatrix}0 & a_{12} & a_{13} & 1 & 0 & 0\\ 1 & \widetilde{a}_{22} & \widetilde{a}_{23} & 0 & b_{22} & 0\\ 0 & \widetilde{a}_{32} & \widetilde{a}_{33} & 0 & b_{32} & 1\end{pmatrix}\overset{rewrite}{\rightarrow}\begin{pmatrix}0 & a_{12} & a_{13} & 1 & 0 & 0\\ 1 & a_{22} & a_{23} & 0 & b_{22} & 0\\ 0 & a_{32} & a_{33} & 0 & b_{32} & 1\end{pmatrix}.$$

This is the case (ii).
(iii). If $a_{11}=a_{21}=0$, then (3.1) has the following form

$$(3.1)\rightarrow\begin{pmatrix}0 & a_{12} & a_{13} & 1 & 0 & 0\\ 0 & a_{22} & a_{23} & 0 & 1 & 0\\ 1 & \widetilde{a}_{32} & \widetilde{a}_{33} & 0 & 0 & b_{33}\end{pmatrix}\overset{rewrite}{\rightarrow}\begin{pmatrix}0 & a_{12} & a_{13} & 1 & 0 & 0\\ 0 & a_{22} & a_{23} & 0 & 1 & 0\\ 1 & a_{32} & a_{33} & 0 & 0 & b_{33}\end{pmatrix}.$$

This is the case (iii).

Obviously, given $A,B\in M_3(\mathbb{C})$ satisfy $rank(A)=rank(B)=3$, $(A:B)$ can be obtained from one of the three cases given above by using the elementary row transformation. We calculate canonical forms for the coupled boundary conditions by using this classification.

THEOREM 2. *For the third order regular differential operator, every coupled self-adjoint boundary condition is equivalent to one of the following two canonical forms:*
(i)

$$A=z\begin{pmatrix}1 & -iz_1 & r_1\\ 0 & 1 & 2\overline{z}_1\\ 0 & -iz_3 & z_2\end{pmatrix};\ B=\overline{z}\begin{pmatrix}\overline{z}_2 & iz_1 & 0\\ 2\overline{z}_3 & 1 & 0\\ r_2 & iz_3 & 1\end{pmatrix}, \tag{3.5}$$

where

$$z,z_1,z_2,z_3\in\mathbb{C}\quad and\quad r_1,r_2\in\mathbb{R};$$

(ii)

$$A = z\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 1 & 2\bar{z}_2 \\ 1 & -iz_2 & r_1 \end{pmatrix};\ B = \bar{z}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & iz_2 & -\bar{z}_1 \end{pmatrix}, \tag{3.6}$$

where

$$z, z_1, z_2 \in \mathbb{C} \quad \textit{and} \quad r_1 \in \mathbb{R}.$$

Proof. (i): If $(A:B)$ is in the case (i) of Lemma 3, we have the following equations by a calculation

$$AEA^* = \begin{pmatrix} ia_{12}\bar{a}_{12} + a_{13} - \bar{a}_{13} & ia_{12}\bar{a}_{22} - \bar{a}_{23} & ia_{12}\bar{a}_{32} - \bar{a}_{33} \\ ia_{22}\bar{a}_{12} + a_{23} & ia_{22}\bar{a}_{22} & ia_{22}\bar{a}_{32} \\ ia_{32}\bar{a}_{12} + a_{33} & ia_{32}\bar{a}_{22} & ia_{32}\bar{a}_{32} \end{pmatrix}; \tag{3.7}$$

$$BEB^* = \begin{pmatrix} 0 & 0 & -b_{11} \\ 0 & i & -b_{21} \\ \bar{b}_{11} & \bar{b}_{21} & \bar{b}_{31} - b_{31} \end{pmatrix}. \tag{3.8}$$

Assume the self-adjointness conditions (1.12) can be satisfied, then

$$\begin{pmatrix} ia_{12}\bar{a}_{12} + a_{13} - \bar{a}_{13} & ia_{12}\bar{a}_{22} - \bar{a}_{23} & ia_{12}\bar{a}_{32} - \bar{a}_{33} \\ ia_{22}\bar{a}_{12} + a_{23} & ia_{22}\bar{a}_{22} & ia_{22}\bar{a}_{32} \\ ia_{32}\bar{a}_{12} + a_{33} & ia_{32}\bar{a}_{22} & ia_{32}\bar{a}_{32} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -b_{11} \\ 0 & i & -b_{21} \\ \bar{b}_{11} & \bar{b}_{21} & \bar{b}_{31} - b_{31} \end{pmatrix}. \tag{3.9}$$

it means $a_{22} = e^{i\varphi}$, where $\varphi \in [0, 2\pi]$. In order to make our calculations more transparent, let $a_{12} = -2i\widetilde{a_{12}}$, $a_{32} = -2i\widetilde{a_{32}}e^{i\varphi}$ and $a_{33} = \widetilde{a_{33}}e^{i\varphi}$, where $\widetilde{a_{12}}, \widetilde{a_{32}}, \widetilde{a_{33}} \in \mathbb{C}$. By (3.9), we have $a_{23} = 2\bar{\widetilde{a_{12}}}e^{i\varphi}, b_{11} = (\bar{\widetilde{a_{33}}} - 4i\widetilde{a_{12}}\bar{\widetilde{a_{32}}})e^{-i\varphi}, b_{21} = 2\bar{\widetilde{a_{32}}}, a_{13} = r_1 - 2i|\widetilde{a_{12}}|^2, b_{31} = r_2 - 2i|\widetilde{a_{32}}|^2$, where $r_1, r_2 \in \mathbb{R}$. Putting these parameters into (3.2), by a transformation of rows, we have

$$\begin{pmatrix} 1 & -2i\widetilde{a_{12}} & r_1 - 2i|\widetilde{a_{12}}|^2 & (\bar{\widetilde{a_{33}}} - 4i\widetilde{a_{12}}\bar{\widetilde{a_{32}}})e^{-i\varphi} & 0 & 0 \\ 0 & e^{i\varphi} & 2\bar{\widetilde{a_{12}}}e^{i\varphi} & 2\bar{\widetilde{a_{32}}} & 1 & 0 \\ 0 & -2i\widetilde{a_{32}}e^{i\varphi} & \widetilde{a_{33}}e^{i\varphi} & r_2 - 2i|\widetilde{a_{32}}|^2 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & -i\widetilde{a_{12}} & r_1 & (\bar{\widetilde{a_{33}}} - 2i\widetilde{a_{12}}\bar{\widetilde{a_{32}}})e^{-i\varphi} & i\widetilde{a_{12}}e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} & 2\bar{\widetilde{a_{12}}}e^{i\varphi} & 2\bar{\widetilde{a_{32}}} & 1 & 0 \\ 0 & -i\widetilde{a_{32}}e^{i\varphi} & (\widetilde{a_{33}} + 2i\bar{\widetilde{a_{12}}}\widetilde{a_{32}})e^{i\varphi} & r_2 & i\widetilde{a_{32}} & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} e^{i\frac{\varphi}{2}} & -i\widetilde{a_{12}}e^{i\frac{\varphi}{2}} & r_1 e^{i\frac{\varphi}{2}} & (\bar{\widetilde{a_{33}}} - 2i\widetilde{a_{12}}\bar{\widetilde{a_{32}}})e^{-i\frac{\varphi}{2}} & i\widetilde{a_{12}}e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} & 2\bar{\widetilde{a_{12}}}e^{i\frac{\varphi}{2}} & 2\bar{\widetilde{a_{32}}}e^{-i\frac{\varphi}{2}} & e^{-i\frac{\varphi}{2}} & 0 \\ 0 & -i\widetilde{a_{32}}e^{i\frac{\varphi}{2}} & (\widetilde{a_{33}} + 2i\bar{\widetilde{a_{12}}}\widetilde{a_{32}})e^{i\frac{\varphi}{2}} & r_2 e^{-i\frac{\varphi}{2}} & i\widetilde{a_{32}}e^{-i\frac{\varphi}{2}} & e^{-i\frac{\varphi}{2}} \end{pmatrix}.$$

Let $z = e^{i\frac{\varphi}{2}}, z_1 = \widetilde{a_{12}}, z_2 = \widetilde{a_{33}} + 2i\bar{\widetilde{a_{12}}}\widetilde{a_{32}}, z_3 = \widetilde{a_{32}}$, the canonical form of this case is

$$A = z\begin{pmatrix} 1 & -iz_1 & r_1 \\ 0 & 1 & 2\bar{z}_1 \\ 0 & -iz_3 & z_2 \end{pmatrix};\ B = \bar{z}\begin{pmatrix} \bar{z}_2 & iz_1 & 0 \\ 2\bar{z}_3 & 1 & 0 \\ r_2 & iz_3 & 1 \end{pmatrix},$$

where

$$z, z_1, z_2, z_3 \in \mathbb{C} \quad \textit{and} \quad r_1, r_2 \in \mathbb{R}.$$

(ii): If $(A : B)$ is in the case (ii) of Lemma 3, then we have the following equation by a calculation

$$AEA^* = \begin{pmatrix} ia_{12}\bar{a}_{12} & ia_{12}\bar{a}_{22} + a_{13} & ia_{12}\bar{a}_{32} \\ ia_{22}\bar{a}_{12} - i\bar{a}_{13} & ia_{22}\bar{a}_{22} + a_{23} - \bar{a}_{23} & ia_{22}\bar{a}_{32} - \bar{a}_{33} \\ ia_{32}\bar{a}_{12} & ia_{32}\bar{a}_{22} + a_{33} & ia_{32}\bar{a}_{32} \end{pmatrix}; \tag{3.10}$$

$$BEB^* = \begin{pmatrix} 0 & 0 & -1 \\ 0 & ib_{22}\bar{b}_{22} & ib_{22}\bar{b}_{23} \\ 1 & ib_{32}\bar{b}_{22} & ib_{32}\bar{b}_{32} \end{pmatrix}. \tag{3.11}$$

Assume the self-adjointness conditions (1.12) can be satisfied, then

$$\begin{pmatrix} ia_{12}\bar{a}_{12} & ia_{12}\bar{a}_{22} + a_{13} & ia_{12}\bar{a}_{32} \\ ia_{22}\bar{a}_{12} - i\bar{a}_{13} & ia_{22}\bar{a}_{22} + a_{23} - \bar{a}_{23} & ia_{22}\bar{a}_{32} - \bar{a}_{33} \\ ia_{32}\bar{a}_{12} & ia_{32}\bar{a}_{22} + a_{33} & ia_{32}\bar{a}_{32} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & ib_{22}\bar{b}_{22} & ib_{22}\bar{b}_{23} \\ 1 & ib_{32}\bar{b}_{22} & ib_{32}\bar{b}_{32} \end{pmatrix}.$$

i.e., $a_{12} = a_{13} = 0$. It contradicts $rank(A) = 3$.

(iii): If $(A : B)$ is in the case (iii) of Lemma 3, then we have the following equations by a calculation

$$AEA^* = \begin{pmatrix} ia_{12}\bar{a}_{12} & ia_{12}\bar{a}_{22} & ia_{12}\bar{a}_{32} + a_{13} \\ ia_{22}\bar{a}_{12} & ia_{22}\bar{a}_{22} & ia_{22}\bar{a}_{32} + a_{23} \\ ia_{32}\bar{a}_{12} - \bar{a}_{13} & ia_{32}\bar{a}_{22} - \bar{a}_{23} & ia_{32}\bar{a}_{32} + a_{33} - \bar{a}_{33} \end{pmatrix}; \tag{3.12}$$

$$BEB^* = \begin{pmatrix} 0 & 0 & -\bar{b}_{33} \\ 0 & i & 0 \\ b_{33} & 0 & 0 \end{pmatrix}. \tag{3.13}$$

Similar to the case (i), the canonical form of this case is

$$A = z\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 1 & 2\bar{z}_2 \\ 1 & -iz_2 & r_1 \end{pmatrix}; \; B = \bar{z}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & iz_2 & -\bar{z}_1 \end{pmatrix},$$

where

$$z, z_1, z_2 \in \mathbb{C} \quad \textit{and} \quad r_1 \in \mathbb{R}.$$

Using Theorem 2 the canonical form for the real coupled self-adjoint boundary conditions can be obtained simply in the following corollary.

COROLLARY 1. *For the third order regular differential operator, every real coupled self-adjoint boundary condition is equivalent to the following canonical form*

$$Y(b) = KY(a), \quad K = \begin{pmatrix} \gamma_1 & 0 & \gamma_2 \\ 0 & 1 & 0 \\ \gamma_3 & 0 & \gamma_4 \end{pmatrix}, \tag{3.14}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{R}$, $det(K) = 1$.

Proof. Given a coupled self-adjoint boundary condition, it is equivalent to one of the two canonical forms given by Theorem 2. Without loss of generality, assume that the coupled self-adjoint is equivalent to (3.5). Since the coupled self-adjoint boundary condition is real, it is easy to know $z, z_2 \in \mathbb{R}$, $z_1 = z_3 = 0$. In this case, (3.5) has the following transformation

$$\begin{pmatrix} z & 0 & zr_1 & zz_2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & zz_2 & zr_2 & 0 & z \end{pmatrix} \to \begin{pmatrix} z_2^{-1} & 0 & z_2^{-1}r_1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & z_2 & r_2 & 0 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} z_2^{-1} & 0 & z_2^{-1}r_1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -r_2z_2^{-1} & 0 & z_2 - z_2^{-1}r_1r_2 & 0 & 0 & 1 \end{pmatrix} \overset{rewrite}{\to} \begin{pmatrix} \gamma_1 & 0 & \gamma_2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \gamma_3 & 0 & \gamma_4 & 0 & 0 & 1 \end{pmatrix}, \tag{3.15}$$

where $\gamma_1\gamma_4 - \gamma_2\gamma_3 = 1$.

In addition, if $r_1 = 0$, then (3.15) is the real coupled canonical form corresponding to (3.6).

4. Mixed self-adjoint boundary conditions

In this section, we prove that the mixed self-adjoint boundary conditions can be realized, furthermore, every mixed self-adjoint boundary condition is equivalent to one of the four canonical forms given below.

By Definition 1, if the mixed self-adjoint boundary conditions can be realized, then $(A:B)$ can be transformed to

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{4.1}$$

and the form $(A:B)$ given by (4.1) satisfies the self-adjointness condition (1.12). Similar to the coupled case, we give the following classification to verify whether $(A:B)$ satisfies the self-adjointness condition (1.12).

LEMMA 4. *Assume $A, B \in M_3(\mathbb{C})$ satisfy $rank(A:B) = 3$, $rank(A) = rank(B) = 2$. Then (4.1) is equivalent to one of the following four cases:*
Case (i):

$$(A:B) = \begin{pmatrix} 1 & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & b_{21} & b_{22} & 0 \\ 0 & 0 & 0 & b_{31} & b_{32} & 1 \end{pmatrix}; \tag{4.2}$$

Case (ii):

$$(A:B) = \begin{pmatrix} 1 & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & 0 \end{pmatrix}; \tag{4.3}$$

Case (iii):

$$(A:B) = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & b_{21} & b_{22} & 0 \\ 0 & 0 & 0 & b_{31} & b_{32} & 1 \end{pmatrix}; \tag{4.4}$$

Case (iv):

$$(A:B)=\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & 0 \end{pmatrix}. \tag{4.5}$$

Proof. 1. Assume $a_{11}\neq 0$, by a transformation of rows, (4.1) is equivalent to

$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix}. \tag{4.6}$$

Then, we give a classification of (4.6) by discussion the last column of B.
(i) If $b_{33}\neq 0$, (4.6) has the following form by a transformation of rows

$$(4.6)\rightarrow\begin{pmatrix} 1 & \widetilde{a}_{12} & \widetilde{a}_{13} & 0 & 0 & 0 \\ 0 & \widetilde{a}_{22} & \widetilde{a}_{23} & \widetilde{b}_{21} & \widetilde{b}_{22} & 0 \\ 0 & 0 & 0 & \widetilde{b}_{31} & \widetilde{b}_{32} & 1 \end{pmatrix} \overset{rewrite}{\rightarrow} \begin{pmatrix} 1 & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & b_{21} & b_{22} & 0 \\ 0 & 0 & 0 & b_{31} & b_{32} & 1 \end{pmatrix}.$$

This is the case (i).
(ii) If $b_{33}=0$, then (4.6) has the following form

$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & 0 \end{pmatrix}.$$

This is the case (ii).

2. Assume $a_{11}=0$, then (4.1) has the following form

$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix}. \tag{4.7}$$

Similarly, we obtain case (iii) and case (iv) depending on whether $b_{33}\neq 0$ or $b_{33}=0$ respectively.

THEOREM 3. *For the third order regular differential operator, every mixed self-adjoint boundary condition is equivalent to one of the following four canonical forms:*

(i)

$$A=z\begin{pmatrix} 1 & -iz_1 & r_1 \\ 0 & 1 & 2\overline{z}_1 \\ 0 & -iz_2 & -iz_2(2\overline{z}_1) \end{pmatrix};\ B=\overline{z}\begin{pmatrix} iz_1(2\overline{z}_2) & iz_1 & 0 \\ 2\overline{z}_2 & 1 & 0 \\ r_2 & iz_2 & 1 \end{pmatrix}, \tag{4.8}$$

where

$$z,z_1,z_2\in\mathbb{C}\ \ and\ \ r_1,r_2\in\mathbb{R};$$

(ii)

$$A=z\begin{pmatrix} 1 & -iz_1 & r_1 \\ 0 & 1 & 2\overline{z}_1 \\ 0 & 0 & 0 \end{pmatrix};\ B=\overline{z}\begin{pmatrix} 0 & iz_1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{4.9}$$

where

$$z, z_1 \in \mathbb{C} \text{ and } r_1 \in \mathbb{R};$$

(iii)

$$A = z\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -iz_1 & 0 \end{pmatrix}; \ B = \bar{z}\begin{pmatrix} 0 & 0 & 0 \\ 2\bar{z}_1 & 1 & 0 \\ r_1 & iz_1 & 1 \end{pmatrix}, \tag{4.10}$$

where

$$z, z_1 \in \mathbb{C} \text{ and } r_1 \in \mathbb{R};$$

(iv)

$$A = z\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ B = \bar{z}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.11}$$

where

$$z \in \mathbb{C}.$$

Proof. (i) If $(A:B)$ is in the case (i) of Lemma 4, we have the following equations by a direct calculation

$$AE_3A^* = \begin{pmatrix} ia_{12}\bar{a}_{12} - \bar{a}_{13} + a_{13} & ia_{12}\bar{a}_{22} - \bar{a}_{23} & 0 \\ ia_{22}\bar{a}_{12} + a_{23} & ia_{22}\bar{a}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{4.12}$$

$$BE_3B^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & ib_{22}\bar{b}_{22} & ib_{22}\bar{b}_{32} - b_{21} \\ 0 & ib_{32}\bar{b}_{22} + \bar{b}_{21} & ib_{32}\bar{b}_{32} - b_{31} + \bar{b}_{31} \end{pmatrix}. \tag{4.13}$$

Assume the self-adjointness conditions (1.12) can be satisfied, then

$$\begin{pmatrix} ia_{12}\bar{a}_{12} - \bar{a}_{13} + a_{13} & ia_{12}\bar{a}_{22} - \bar{a}_{23} & 0 \\ ia_{22}\bar{a}_{12} + a_{23} & ia_{22}\bar{a}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & ib_{22}\bar{b}_{22} & ib_{22}\bar{b}_{32} - b_{21} \\ 0 & ib_{32}\bar{b}_{22} + \bar{b}_{21} & ib_{32}\bar{b}_{32} - b_{31} + \bar{b}_{31} \end{pmatrix}. \tag{4.14}$$

In order to make our calculations more transparent, let $a_{12} = -2i\widetilde{a_{12}}$, $b_{32} = 2i\widetilde{b_{32}}$, $\widetilde{a_{12}}, \widetilde{b_{32}} \in \mathbb{C}$. By (4.14), we have $a_{23} = 2r\overline{\widetilde{a_{12}}}e^{i\varphi_1}, b_{21} = 2r\overline{\widetilde{b_{32}}}e^{i\varphi_2}, a_{13} = r_1 - 2i|\widetilde{a_{12}}|^2, b_{31} = r_2 + 2i|\widetilde{b_{32}}|^2$, where $r_1, r_2 \in \mathbb{R}$. Putting these parameters into (4.2), by a transformation of rows, we have

$$\begin{pmatrix} 1 & -2i\widetilde{a_{12}} & r_1 - 2i|\widetilde{a_{12}}|^2 & 0 & 0 & 0 \\ 0 & re^{i\varphi_1} & 2r\overline{\widetilde{a_{12}}}e^{i\varphi_1} & 2r\overline{\widetilde{b_{32}}}e^{i\varphi_2} & re^{i\varphi_2} & 0 \\ 0 & 0 & 0 & r_2 + 2i|\widetilde{b_{32}}|^2 & 2i\widetilde{b_{32}} & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & -i\widetilde{a_{12}} & r_1 & 2i\widetilde{a_{12}}\overline{\widetilde{b_{32}}}e^{i(\varphi_2-\varphi_1)} & i\widetilde{a_{12}}e^{i(\varphi_2-\varphi_1)} & 0 \\ 0 & re^{i\varphi_1} & 2r\overline{\widetilde{a_{12}}}e^{i\varphi_1} & 2r\overline{\widetilde{b_{32}}}e^{i\varphi_2} & re^{i\varphi_2} & 0 \\ 0 & -i\widetilde{b_{32}}e^{i(\varphi_1-\varphi_2)} & -2i\widetilde{b_{32}}\overline{\widetilde{a_{12}}}e^{i(\varphi_1-\varphi_2)} & r_2 & i\widetilde{b_{32}} & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} e^{i\frac{\varphi_1-\varphi_2}{2}} & -i\widetilde{a_{12}}e^{i\frac{\varphi_1-\varphi_2}{2}} & r_1e^{i\frac{\varphi_1-\varphi_2}{2}} & 2i\widetilde{a_{12}}\overline{\widetilde{b_{32}}}e^{i\frac{\varphi_2-\varphi_1}{2}} & i\widetilde{a_{12}}e^{i\frac{\varphi_2-\varphi_1}{2}} & 0 \\ 0 & e^{i\frac{\varphi_1-\varphi_2}{2}} & 2\overline{\widetilde{a_{12}}}e^{i\frac{\varphi_1-\varphi_2}{2}} & 2\overline{\widetilde{b_{32}}}e^{i\frac{\varphi_2-\varphi_1}{2}} & e^{i\frac{\varphi_2-\varphi_1}{2}} & 0 \\ 0 & -i\widetilde{b_{32}}e^{i\frac{\varphi_1-\varphi_2}{2}} & -2i\widetilde{b_{32}}\overline{\widetilde{a_{12}}}e^{i\frac{\varphi_1-\varphi_2}{2}} & r_2e^{i\frac{\varphi_2-\varphi_1}{2}} & i\widetilde{b_{32}}e^{i\frac{\varphi_2-\varphi_1}{2}} & e^{i\frac{\varphi_2-\varphi_1}{2}} \end{pmatrix}.$$

Let $z = e^{\frac{\varphi_1-\varphi_2}{2}}, \widetilde{a_{12}} = z_1, \widetilde{b_{32}} = z_2$, the canonical form of this case is

$$A = z\begin{pmatrix} 1 & -iz_1 & r_1 \\ 0 & 1 & 2\bar{z}_1 \\ 0 & -iz_2 & -iz_2(2\bar{z}_1) \end{pmatrix}; \; B = \bar{z}\begin{pmatrix} iz_1(2\bar{z}_2) & iz_1 & 0 \\ 2\bar{z}_2 & 1 & 0 \\ r_2 & iz_2 & 1 \end{pmatrix},$$

where

$$z, z_1, z_2 \in \mathbb{C} \; and \; r_1, r_2 \in \mathbb{R}.$$

(ii) If $(A : B)$ is in the case (ii) of Lemma 4, we have the following equation by a calculation

$$BE_3B^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & ib_{22}\bar{b}_{22} + b_{23}\bar{b}_{21} - b_{21}\bar{b}_{23} & ib_{22}\bar{b}_{32} + b_{23}\bar{b}_{31} \\ 0 & ib_{32}\bar{b}_{22} - b_{31}\bar{b}_{23} & i\bar{b}_{32}b_{32} \end{pmatrix}. \tag{4.15}$$

Similar to the case (i), the canonical form of this case is

$$A = z\begin{pmatrix} 1 & -iz_1 & r_1 \\ 0 & 1 & 2\bar{z}_1 \\ 0 & 0 & 0 \end{pmatrix}; \; B = \bar{z}\begin{pmatrix} 0 & iz_1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where

$$z, z_1 \in \mathbb{C} \; and \; r_1 \in \mathbb{R}.$$

(iii) If $(A : B)$ is in the case (iii) of Lemma 4, we have the following equation by a calculation

$$AE_3A^* = \begin{pmatrix} ia_{12}\bar{a}_{12} & ia_{12}\bar{a}_{22} + a_{13}\bar{a}_{21} & 0 \\ ia_{22}\bar{a}_{12} - a_{21}\bar{a}_{13} & ia_{22}\bar{a}_{22} + a_{23}\bar{a}_{21} - a_{21}\bar{a}_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

Similar to the case (i), the canonical form of this case is

$$A = z\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -iz_1 & 0 \end{pmatrix}; \; B = \bar{z}\begin{pmatrix} 0 & 0 & 0 \\ 2\bar{z}_1 & 1 & 0 \\ r_1 & iz_1 & 1 \end{pmatrix},$$

where

$$z, z_1 \in \mathbb{C} \; and \; r_1 \in \mathbb{R}.$$

(iv) If $(A : B)$ is in the case (iv) of Lemma 4, then by (4.15) and (4.16), it is easy to know that $a_{12} = b_{32} = 0$. Since $ib_{22}\bar{b}_{32} + b_{23}\bar{b}_{31} = 0, b_{32} = 0, rank(B) = 2$, then $b_{31} \neq 0, b_{23} = 0$. The case of $a_{23} = 0$ can be proved in the same way. In summary, we can get $a_{12} = a_{23} = b_{32} = b_{23} = 0$. Similar to the case (i), the canonical form of this case is

$$A = z\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; B = \bar{z}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

where $z \in \mathbb{C}$.

According to Theorem 3, we can obtain canonical form for the real mixed self-adjoint boundary conditions simply.

COROLLARY 2. *For the third order regular differential operator, every real mixed self-adjoint boundary condition is equivalent to the following canonical form*

$$AY(a)+BY(b)=0, \tag{4.17}$$

$$(A:B)=\begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cos\beta & 0 & \sin\beta \end{pmatrix}, \tag{4.18}$$

where $-\frac{\pi}{2}<\alpha,\beta\leqslant\frac{\pi}{2}$.

Proof. Given a mixed self-adjoint boundary condition, it is equivalent to one of the four canonical forms given by Theorem 3. Without loss of generality, assume that the mixed self-adjoint is equivalent to (4.8). Since the mixed self-adjoint boundary condition is real, it is easy to know $z\in\mathbb{R}$, $z_1=z_2=0$. In this case, (4.8) has the following transformation

$$\begin{aligned} &\begin{pmatrix} z & 0 & zr_1 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & z & 0 \\ 0 & 0 & 0 & zr_2 & 0 & z \end{pmatrix} \to \begin{pmatrix} 1 & 0 & r_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r_2 & 0 & 1 \end{pmatrix} \to \\ &\begin{pmatrix} \frac{1}{1+r_1^2} & 0 & \frac{r_1}{1+r_1^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{r_2}{1+r_2^2} & 0 & \frac{1}{1+r_2^2} \end{pmatrix} \overset{rewrite}{\longrightarrow} \begin{pmatrix} cos\alpha & 0 & sin\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & sin\beta & 0 & cos\beta \end{pmatrix}. \end{aligned} \tag{4.19}$$

where $-\frac{\pi}{2}<\alpha,\beta<\frac{\pi}{2}$.

In addition, if $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, $\beta=\frac{\pi}{2}$. Then (4.18) is the real mixed canonical form corresponding to (4.9).

If $-\frac{\pi}{2}<\beta<\frac{\pi}{2},\alpha=\frac{\pi}{2}$. Then (4.18) is the real mixed canonical form corresponding to (4.10).

If $\beta=\alpha=0$. Then (4.18) is the real mixed canonical form corresponding to (4.11).

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