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ON THE NUMERICAL RANGE AND OPERATOR NORM OF V^2

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Abstract. Let V denote the classical Volterra operator on $L^2(0,1)$. We investigate the numerical range and operator norm of V^2. In particular, we obtain the numerical range, numerical radius and norm of $\mathrm{Re}V^2$ and $\mathrm{Im}V^2$.

1. Introduction

Let H be a complex Hilbert space equipped with the inner product $(\cdot,\cdot)$, which induces the norm $\|\cdot\|$. Denote by $B(H)$ the Banach algebra of bounded linear operators acting on H with the operator norm defined by

$$\|A\| = \sup\{\|Ax\| : x \in H, \|x\|| = 1\}, A \in B(H).$$

For a bounded linear operator A on a complex Hilbert space H, the numerical range $W(A)$ is the image of the unit sphere of H under the quadratic form $x \longrightarrow (Ax,x)$ associated with the operator. More precisely,

$$W(A) = \{(Ax,x) : x \in H, \|x\| = 1\}.$$

It is well known that numerical range of an operator is convex (The Toeplitz-Hausdorf theorem) and the spectrum is contained in the closure of its numerical range. Note that A is a self-adjoint if and only if $W(A) \subset \mathbb{R}$.

The numerical radius of an operator A is defined by

$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$$

and the following inequalities

$$\frac{\|A\|}{2} \leqslant \omega(A) \leqslant \|A\|$$

hold. (see [1])

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Denote by V the classical Volterra operator

$$(Vf)(x) = \int_0^x f(t)dt, \quad f \in L^2(0,1).$$

The adjoint of the Volterra operator is

$$(V^* f)(x) = \int_x^1 f(t)dt.$$

We recall the well-known formula

$$(V^n f)(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt$$

and

$$(V^{*n} f)(x) = \int_x^1 \frac{(t-x)^{n-1}}{(n-1)!} f(t)dt$$

for $n \in \mathbb{N}$.

The Volterra operator is compact, quasinilpotent and accretive. (see [1], [2])

Recall that $W(V)$ is bounded by the curve

$$t \mapsto \frac{1-\cos t}{t^2} \pm \frac{t-\sin t}{t^2}, \quad 0 \leqslant t \leqslant 2\pi$$

and $||V|| = \frac{2}{\pi}$. (see [2], [4])

The aim of this paper is to study the numerical range and operator norm of V^2 on $L^2(0,1)$. In particular, we will obtain the numerical range, numerical radius and operator norm of $\mathrm{Re}V^2$ and $\mathrm{Im}V^2$.

We will need the following theorem.

THEOREM 1.1. [1, page 268], [3, page 37] *If A is a bounded operator on H and $\theta \in [-\pi,\pi]$, put $\lambda_\theta = \max\sigma(B_\theta)$, where $B_\theta = \frac{1}{2}(e^{-i\theta}A + e^{i\theta}A^*) = B_\theta^*$. Then*

$$\overline{W(A)} = \bigcap_{\theta \in [-\pi,\pi]} H_\theta,$$

where the half-space H_θ is defined by

$$H_\theta = \{z \in \mathbb{C} : \mathrm{Re}(e^{-i\theta}z) \leqslant \lambda_\theta\}, \quad z = x + iy.$$

2. The results

PROPOSITION 2.1. *According to Theorem 1.1 and under the assumption* $\lambda_\theta \in C^1[-\pi,\pi]$, *we have*

$$\begin{cases} x = \lambda_\theta \cos\theta - \lambda_\theta' \sin\theta \\ y = \lambda_\theta \sin\theta + \lambda_\theta' \cos\theta, \end{cases} \tag{2.1}$$

which is an envelope curve.

Proof. Fix θ and consider the supporting line L_θ of $W(A)$ given by

$$x\cos\theta + y\sin\theta = \lambda_\theta.$$

Now fix x,y such that $x+iy \in L_\theta \cap W(A)$. So we have

$$x\cos\theta + y\sin\theta = \lambda_\theta \tag{2.2}$$

and also, since $x+iy \in W(A)$,

$$x\cos\xi + y\sin\xi \leqslant \lambda_\xi, \ \ \xi \in [-\pi,\pi]. \tag{2.3}$$

Now, for any $h > 0$, $x\cos(\theta+h) + y\sin(\theta+h) \leqslant \lambda_{\theta+h}$ by (2.3). Together with the equality (2.2) and that $h > 0$, we get

$$x\frac{\cos(\theta+h)-\cos\theta}{h} + y\frac{\sin(\theta+h)-\sin\theta}{h} \leqslant \frac{\lambda_{\theta+h}-\lambda_\theta}{h}. \tag{2.4}$$

Taking limit as $h \to 0$, we obtain

$$-x\sin\theta + y\cos\theta \leqslant \lambda_\theta'. \tag{2.5}$$

If we now repeat the argument with $h < 0$, we obtain

$$-x\sin\theta + y\cos\theta \geqslant \lambda_\theta'. \tag{2.6}$$

Combining (2.5) and (2.6) we have

$$-x\sin\theta + y\cos\theta = \lambda_\theta'. \tag{2.7}$$

Now (2.7) and (2.2) give a linear system on x,y that allows us to obtain (2.1). ☐

PROPOSITION 2.2. *If* $z \in W(V^n)$, *then implies that* $\overline{z} \in W(V^n)$ *for all* $n \in \mathbb{N}$.

Proof. Suppose that $\forall z \in W(V^n)$, $\exists f$ such that $||f|| = 1$ and $z = (V^n f, f)$ for $n \in \mathbb{N}$. Then

$$\overline{z} = \overline{\int_0^1 (V^n f(x))\overline{f}(x)dx} = \int_0^1 (V^n \overline{f}(x)) f(x)dx = (V^n\overline{f},\overline{f}), ||\overline{f}|| = 1. \quad \square$$

We consider the numerical range of V^2.

THEOREM 2.1. *The boundary of $W(V^2)$ is union of the curves γ_1 and γ_2, where*

$$\gamma_1 : \begin{cases} x = x(\mu) = \frac{1}{\mu^2} - \frac{16-\varphi^2(\mu)}{2\mu^3\varphi'(\mu)} \\ y = y(\mu) = \pm\frac{4+\varphi(\mu)}{2\mu^3\varphi'(\mu)}\sqrt{16-\varphi^2(\mu)} \end{cases} \tag{2.8}$$

$$\varphi(\mu) = (\mu-2)e^{\mu} - (\mu+2)e^{-\mu},\ \mu \in (0,\mu_0],\ e^{\mu_0} = \frac{\mu_0+2}{\mu_0-2}$$

and

$$\gamma_2 : \begin{cases} x = x(\mu) = -\frac{1}{\mu^2} + \frac{4-\psi^2(\mu)}{\mu^3\psi'(\mu)} \\ y = y(\mu) = \pm\frac{2+\psi(\mu)}{\mu^3\psi'(\mu)}\sqrt{4-\psi^2(\mu)} \end{cases} \tag{2.9}$$

$$\psi(\mu) = -2\cos\mu - \mu\sin\mu,\ \mu \in (0,\pi],\ x(0) = \frac{1}{30},\ y(0) = \frac{\sqrt{3}}{12}.$$

Proof. Put $A = V^2$ into Theorem 1.1. Let $0 \leqslant \theta \leqslant \pi$. The spectral problem is

$$e^{-i\theta}\int_0^x (x-t)f(t)dt + e^{i\theta}\int_x^1 (t-x)f(t)dt = 2\lambda f(x). \tag{2.10}$$

We proceed from this integral equation to a differential equation by applying the operator $D = \frac{d}{dx}$ twice. Thus,

$$e^{-i\theta}\int_0^x f(t)dt - e^{i\theta}\int_x^1 f(t)dt = 2\lambda f'(x) \tag{2.11}$$

and

$$f''(x) - \frac{\cos\theta}{\lambda}f(x) = 0.$$

(if $\lambda = 0$, then $f = 0$)

Substituting $x = 0$ and $x = 1$ into (2.10), we obtain

$$e^{i\theta}\int_0^1 tf(t)dt = 2\lambda f(0),$$

$$e^{-i\theta}\int_0^1 (1-t)f(t)dt = 2\lambda f(1).$$

Eliminating above the boundary conditions, we get

$$\int_0^1 f(t)dt = 2\lambda(e^{-i\theta}f(0) + e^{i\theta}f(1)).$$

Similarly, substituting $x = 0$ and $x = 1$ into (2.11), we obtain

$$-e^{i\theta}\int_0^1 f(x)dx = 2\lambda f'(0)$$

and

$$e^{-i\theta}\int_0^1 f(x)dx = 2\lambda f^{'}(1)$$

which implies

$$e^{-2i\theta}f(0)+f(1) = -e^{-2i\theta}f^{'}(0) = f^{'}(1).$$

The spectral problem (2.8) is equivalent to the following differential equation with the following boundary conditions

$$f^{''}(x) - \frac{\cos\theta}{\lambda}f(x) = 0, \quad (\lambda \neq 0, \theta \in [-\pi,\pi]) \tag{2.12}$$

$$-e^{-2i\theta}f^{'}(0) = f^{'}(1), \tag{2.13}$$

$$e^{-2i\theta}f(0)+f(1) = -e^{-2i\theta}f^{'}(0). \tag{2.14}$$

The case $\frac{\cos\theta}{\lambda} = \mu^2 \ (\mu > 0)$. The solution of (2.12) is

$$f(x) = e^{\mu x} + \alpha e^{-\mu x}. \ (\alpha = const)$$

From (2.13) and (2.14), we obtain

$$\alpha = \frac{1+e^{\mu}e^{2i\theta}}{1+e^{-\mu}e^{2i\theta}}$$

and

$$\alpha = \frac{e^{-2i\theta}+e^{\mu}+\mu e^{-2i\theta}}{\mu e^{-2i\theta}-e^{-2i\theta}-e^{-\mu}},$$

respectively. Then

$$4\cos 2\theta = (\mu-2)e^{\mu} - (\mu+2)e^{-\mu}. \tag{2.15}$$

The case $\frac{\cos\theta}{\lambda} = -\mu^2. \ (\mu > 0)$ The solution of (2.12) is

$$f(x) = \cos\mu x + \alpha\sin\mu x \ (\alpha = \text{const})$$

From (2.13) and (2.14), we obtain

$$\alpha = \frac{e^{2i\theta}\sin\mu}{1+e^{2i\theta}\cos\mu}$$

and

$$\alpha = -\frac{e^{-2i\theta}+\cos\mu}{\mu e^{-2i\theta}+\sin\mu},$$

respectively. Then

$$2\cos 2\theta = -2\cos\mu - \mu\sin\mu. \tag{2.16}$$

Thus,

$$\lambda_\theta = \max\left\{\frac{\cos\theta}{\mu^2}, -\frac{\cos\theta}{\mu^2}\right\} = \begin{cases} \frac{\cos\theta}{\mu^2}, & 0 < \theta \leqslant \frac{\pi}{2} \\ -\frac{\cos\theta}{\mu^2}, & \frac{\pi}{2} < \theta \leqslant \pi. \end{cases} \tag{2.17}$$

Suppose that $0 < \theta \leqslant \frac{\pi}{2}$. Denote

$$\varphi(\mu) = (\mu - 2)e^{\mu} - (\mu + 2)e^{-\mu}, \mu \geqslant 0.$$

It is easy to see that φ is an increasing function, that is

$$\min_{\mu \geqslant 0} \varphi(\mu) = \varphi(0) = -4.$$

Denote by μ_0 the solution of the equation

$$\varphi(\mu) = 4.$$

Thus,

$$\varphi(\mu_0) = 4 \Leftrightarrow \frac{\mu_0}{2} = \coth\left(\frac{\mu_0}{2}\right) \Leftrightarrow e^{\mu_0} = \frac{\mu_0 + 2}{\mu_0 - 2}. \quad (\mu_0 \approx 2.399)$$

Therefore, $\mu \in (0, \mu_0]$. By (2.15) implies that if $\theta \to 0$ and $\theta \to \frac{\pi}{2}$, then $\mu \to \mu_0$ and $\mu \to 0$, respectively.

From (2.15), we obtain

$$-8(\sin 2\theta)\theta'_\mu = \varphi'(\mu) \Rightarrow \theta'_\mu = -\frac{\varphi'(\mu)}{8\sin 2\theta}.$$

Recall that $\lambda = \frac{\cos\theta}{\mu^2}$. Then

$$\lambda'_\mu = -\frac{\mu^2(\sin\theta)\theta'_\mu + 2\mu\cos\theta}{\mu^4} = -\frac{\sin\theta}{\mu^2}\theta'_\mu - \frac{2\cos\theta}{\mu^3}$$

implies that

$$\lambda'_\theta = \frac{\lambda'_\mu}{\theta'_\mu} = -\frac{\sin\theta}{\mu^2} - \frac{2\cos\theta}{\mu^3\theta'_\mu} = -\frac{\sin\theta}{\mu^2} + \frac{16\cos\theta\sin 2\theta}{\mu^3\varphi'(\mu)}.$$

By (2.1) and (2.15), we get

$$\gamma_1 : \begin{cases} x = \lambda\cos\theta - \lambda'_\theta\sin\theta = \frac{1}{\mu^2} - \frac{16 - \varphi^2(\mu)}{2\mu^3\varphi'(\mu)} \\ y = \lambda\sin\theta + \lambda'_\theta\cos\theta = \frac{4 + \varphi(\mu)}{2\mu^3\varphi'(\mu)}\sqrt{16 - \varphi^2(\mu)}. \end{cases}$$

Suppose $\frac{\pi}{2} < \theta \leqslant \pi$. Denote

$$\psi(\mu) = -2\cos\mu - \mu\sin\mu. \ (\mu \in (0, \pi])$$

Similarly, to the previous case ψ is increasing function and $-2 = \psi(0) \leqslant \psi(\mu) \leqslant \psi(\pi) = 2$. If $\theta \to \frac{\pi}{2}$ and $\theta \to \pi$ then $\mu \to 0$ and $\mu \to \pi$, respectively.

From (2.16), we obtain

$$\theta_\mu' = -\frac{\psi'(\mu)}{4\sin 2\theta},$$
$$\lambda_\mu' = \left(-\frac{\cos\theta}{\mu^2}\right)_\mu' = \frac{\sin\theta}{\mu^2}\theta_\mu' + \frac{2\cos\theta}{\mu^3}$$

and

$$\lambda_\theta' = \frac{\lambda_\mu'}{\theta_\mu'} = \frac{\sin\theta}{\mu^2} - \frac{8\cos\theta\sin 2\theta}{\mu^3\psi'(\mu)}.$$

By (2.1) and (2.16), we get

$$\gamma_2 : \begin{cases} x = x\cos\theta - \lambda_\theta'\sin\theta = -\frac{1}{\mu^2} + \frac{4-\psi^2(\mu)}{\mu^3\psi'(\mu)} \\ y = x\sin\theta + \lambda_\theta'\cos\theta = \pm\frac{2+\psi(\mu)}{\mu^3\psi'(\mu)}\sqrt{4-\psi^2(\mu)}. \end{cases}$$

It is easy to see that

$$x = \frac{1}{\mu^2} - \frac{16-\varphi^2(\mu)}{2\mu^3\varphi'(\mu)} = \frac{\frac{1}{30} + \frac{19}{840}\mu^2 + o(\mu^4)}{1 + \frac{1}{10}\mu^2 + o(\mu^4)} \Rightarrow \lim_{\mu\to 0} x = \frac{1}{30}$$

and

$$y = \frac{4+\varphi(\mu)}{2\mu^3\varphi'(\mu)}\sqrt{16-\varphi^2(\mu)} = \frac{\frac{\mu^4}{6}\left(1 + \frac{1}{15}\mu^2 + o(\mu^4)\right)}{\frac{4}{3}\mu^6\left(1 + \frac{1}{10}\mu^2 + o(\mu^4)\right)}\frac{2\mu^2}{\sqrt{3}}\sqrt{1 + \frac{\mu^2}{15} + o(\mu^4)}$$
$$\Rightarrow \lim_{\mu\to 0} y = \frac{\sqrt{3}}{12}$$

for γ_1.

In the case of γ_2,

$$x = -\frac{1}{\mu^2} + \frac{4-\psi^2(\mu)}{\mu^3\psi'(\mu)} = \frac{\frac{1}{30} + o(\mu^2)}{1 - \frac{\mu^2}{10} + o(\mu^4)} \Rightarrow \lim_{\mu\to 0} x = \frac{1}{30}$$

and

$$y = \frac{2+\psi(\mu)}{\mu^3\psi'(\mu)}\sqrt{4-\psi^2(\mu)} = \frac{\frac{\mu^4}{12}\left(1 - \frac{\mu^2}{15} + o(\mu^4)\right)}{\frac{\mu^6}{3}\left(1 - \frac{\mu^2}{10} + o(\mu^4)\right)}\frac{\mu^2}{\sqrt{3}}\sqrt{1 - \frac{\mu^2}{15} + o(\mu^4)}$$
$$\Rightarrow \lim_{\mu\to 0} y = \frac{\sqrt{3}}{12}$$

for γ_2.

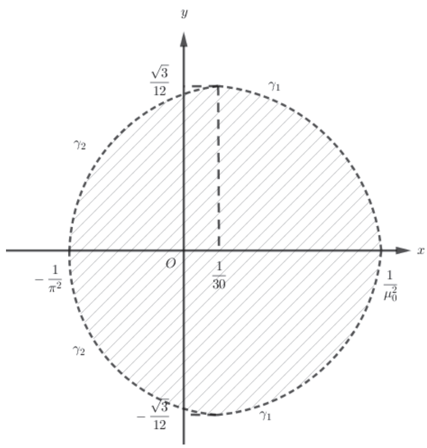
If $\mu = \mu_0$ then $x = \frac{1}{\mu_0^2}, y = 0$ for γ_1. Similarly, if $\mu = \pi$ then $x = -\frac{1}{\pi^2}, y = 0$ for γ_2. This completes the proof. □

COROLLARY 2.1. *We have*

$$a)\ W(ReV^2) = \left[-\frac{1}{\pi^2}, \frac{1}{\mu_0^2}\right],\ \omega(ReV^2) = \frac{1}{\mu_0^2},\ ||ReV^2|| = \frac{1}{\mu_0^2}.$$

$$b)\ W(ImV^2) = \left[-\frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{12}\right],\ \omega(ImV^2) = \frac{\sqrt{3}}{12},\ ||ImV^2|| = \frac{\sqrt{3}}{12}.$$

We consider numerical range of V^2 on the following picture.



THEOREM 2.2.

$$||V^2|| = \frac{1}{\tau^2},\ \left(\frac{1}{\tau^2} \approx 0.2844\right)$$

where τ *is the smallest positive root of the equation*

$$1 + \cos v \cosh v = 0. \tag{2.18}$$

Proof. It is easy to see that

$$\begin{aligned}(V^{*2}V^2 f)(x) &= \int_x^1 (t-x)dt \int_0^t (t-s)f(s)ds \\ &= \int_0^x f(s)ds \int_x^1 (t-x)(t-s)dt + \int_x^1 f(s)ds \int_s^1 (t-x)(t-s)dt.\end{aligned}$$

We will determine the norm of V^2 by calculating the largest eigenvalue of $V^{*2}V^2$, that is we looking for the largest $\lambda > 0$ such that $V^{*2}V^2 f = \lambda f$ for some non-zero $f \in L^2(0,1)$. We can rewrite the spectral problem

$$\int_0^x f(s)ds \int_x^1 (t-x)(t-s)dt + \int_x^1 f(s)ds \int_s^1 (t-x)(t-s)dt = \lambda f(x). \tag{2.19}$$

We proceed from this integral equation to a differential equation by applying the operator $D = \frac{d}{dx}$ four times,

$$-\frac{1}{2}\int_0^1 (1-s)^2 f(s)ds + \frac{1}{2}\int_0^x (x-s)^2 f(s)ds = \lambda f^{'}(x), \tag{2.20}$$

$$\int_0^x (x-s)f(s)ds = \lambda f^{''}(x), \tag{2.21}$$

$$\int_0^x f(s)ds = \lambda f^{'''}(x) \tag{2.22}$$

and

$$f(x) = \lambda f^{iv}(x). \tag{2.23}$$

Substituting $x = 1$ into (2.19), (2.20) and $x = 0$ into (2.21), (2.22), we obtain the following boundary conditions (2.24)

$$f(1) = f^{'}(1) = f^{''}(0) = f^{'''}(0) = 0. \tag{2.24}$$

The solution of (2.23) is

$$f(x) = e^{\nu x} + ae^{-\nu x} + b\cos\nu x + c\sin\nu x$$

where $\nu = \frac{1}{\sqrt[4]{\lambda}}$, $a,b,c-$const. Accounting the boundary conditions (2.24), we obtain (2.18). The completes the proof. □

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