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A NOTE ON A CONJECTURED SINGULAR VALUE INEQUALITY RELATED TO A LINEAR MAP

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Abstract. If $\begin{pmatrix} A & D \\ D^* & C \end{pmatrix}$ is positive semidefinite with each block $n\times n$, Lin conjectured that

$$s_j(\Phi(D)) \leqslant s_j(\Phi(A)\sharp\Phi(C)),\ \ j=1,\ldots,n,$$

where Φ is the linear map: $D\mapsto D+(\mathrm{tr}D)I_n$ and $s_j(D)$ denotes the j-th largest singular value of the matrix D. In this note, we confirm this conjecture when $n=2$.

1. Introduction

Throughout this paper, we let $\mathbb{M}_n$ and $\mathbb{M}_n^+$ denote the set of $n\times n$ complex matrices and the set of $n\times n$ positive semidefinite matrices, respectively. For any $A\in\mathbb{M}_n$, the conjugate transpose of A is denoted by A^*. The singular values of A, denoted by $s_1(A),s_2(A),\ldots,s_n(A)$, are the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{1/2}$, arranged in non-increasing order and repeated according to multiplicity as $s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant s_n(A)$. For a Hermitian matrix A, we write $A\geqslant 0$ to mean A is positive semidefinite. If $A\in\mathbb{M}_n^+$, then it has a unique positive semidefinite square root, which is denoted by $A^{\frac{1}{2}}$. I_n denotes the $n\times n$ identity matrix.

The geometric mean of two positive definite matrices $A,B\in\mathbb{M}_n$ is defined by $A\sharp B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$. More details on the matrix geometric mean can be found in [1, Chapter 4].

Consider

$$A=\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ldots & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix}, \tag{1}$$

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with each block in $\mathbb{M}_n$. For convenience, we use $A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ for (1). A linear map Φ on $\mathbb{M}_n$ is said to be m-positive if

$$[A_{i,j}]_{i,j=1}^m \geqslant 0 \quad \Rightarrow \quad [\Phi(A_{i,j})]_{i,j=1}^m \geqslant 0. \tag{2}$$

It is said to be completely positive if (2) is true for any integer $m \geqslant 1$.

Now we define a linear map on $\mathbb{M}_n$

$$\Phi : X \mapsto X + (\mathrm{tr}X)I_n \tag{3}$$

as in [2]. Many properties of this linear map Φ are given in [2]. For example, [2] showed that Φ is completely positive.

In [4], Lin proved the following result which is unsolved in [2].

THEOREM 1.1. *If* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$*, where* $A,B,C \in \mathbb{M}_n$*, is positive semidefinite, then*

$$2s_j(\Phi(B)) \leqslant s_j(\Phi(A+C)), \;\; j = 1,\ldots,n, \tag{4}$$

where Φ *is the linear map defined in* (3)*.*

Moreover, Lin also presented a conjecture in [4]:

CONJECTURE 1.2. If $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$, where $A,B,C \in \mathbb{M}_n$, is positive semidefinite, then

$$s_j(\Phi(B)) \leqslant s_j(\Phi(A)\sharp\Phi(C)), \;\; j = 1,\ldots,n,$$

where Φ is the linear map defined in (3).

In this note, the conjecture is confirmed when $n = 2$.

2. Main result and proofs

Now we present our result and prove it.

THEOREM 2.1. *If* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$*, where* $A,B,C \in \mathbb{M}_2$*, is positive semidefinite, then*

$$s_j(\Phi(B)) \leqslant s_j(\Phi(A)\sharp\Phi(C)), \;\; j = 1,2,$$

where linear map Φ *is defined in* (3) *and* $\Phi(A)$, $\Phi(C)$ *have the same eigenvalues.*

Proof. Since the linear map Φ is completely positive, then

$$\begin{pmatrix} \Phi(A) & \Phi(B) \\ \Phi(B^*) & \Phi(C) \end{pmatrix}$$

is positive semidefinite.

Without loss of generality, we suppose that $\Phi(A)$ and $\Phi(C)$ are 2×2 positive definite matrices with $\det\Phi(A)=\alpha^2>0$, $\det\Phi(C)=\beta^2>0$.

Thanks to [1, p. 111], we have

$$\Phi(A)\sharp\Phi(C)=\frac{\sqrt{\alpha\beta}(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}{\sqrt{\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}}, \tag{5}$$

which means that

$$s_j(\Phi(A)\sharp\Phi(C))=s_j\left(\frac{\sqrt{\alpha\beta}(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}{\sqrt{\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}}\right),\quad j=1,2.$$

Clearly,

$$\begin{pmatrix} \frac{1}{\alpha}\Phi(A) & \frac{1}{\sqrt{\alpha\beta}}\Phi(B) \\ \frac{1}{\sqrt{\alpha\beta}}\Phi(B^*) & \frac{1}{\beta}\Phi(C) \end{pmatrix}$$

is also a positive semidefinite matrix.
Hence by (4) and (5), we have

$$\begin{aligned} s_j\left(\frac{\sqrt{\alpha\beta}(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}{\sqrt{\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}}\right) &\geqslant \frac{\sqrt{\alpha\beta}s_j\left(\frac{2}{\sqrt{\alpha\beta}}\Phi(B)\right)}{\sqrt{\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}} \\ &= \frac{2s_j(\Phi(B))}{\sqrt{\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}}. \end{aligned} \tag{6}$$

Now we assume that $\Phi(A)$ has two positive eigenvalues λ_1, λ_2 and the eigenvalues of $\Phi(C)$ are positive numbers μ_1 and μ_2. Then,

$$\begin{aligned} \sqrt{\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))} &\leqslant \left(2+\frac{\lambda_1^2}{\alpha^2}+\frac{\lambda_2^2}{\alpha^2}\right)^{\frac{1}{4}}\left(2+\frac{\mu_1^2}{\beta^2}+\frac{\mu_2^2}{\beta^2}\right)^{\frac{1}{4}} \\ &= \sqrt{\frac{\lambda_1}{\alpha}+\frac{\lambda_2}{\alpha}}\sqrt{\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}} \leqslant \frac{\frac{\lambda_1}{\alpha}+\frac{\lambda_2}{\alpha}+\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}}{2}, \end{aligned} \tag{7}$$

where the first inequality follows from the conclusion [5, p.232]:

$$(\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C)))^2\leqslant\det(I+\alpha^{-2}\Phi(A)^2)\det(I+\beta^{-2}\Phi(C)^2),$$

and the second one follows by the mean value inequality.
By the inequalities (6) and (7), we have

$$\begin{aligned} s_j(\Phi(A)\sharp\Phi(C)) &= s_j\left(\frac{\sqrt{\alpha\beta}(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}{\sqrt{\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}}\right) \\ &\geqslant \frac{2s_j(\Phi(B))}{\sqrt{\det(\alpha^{-1}\Phi(A)+\beta^{-1}\Phi(C))}} \\ &\geqslant \frac{4s_2(\Phi(B))}{\frac{\lambda_1}{\alpha}+\frac{\lambda_2}{\alpha}+\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}}, \qquad j=1,2. \end{aligned}$$

Note that

$$\frac{\lambda_1}{\alpha}+\frac{\lambda_2}{\alpha}+\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta} \geqslant 4\sqrt[4]{\frac{\lambda_1}{\alpha}\cdot\frac{\lambda_2}{\alpha}\cdot\frac{\mu_1}{\beta}\cdot\frac{\mu_2}{\beta}} = 4$$

for all $\lambda_1,\lambda_2,\mu_1,\mu_2 > 0$ and the same eigenvalues of $\Phi(A)$ and $\Phi(C)$ we can take

$$\min_{\lambda_1,\lambda_2,\mu_1,\mu_2>0}\left(\frac{\lambda_1}{\alpha}+\frac{\lambda_2}{\alpha}+\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}\right) = 4.$$

Thus,

$$\begin{aligned} s_j(\Phi(A)\sharp\Phi(C)) &\geqslant \frac{4s_j(\Phi(B))}{\min\limits_{\lambda_1,\lambda_2,\mu_1,\mu_2>0}\left(\frac{\lambda_1}{\alpha}+\frac{\lambda_2}{\alpha}+\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}\right)} \\ &= s_j(\Phi(B)), \qquad j=1,2. \end{aligned}$$

Now if $\Phi(A)$ and $\Phi(C)$ are 2×2 positive semidefinite matrices, the definition of the geometric mean of $\Phi(A)$ and $\Phi(C)$ can be uniquely depicted by limit as follows:

$$\Phi(A)\sharp\Phi(C) := \lim_{\varepsilon\mapsto 0}(\Phi(A)+\varepsilon I_2)\sharp(\Phi(C)+\varepsilon I_2).$$

So the desired inequality can be proved similarly by replacing $\Phi(A)\sharp\Phi(C)$ with $\lim\limits_{\varepsilon\mapsto 0}((\Phi(A)+\varepsilon I_2)\sharp(\Phi(C)+\varepsilon I_2))$.

REMARK 2.2. Although we have not solved Lin's conjecture, our result is a step closer to the conjecture. When $n\geqslant 3$, we cannot prove the conjecture, because we don't know how to derive the expression of $\Phi(A)\sharp\Phi(C)$ as in (5). This is left for our future research.

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