Operators and Matrices
Volume 14, Number 1 (2020), 271–281 doi:10.7153/oam-2020-14-21

ON THE NORM OF HANKEL OPERATOR RESTRICTED TO FOCK SPACE

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(*Communicated by G. Misra*)

Abstract. In this note, we characterize the norm of Hankel operator $H_{\bar{z}}$. Then we find the formula of the norm of $H_{\bar{z}^n}(g)$ and give an upper bound of the norm of H_n on Fock space. Lastly, we prove the concomitant operator P_n of $H_{\bar{z}^n}$ is quasi-affine to the direct sum of n copies of the concomitant operator P_1 of $H_{\bar{z}}$.

1. Introduction

Let $\mathbb{C}$ be the complex plane. The Fock space F_α^2 (see [14]) consists of all entire functions f in $L^2(\mathbb{C}, d\lambda_\alpha)$, where $\alpha > 0$ and the Gaussian measure

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z),$$

dA is the Euclidean area measure on $\mathbb{C}$. It is easy to show that F_α^2 is a closed subspace of $L^2(\mathbb{C}, d\lambda_\alpha)$. F_α^2 is a Hilbert space. The inner product is defined by

$$\langle f, g\rangle_\alpha = \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda_\alpha(z).$$

The reproducing kernel of F_α^2 is given by $K_\alpha(z,w) = e^{\alpha z\bar{w}}$, $z, w \in \mathbb{C}$. For any $z \in \mathbb{C}$, we let

$$k_z(w) = \frac{K_\alpha(w,z)}{\sqrt{K_\alpha(z,z)}} = e^{\alpha \bar{z} w - \frac{\alpha}{2}|z|^2}$$

denote the normalized reproducing kernel at z. The Fock projection $P : L^2(\mathbb{C}, d\lambda_\alpha) \to F_\alpha^2$ is an integral operator defined by

$$Pf(z) = \int_{\mathbb{C}} K_\alpha(z,w) f(w) d\lambda_\alpha(w) \text{ for } f(z) \in L^2(\mathbb{C}, d\lambda_\alpha).$$

In [5], Haslinger researched the canonical solution operator to $\bar{\partial}$ restricted to Bergman spaces. He proved that in the case of the unit disc in $\mathbb{C}$ the canonical solution operator to $\bar{\partial}$ restricted to (0,1)-forms with holomorphic coefficients is a Hilbert-Schmidt operator. In 2002, Haslinger researched the canonical solution operator to $\bar{\partial}$

Mathematics subject classification (2010): 32A36, 47B35.
Keywords and phrases: Fock space, Hankel operator, norm, quasi-affinity.
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restricted to spaces of entire functions (see [6]). In 2006, Knirsch and Schneider researched generalized Hankel operators and the generalized solution operator to $\bar{\partial}$ on the Fock space and on the Bergman space of the unit disc (see [9]). Fu and Straube proved in [4] that compactness of the solution operator to $\bar{\partial}$ on (0,1)-forms implies that the boundary of a bounded domain Ω in $\mathbb{C}^n$ does not contain any analytic variety of dimension greater than or equal to 1.

It is well known that the canonical solution operator to $\bar{\partial}$-equation restricted to (0,1)-forms with holomorphic coefficients in the Bergman space can be interpreted by the Hankel operator

$$H_{\bar{z}}(g) = (I-P)(\bar{z}g),$$

where $P : L^2(\Omega) \to A^2(\Omega)$ denotes the Bergman projection, and Ω is a bounded domain in $\mathbb{C}^n$. See [1], [2], [3], [6], [7], [8], [9], [10], [12], [13] for details.

Unfortunately there exists $f \in F^2_\alpha$ such that $\bar{z}^n f \notin L^2(\mathbb{C}, d\lambda_\alpha)$. In the sequel, for fixed positive integer n, we consider the space

$$A^2_n(\mathbb{C}) = \left\{ f : f \text{ entire}, \sum_{k=0}^{\infty} \frac{(k+n)!}{\alpha^{k+n}} \frac{|f^{(k)}(0)|^2}{(k!)^2} < \infty \right\}$$

as the Hankel operator's domain. It is easy to see that $A^2_n(\mathbb{C})$ is dense in F^2_α, because the polynomials z^n belong to $A^2_n(\mathbb{C})$. In this note, we compute the norm of $H_{\bar{z}}$. Then we find the formula of the norm of $H_{\bar{z}^n}(g)$ and give an upper bound of the norm of H_n on Fock space. Lastly, we prove the concomitant operator P_n of $H_{\bar{z}^n}$ is quasi-affine to the direct sum of n copies of the concomitant operator P_1 of $H_{\bar{z}}$.

2. The norm of Hankel operators

The following example indicates $g(z) \in F^2_\alpha$ but $\bar{z}^n g(z) \notin L^2(\mathbb{C}, d\lambda_\alpha)$.

EXAMPLE 1. For fixed positive integer n, let $g(z) = \sum_{k=0}^{\infty} \frac{\sqrt{\alpha}^{k+n}}{(k+n)\sqrt{(k+n-1)!}} z^k$. Then $g(z) \in F^2_\alpha(\mathbb{C})$, but $\bar{z}^n g(z) \notin L^2(\mathbb{C}, d\lambda_\alpha)$.

From the definition of $A^2_n(\mathbb{C})$, we know that $g(z) \in A^2_n(\mathbb{C})$ implies that $\bar{z}^n g(z) \in L^2(\mathbb{C}, d\lambda_\alpha)$.

LEMMA 1. *If $g(z) \in A^2_n(\mathbb{C})$, then:*

(1) $g^{(n)}(z) \in F^2_\alpha(\mathbb{C})$;

(2) $g(z) \in F^2_\alpha(\mathbb{C})$.

Proof.

(1) Suppose $g(z)=\sum\limits_{k=0}^{\infty}a_kz^k$. Then we have $\sum\limits_{k=0}^{\infty}\frac{|a_k|^2(k+n)!}{\alpha^{k+n}}=C<+\infty$. Note that $g^{(n)}(z)=\sum\limits_{k=0}^{\infty}\frac{(n+k)!}{k!}a_{n+k}z^k$, thus

$$\int_{\mathbb{C}}|\sum_{k=0}^{\infty}\frac{(n+k)!}{k!}a_{n+k}z^k|^2\frac{\alpha}{\pi}e^{-\alpha|z|^2}dA(z)=\int_0^{+\infty}\sum_{k=0}^{\infty}\frac{[(n+k)!]^2}{(k!)^2}|a_{n+k}|^2\rho^k\alpha e^{-\alpha\rho}d\rho$$
$$=\sum_{k=0}^{\infty}\frac{[(n+k)!]^2|a_{n+k}|^2}{k!\alpha^k}.$$

Applying the inequality

$$\frac{[(n+k)!]^2}{k!}<(2n+k)! \tag{1}$$

we obtain

$$\sum_{k=0}^{\infty}\frac{[(n+k)!]^2|a_{n+k}|^2}{k!\alpha^{k+n}}<\sum_{k=0}^{\infty}\frac{(2n+k)!|a_{n+k}|^2}{\alpha^{k+n}}=\sum_{k=n}^{\infty}\frac{(n+k)!|a_k|^2}{\alpha^k}$$
$$<\sum_{k=0}^{\infty}\frac{(n+k)!|a_k|^2}{\alpha^k}=\alpha^nC.$$

This implies that $g^{(n)}(z)\in F_\alpha^2(\mathbb{C})$.

(2) Note that $\|g(z)\|^2=\sum\limits_{k=0}^{\infty}\frac{k!|a_k|^2}{\alpha^k}<\alpha^nC$. This implies that $g(z)\in F_\alpha^2(\mathbb{C})$. □

LEMMA 2. *Let* $g(z)\in A_n^2(\mathbb{C})$ *and* $P:L^2(\mathbb{C},d\lambda_\alpha)\to F_\alpha^2(\mathbb{C})$. *Then* $P(\bar{z}^ng(z))=\frac{1}{\alpha^n}g^{(n)}(z)$.

Proof. Suppose $g(z)=\sum\limits_{k=0}^{\infty}a_kz^k$. Then we have

$$\begin{aligned}
&P(\bar{z}^ng(z))\\
&=\int_{\mathbb{C}}\bar{w}^n\sum_{k=0}^{\infty}a_kw^ke^{\alpha z\bar{w}}d\lambda_\alpha(w)=\frac{\alpha}{\pi}\int_{\mathbb{C}}\bar{w}^n\sum_{k=0}^{\infty}a_kw^k\sum_{m=0}^{\infty}\frac{(\alpha z)^m}{m!}\bar{w}^me^{-\alpha|w|^2}dA(w)\\
&=\frac{\alpha}{\pi}\int_{\mathbb{C}}\left(a_n|w|^{2n}+a_{n+1}\frac{\alpha z}{1!}|w|^{2(n+1)}+a_{n+2}\frac{(\alpha z)^2}{2!}|w|^{2(n+2)}+\cdots\right)e^{-\alpha|w|^2}dA(w)\\
&=\frac{1}{\alpha^n}\int_0^{\infty}\sum_{k=0}^{\infty}\frac{z^k}{k!}a_{n+k}x^{n+k}e^{-x}dx=\frac{1}{\alpha^n}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!}a_{n+k}z^k=\frac{1}{\alpha^n}g^{(n)}(z). \quad \square
\end{aligned} \tag{2}$$

For simpleness, we denote $H_{\bar{z}^n}$ by H_n. In [6, 9], Haslinger, Knirsch and Schneider proved in their paper that Hankel operator H_n fails to be compact on the Fock space. Now we prove that H_n is a bounded linear operator on $A_n^2(\mathbb{C})$.

PROPOSITION 1. *Let* $H_1 : A_1^2(\mathbb{C}) \to L^2(\mathbb{C}, d\lambda_\alpha)$. *Then* H_1 *is a bounded linear operator, and* $\|H_1\| = \sqrt{\frac{1}{\alpha}}$.

Proof. For $g,h \in A_1^2(\mathbb{C}), a,b \in \mathbb{C}$, it is easy to show that

$$H_1(ag+bh)(z) = aH_1(g) + bH_1(h).$$

Suppose that $g(z) = \sum_{k=0}^{\infty} a_k z^k$. Then applying Lemma 2, we have

$$\begin{aligned}\|H_1(g)\|^2 &= \langle H_1(g), H_1(g)\rangle = \langle \bar{z}g - P(\bar{z}g), \bar{z}g - P(\bar{z}g)\rangle \\ &= \langle \bar{z}g, \bar{z}g\rangle - \langle P(\bar{z}g), \bar{z}g\rangle - \langle \bar{z}g, P(\bar{z}g)\rangle + \langle P(\bar{z}g), P(\bar{z}g)\rangle \\ &= \langle \bar{z}g, \bar{z}g\rangle - \frac{1}{\alpha}\langle g'(z), \bar{z}g\rangle - \frac{1}{\alpha}\langle \bar{z}g, g'(z)\rangle + \frac{1}{\alpha^2}\langle g'(z), g'(z)\rangle \\ &= I_1 - I_2 - I_3 + I_4.\end{aligned}$$

$$\begin{aligned}I_1 &= \int_{\mathbb{C}} |z|^2 \sum_{k=0}^{\infty} a_k z^k \sum_{m=0}^{\infty} \bar{a}_m \bar{z}^m \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z) = \alpha \int_0^{\infty} \sum_{k=0}^{\infty} |a_k|^2 r^{k+1} e^{-\alpha r} dr \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{|a_k|^2}{\alpha^{k+1}} x^{k+1} e^{-x} dx = \sum_{k=0}^{\infty} \frac{|a_k|^2 (k+1)!}{\alpha^{k+1}}. \\ I_2 &= \frac{1}{\alpha} \int_{\mathbb{C}} \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k \sum_{m=0}^{\infty} \bar{a}_m \bar{z}^m z \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z) = \int_0^{\infty} \sum_{k=1}^{\infty} k|a_k|^2 r^k e^{-\alpha r} dr \\ &= \sum_{k=1}^{\infty} \frac{k|a_k|^2 k!}{\alpha^{k+1}}. \\ I_3 &= \bar{I}_2 = \sum_{k=1}^{\infty} \frac{k|a_k|^2 k!}{\alpha^{k+1}}. \\ I_4 &= \frac{1}{\alpha^2} \int_{\mathbb{C}} \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k \sum_{m=0}^{\infty} (m+1) \bar{a}_{m+1} \bar{z}^m z \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z) \\ &= \frac{1}{\alpha} \int_0^{\infty} \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{k-1} e^{-\alpha r} dr = \sum_{k=1}^{\infty} \frac{k^2 |a_k|^2 (k-1)!}{\alpha^{k+1}} = \sum_{k=1}^{\infty} \frac{k|a_k|^2 k!}{\alpha^{k+1}}.\end{aligned}$$

Therefore,

$$\|H_1(g)\|^2 = \sum_{k=0}^{\infty} \frac{|a_k|^2 (k+1)!}{\alpha^{k+1}} - \sum_{k=1}^{\infty} \frac{k|a_k|^2 k!}{\alpha^{k+1}} = \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{|a_k|^2 k!}{\alpha^k}. \tag{3}$$

Note that $\|g(z)\|^2 = \sum_{k=0}^{\infty} \frac{k!}{\alpha^k} |a_k|^2$. So $\|H_1(g)\|^2 = \frac{1}{\alpha}\|g\|^2$.

This implies that $\|H_1\| = \sqrt{\frac{1}{\alpha}}$. □

PROPOSITION 2. *Let* $H_1 : A_1^2(\mathbb{C}) \to L^2(\mathbb{C}, d\lambda_\alpha)$. *Then* $\ker H_1 = \{0\}$.

Proof. Note that

$$H_1(g) = \bar{z}g - P(\bar{z}g) = \bar{z}g(z) - \tfrac{1}{\alpha}g'(z).$$

From $H_1(g) = 0$, we obtain $g'(z) = \alpha\bar{z}g(z)$. So $g(z) = ce^{\alpha|z|^2}$.

Observe that $g(z)$ is an entire function, applying the Cauchy-Riemann equation, we get $c = 0$. Hence $\ker H_1 = \{0\}$. □

In order to estimate the norm of H_n, we need the following lemma.

LEMMA 3. *Let $p(k,1) = 1$, and $p(k,n) = \prod_{j=1}^{n}(k+j) - \prod_{j=1}^{n}(k+j-n)$, $n \geqslant 2$. Then*

$$p(k,n) = n! + \sum_{j=2}^{n} \frac{C_n^{j-1}n!}{(j-1)!}k(k-1)\cdots(k-j+2),$$

where $C_n^{j-1} = \frac{n!}{(j-1)!(n-j+1)!}$.

Proof. We prove the lemma by mathematics induction.

Step 1 When $n = 2$, it is easy to see the equality holds.

Step 2 Assume that the equality holds for $n = l$. That is,

$$p(k,l) = \prod_{j=1}^{l}(k+j) - \prod_{j=1}^{l}(k+j-l) = l! + \sum_{j=2}^{l} \frac{C_l^{j-1}l!}{(j-1)!}k(k-1)\cdots(k-j+2).$$

When $n = l+1$, we have

$$\begin{aligned}
&p(k,l+1)\\
=&\prod_{j=1}^{l+1}(k+j) - \prod_{j=1}^{l+1}(k+j-l-1)\\
=&(k+l+1)\prod_{j=1}^{l}(k+j) - (k+l+1)\prod_{j=1}^{l}(k+j-l)\\
&+(k+l+1)\prod_{j=1}^{l}(k+j-l) - \prod_{j=1}^{l+1}(k+j-l-1)\\
=&(k+l+1)p(k,l) + \prod_{j=1}^{l}(k+j-l)(2l+1)\\
=&(k+l+1)[l! + \sum_{j=2}^{l}\frac{C_l^{j-1}l!}{(j-1)!}k(k-1)\cdots(k-j+2)] + \prod_{j=1}^{l}(k+j-l)(2l+1)\\
=&(l+1)! + kl! + \sum_{j=2}^{l}\frac{C_l^{j-1}l!}{(j-1)!}k(k-1)\cdots(k-j+2)(k+l+1) + \prod_{i=1}^{l}(k+i-l)(2l+1).
\end{aligned}$$

Note that $p(k,l+1)=(l+1)!+\sum_{j=2}^{l+1}\frac{C_{l+1}^{j-1}(l+1)!}{(j-1)!}k(k-1)\cdots(k-j+2)$.

We need only to show that

$$\begin{aligned} &kl!+\sum_{j=2}^{l}\frac{C_l^{j-1}l!}{(j-1)!}k(k-1)\cdots(k-j+2)(k+l+1)+\prod_{i=1}^{l}(k+i-l)(2l+1) \\ &=\sum_{j=2}^{l+1}\frac{C_{l+1}^{j-1}(l+1)!}{(j-1)!}k(k-1)\cdots(k-j+2). \end{aligned} \tag{4}$$

We rewrite the above equality as the following form

$$\begin{aligned} &kl!+\sum_{j=2}^{l}\frac{C_l^{j-1}l!}{(j-1)!}k(k-1)\cdots(k-j+2)[(k-j+1)+(l+j)]+\prod_{i=1}^{l}(k+i-l)(2l+1) \\ &=\sum_{j=2}^{l+1}\frac{C_{l+1}^{j-1}(l+1)!}{(j-1)!}k(k-1)\cdots(k-j+2). \end{aligned} \tag{5}$$

Now by comparing the coefficient of the form polynomial $k(k-1)\cdots(k-j+2)$ $(j=2,\cdots l+1)$ in the two sides of (5), we obtain the following facts.
When $j=2$, we have

$$l!+\frac{C_l^1 l!}{1!}(l+2)=\frac{C_{l+1}^1(l+1)!}{1!}. \tag{6}$$

When $2<j<l$, we have

$$\frac{C_l^{j-1}l!}{(j-1)!}+\frac{C_l^j l!(l+j+1)}{j!}=\frac{C_{l+1}^j(l+1)!}{j!}. \tag{7}$$

When $j=l$, the coefficient of $k(k-1)\cdots(k-l+1)$ in the left hand side of (5) is $\frac{C_l^{l-1}l!}{(l-1)!}+(2l+1)$. The coefficient of $k(k-1)\cdots(k-l+1)$ in the right hand side of (5) is $\frac{C_{l+1}^l(l+1)!}{l!}$ (when $j=l+1$). Simple observation shows that

$$\frac{C_l^{l-1}l!}{(l-1)!}+(2l+1)=\frac{C_{l+1}^l(l+1)!}{l!}. \tag{8}$$

Therefore, the lemma is true, as desired. □

In the following proposition, we give the norm characterization of $H_n(g)$.

PROPOSITION 3. *Let* $g(z)=\sum_{k=0}^{\infty}a_k z^k\in A_n^2(\mathbb{C})$. *Then* $\|H_n(g)\|^2=\sum_{j=1}^{n}\frac{C_n^{j-1}n!}{(j-1)!\alpha^{n+j-1}}\left\|g^{(j-1)}(z)\right\|^2$, *where* $\left\|g^{(j-1)}(z)\right\|^2=\sum_{k=0}^{\infty}\frac{k!k(k-1)\cdots(k-j+2)}{\alpha^{k-j+1}}|a_k|^2$, $(j=2,3,\cdots,n)$.

Proof. Similar to Proposition 1 and applying Lemma 3, we have

$$\begin{aligned}
\|H_n(g)\|^2 &= \sum_{k=0}^{\infty} \frac{|a_k|^2(k+n)!}{\alpha^{k+n}} - \sum_{k=0}^{\infty} \frac{[(k+n)!]^2|a_{k+n}|^2}{k!\alpha^{k+2n}} \\
&= \sum_{k=0}^{\infty} \frac{|a_k|^2(k+n)!}{\alpha^{k+n}} - \sum_{k=n}^{\infty} \frac{(k!)^2|a_k|^2}{(k-n)!\alpha^{k+n}} \\
&= \sum_{k=0}^{n-1} \frac{|a_k|^2(k+n)!}{\alpha^{k+n}} + \sum_{k=n}^{\infty} \frac{|a_k|^2}{\alpha^{k+n}} \left[(k+n)! - \frac{(k!)^2}{(k-n)!}\right] \\
&= \sum_{k=0}^{n-1} \frac{|a_k|^2(k+n)!}{\alpha^{k+n}} + \sum_{k=n}^{\infty} \frac{|a_k|^2 k!}{\alpha^{k+n}} p(k,n) = \sum_{k=0}^{\infty} \frac{|a_k|^2 k!}{\alpha^{k+n}} p(k,n) \\
&= \sum_{k=0}^{\infty} \frac{|a_k|^2 k!}{\alpha^{k+n}} \left[n! + \sum_{j=2}^{n} \frac{C_n^{j-1} n!}{(j-1)!} k(k-1)\cdots(k-j+2)\right] \\
&= \sum_{k=0}^{\infty} \frac{|a_k|^2 k!}{\alpha^{k+n}} n! + \sum_{k=0}^{\infty} \frac{|a_k|^2 k!}{\alpha^{k+n}} \sum_{j=2}^{n} \frac{C_n^{j-1} n!}{(j-1)!} k(k-1)\cdots(k-j+2) \\
&= \frac{n!}{\alpha^n}\|g(z)\|^2 + \sum_{j=2}^{n} \frac{C_n^{j-1} n!}{(j-1)!\alpha^{n+j-1}} \left\|g^{(j-1)}(z)\right\|^2 \\
&= \sum_{j=1}^{n} \frac{C_n^{j-1} n!}{(j-1)!\alpha^{n+j-1}} \left\|g^{(j-1)}(z)\right\|^2.
\end{aligned}$$

Hence, we complete the proof of Proposition 3. □

Now we give an upper bound of the operator H_n.

THEOREM 1. *The norm of the operator H_n is less than or equal to $\sqrt{\frac{n!(2^n-1)}{\alpha^n}}$.*

Proof. Applying (1), we obtain

$$\begin{aligned}
\|g^{(n)}\|^2 &= \sum_{k=0}^{\infty} \frac{[(n+k)!]^2|a_{n+k}|^2}{k!\alpha^k} < \sum_{k=0}^{\infty} \frac{(2n+k)!|a_{n+k}|^2}{\alpha^k} = \alpha^n \sum_{k=0}^{\infty} \frac{(2n+k)!|a_{n+k}|^2}{\alpha^{n+k}} \\
&= \alpha^n \sum_{l=n}^{\infty} \frac{(n+l)!|a_l|^2}{\alpha^l} < \alpha^n \sum_{l=0}^{\infty} \frac{(n+l)!|a_l|^2}{\alpha^l} = \alpha^{2n}\|\bar{z}^n g\|^2 \leqslant \alpha^{2n}\|\bar{z}^n\|^2\|g\|^2 \\
&= \alpha^n n!\|g\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|H_n(g)\|^2 &= \sum_{j=1}^{n} \frac{C_n^{j-1} n!}{(j-1)!\alpha^{n+j-1}} \left\|g^{(j-1)}(z)\right\|^2 < \sum_{j=1}^{n} \frac{C_n^{j-1} n!}{(j-1)!\alpha^{n+j-1}} \alpha^{j-1}(j-1)!\|g\|^2 \\
&= \sum_{j=1}^{n} \frac{C_n^{j-1} n!}{\alpha^n}\|g\|^2 < \frac{n!(2^n-1)}{\alpha^n}\|g\|^2.
\end{aligned}$$

Therefore, we have $\|H_n\| \leqslant \sqrt{\frac{n!(2^n-1)}{\alpha^n}}$. □

3. Some properties of the operator P_n

For $f \in A_n^2(\mathbb{C})$, let $P_n f = P(\bar{z}^n f), n = 1,2,\cdots$, and P is the Fock projection. Now we will consider the relationship of the concomitant operator P_n of $H_{\bar{z}^n}$ and the concomitant operator P_1 of $H_{\bar{z}}$.

PROPOSITION 4. *If $f(z) \in A_n^2(\mathbb{C})$ for fixed $n \geqslant 1$, then $H_n f \in \ker P$.*

Proof. By Lemma 1 and 2, we have

$$H_n f = \bar{z}^n f - P_n f = \bar{z}^n f - \frac{f^{(n)}(z)}{\alpha^n}. \tag{9}$$

Hence

$$PH_n f = P(\bar{z}^n f) - \frac{f^{(n)}(z)}{\alpha^n} = 0.$$

So $H_n f \in \ker P$. □

Recall that for two bounded linear operators T_1 and T_2, T_1 is quasi-affine to T_2, if there exists an intertwining bounded operator X with kernel zero and dense range such that $T_1 X = X T_2$ (see [11]).

Let $e_k(z) = \sqrt{\frac{\alpha^k}{k!}} z^k \, (k = 0,1,\cdots)$ be the orthonormal basis of $F_\alpha^2(\mathbb{C})$. Let $S_j = \overline{span}\{e_{nk+j} | j = 0,1,\cdots,n-1, k = 0,1,\cdots\}$. Clearly, $S_j \, (j = 0,1,\cdots,n-1)$ are the closed subspaces of F_α^2. And $F_\alpha^2 = S_0 \bigoplus S_1 \bigoplus \cdots \bigoplus S_{n-1}$. Denote $L_j = S_j|_{A_n^2(\mathbb{C})}$, Then we have $A_n^2(\mathbb{C}) = L_0 \bigoplus L_1 \bigoplus \cdots \bigoplus L_{n-1}$. Define $X_j : A_n^2(\mathbb{C}) \to L_j$, such that $X_j e_k = c_{k,j} e_{nk+j}$, where the coefficients $c_{k,j}$ are to be determined later. Denote $P_{nj} = P_n|_{L_j} \, (j = 0,1,\cdots,n-1)$. Then we have the following theorem.

THEOREM 2. *The operator $P_n \, (n \geqslant 2)$ is quasi-affine to $\bigoplus_1^n P_1$ on $A_n^2(\mathbb{C})$.*

Proof. It is easy to show $P_{nj} X_j e_0 = X_j P_1 e_0 = 0$. When $k \geqslant 1$, we have

$$\begin{aligned}
& P_{nj} X_j e_k \\
=& P_{nj} c_{k,j} e_{nk+j} = c_{k,j} \sqrt{\frac{\alpha^{nk+j}}{(nk+j)!}} P_{nj}(z^{nk+j}) \\
=& c_{k,j} \sqrt{\frac{\alpha^{nk+j}}{(nk+j)!}} \frac{(nk+j)(nk+j-1)\cdots(n(k-1)+j+1)}{\alpha^n} z^{n(k-1)+j} \\
=& c_{k,j} \sqrt{\frac{\alpha^{nk+j}}{(nk+j)!} \times \frac{(n(k-1)+j)!}{\alpha^{n(k-1)+j}}} \frac{(nk+j)(nk+j-1)\cdots(n(k-1)+j+1)}{\alpha^n} e_{n(k-1)+j} \\
=& c_{k,j} \sqrt{\frac{(nk+j)(nk+j-1)\cdots(n(k-1)+j+1)}{\alpha^n}} e_{n(k-1)+j},
\end{aligned}$$

$$X_jP_1e_k = X_jP_1\left(\sqrt{\frac{\alpha^k}{k!}}z^k\right) = \sqrt{\frac{\alpha^k}{k!}}X_j\left(k\frac{z^{k-1}}{\alpha}\right) = \sqrt{\frac{\alpha^k}{k!}}\frac{k}{\alpha}X_j\left(\sqrt{\frac{(k-1)!}{\alpha^{k-1}}}e_{k-1}\right)$$
$$= \sqrt{\frac{k}{\alpha}}c_{k-1,j}e_{n(k-1)+j}.$$

From $P_{nj}X_je_k = X_jP_1e_k$, we have

$$\frac{c_{k,j}}{c_{k-1,j}} = \frac{\sqrt{\frac{k}{\alpha}}}{\sqrt{\frac{(nk+j)(nk+j-1)\cdots(n(k-1)+j+1)}{\alpha^n}}} = \sqrt{\frac{k\alpha^{n-1}}{(nk+j)(nk+j-1)\cdots(n(k-1)+j+1)}}.$$

So, we obtain

$$c_{k,j} = \sqrt{\frac{\Gamma(k+1)\alpha^{k(n-1)}\Gamma(j+1)}{\Gamma(nk+j+1)}}. \tag{10}$$

Put

$$b_{k,j} = \frac{\Gamma(k+1)\alpha^{k(n-1)}}{\Gamma(nk+j+1)} = \frac{\alpha^{k(n-1)}}{(nk+j)(nk+j-1)\cdots(k+1)}, \tag{11}$$

then $c_{k,j} = \sqrt{b_{k,j}\Gamma(j+1)}$.

In the following, we will analyze the limit of sequence $c_{k,j}$ as $k \to +\infty$.

Case1. When $0 < \alpha < 1$, we have $\lim_{k\to+\infty} c_{k,j} = 0$.

Case2. When $\alpha = 1$, it is easy to see that $\lim_{k\to+\infty} c_{k,j} = 0$.

Case3. When $\alpha > 1$, we will consider the following equality

$$-\ln b_{k,j} = -k(n-1)\ln\alpha + [\ln(nk+j) + \ln(nk+j-1) + \cdots + \ln(k+1)] = A_k - B_k$$
$$= B_k\left(\frac{A_k}{B_k} - 1\right).$$

Note that B_k is a monotone increasing sequence, and $B_k \to +\infty$ as $k \to +\infty$. By Stolz's theorem, we have

$$\begin{aligned}\lim_{k\to+\infty}\frac{A_k}{B_k} &= \lim_{k\to+\infty}\frac{\ln\frac{(nk+j)(nk+j-1)\cdots(k+1)}{(nk-n+j)\cdots k}}{(n-1)\ln\alpha} \\ &= \lim_{k\to+\infty}\frac{\ln\left(n+\frac{j}{k}\right) + \ln((nk+j-1)\cdots(nk-n+j+1))}{(n-1)\ln\alpha} = +\infty.\end{aligned} \tag{12}$$

Hence, there is a positive integer k_0, such that when $k > k_0$, we have $\frac{A_k}{B_k} > 2$. This implies that $\ln\frac{1}{b_{k,j}} \to +\infty$ as $k \to +\infty$. So $\lim_{k\to+\infty} c_{k,j} = 0$.

Suppose that $f \in \ker X_j$, and $f = \sum_{k=0}^{\infty} d_ke_k, d_k \in \mathbb{C}$. Then from

$$0 = \langle X_jf, e_{nk+j}\rangle = \left\langle \sum_{k=0}^{\infty} d_kc_{k,j}e_{nk+j}, e_{nk+j}\right\rangle,$$

we deduce that $d_k = 0\,(k = 0, 1, \cdots)$. So $\ker X_j = \{0\}$.

Next, for $g \in \ker X_j^*$, and $g = \sum\limits_{k=0}^{\infty} m_k e_{nk+j}, m_k \in \mathbb{C}$. From

$$0 = \langle e_k, X_j^* g\rangle = \left\langle c_{k,j} e_{nk+j}, \sum_{k=0}^{\infty} m_k e_{nk+j} \right\rangle,$$

we obtain $m_k = 0\,(k = 0, 1, \cdots)$. So $\ker X_j^* = (RanX_j)^{\perp}$=$\{0\}$, i.e., $\overline{RanX_j} = L_j$. Hence P_{nj} is quasi-affine to P_1.

Moreover, $P_n|_{A_n^2} = P_{n0} \bigoplus P_{n1} \bigoplus \cdots \bigoplus P_{nn-1}$ is quasi-affine to $\bigoplus\limits_{1}^{n} P_1$. $\square$

Acknowledgement. The authors are partially supported by the NNSF of China (11371119). The authors would like to express their sincere appreciation to the referee for his/her valuable suggestions and comments.

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(Received October 30, 2019)

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