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PARSEVAL FRAMES OF PIECEWISE CONSTANT FUNCTIONS

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Abstract. We present a way to construct Parseval frames of piecewise constant functions for $L^2[0,1]$. The construction is similar to the generalized Walsh bases. It is based on iteration of operators that satisfy a Cuntz-type relation, but without the isometry property. We also show how the Parseval frame can be dilated to an orthonormal basis and the operators can be dilated to true Cuntz isometries.

1. Introduction

In [7], Dutkay et al. introduced a method of constructing orthonormal bases from representations of Cuntz algebras. Recall that the *Cuntz algebra* $\mathscr{O}_N$, where N is an integer, $N \geqslant 2$, is the C^*-algebra generated by N isometries $(S_i)_{i=0,\dots,N-1}$ on some Hilbert space $\mathscr{H}$ which satisfy the *Cuntz relations*

$$S_i^*S_j = \delta_{ij}I_{\mathscr{H}}, \quad (i,j \in \{0,\dots,N-1\}), \quad \sum_{i=0}^{N-1} S_iS_i^* = I_{\mathscr{H}}. \tag{1.1}$$

The basic idea was to start with some vector v_0 in $\mathscr{H}$ which is fixed by the first isometry, $S_0v_0 = v_0$, and then apply all the Cuntz isometries $S_{\omega_1}\dots S_{\omega_n}v_0$ where $\omega_1,\dots,\omega_n \in \{0,\dots,N-1\}$. Eliminating the repetitions generated by the fact that $S_0v_0 = v_0$, one can see immediately that the resulting family of vectors is orthonormal. The more delicate issue is, of course, its completeness.

A particular case of this construction yields the classical Walsh basis on $L^2[0,1]$ and some variations of that yield generalized Walsh bases for $L^2[0,1]$ consisting of piecewise constant functions, see [7, 6].

In this paper, we will follow similar ideas, but with some important modifications. We will begin not with a Cuntz algebra representation, but with one where only the relation

$$\sum_{i=0}^{N-1} \tilde{S}_i\tilde{S}_i^* = I_{\mathscr{H}} \tag{1.2}$$

is satisfied. Again we will have a vector v_0 (in our case, the constant function $\mathbf{1}$) with $\tilde{S}_0v_0 = v_0$, and, by iterating the operators $\tilde{S}_i$, we will obtain a family $\tilde{S}_{\omega_1}\dots\tilde{S}_{\omega_n}v_0$ which is a Parseval frame.

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Recall that a *Parseval frame* for a Hilbert space $\mathscr{H}$ is a family of vectors $\{\tilde{e}_j : j \in J\}$ such that

$$\|f\|^2 = \sum_{j\in J} |\langle f, \tilde{e}_j\rangle|^2, \quad (f \in \mathscr{H}). \tag{1.3}$$

To prove that the family $\{\tilde{S}_{\omega_1}\dots\tilde{S}_{\omega_n}v_0 : \omega_1,\dots,\omega_n \in \{0,\dots,N-1\}\}$ is a Parseval frame, we construct a dilation to an orthonormal basis. We recall two important results in dilation theory:

THEOREM 1.1. *A family $\{\tilde{e}_j : j \in J\}$ is a Parseval frame for a Hilbert space $\mathscr{H}$ if and only if there is a larger Hilbert space $\mathscr{K} \supset \mathscr{H}$ and an orthonormal basis $\{e_j : j \in J\}$ such that $P_{\mathscr{H}}e_j = \tilde{e}_j$ for all $j \in J$, where $P_{\mathscr{H}}$ is the orthogonal projection from $\mathscr{K}$ onto the subspace $\mathscr{H}$.*

The second result [8, Theorem 5.1], based on Popescu's dilation theory [2], shows that the relation (1.2) can always be dilated to a representation of the Cuntz algebra.

THEOREM 1.2. *Let $\mathscr{H}$ be a Hilbert space and let $\tilde{S}_0,\dots,\tilde{S}_{N-1}$ be operators on $\mathscr{H}$ satisfying*

$$\sum_{i=0}^{N-1} \tilde{S}_i\tilde{S}_i^* = I_{\mathscr{H}}.$$

Then $\mathscr{H}$ can be embedded into a larger Hilbert space $\mathscr{K}$, carrying a representation $S_0,\dots,S_{N-1}$ of the Cuntz algebra $\mathscr{O}_N$ such that, if $P_{\mathscr{H}}$ is the projection onto $\mathscr{H}$, we have

$$\tilde{S}_i^* = S_i^* P_{\mathscr{H}},$$

(i.e., $S_i^*\mathscr{H} \subset \mathscr{H}$ and $S_i^*P_{\mathscr{H}} = P_{\mathscr{H}}S_i^*P_{\mathscr{H}} = \tilde{S}_i^*$) and $\mathscr{H}$ is cyclic for the representation. The system

$$(\mathscr{K}, S_0,\dots,S_{N-1},P_{\mathscr{H}})$$

is unique up to unitary equivalence.

These are the general lines of our construction. Now we describe the particulars of our construction.

We start with a matrix T of the form

$$T := \frac{1}{\sqrt{N}}(\alpha_{i,j})_{\substack{i=0,\dots,M-1\\ j=0,\dots,N-1}}, \tag{1.4}$$

such that

$$T^*T = I_N, \tag{1.5}$$

i.e., T is an isometry. This means that the columns are orthonormal vectors in $\mathbb{C}^M$, and, equivalently, that the rows form a Parseval frame for $\mathbb{C}^N$ (see, e.g., [3, Lemma 3.8]).

We assume that

$$\alpha_{0,j} = 1 \text{ for } j \in \{0,\dots,N-1\}. \tag{1.6}$$

i.e., the first row of T is $1/\sqrt{N}$. (This is required for the relation $S_0 v_0 = v_0$).

Next we build the piecewise constant functions

$$m_i(x) = \sum_{j=0}^{N-1} \alpha_{i,j} \chi_{[j/N\ ,\ (j+1)/N]},\ \ i \in \{0,\dots,M-1\},$$

where χ_A denotes the characteristic function of the subset A.

Using these functions, we define the operators:

$$\left(\tilde{S}_i f\right)(x) := m_i(x) f(Nx \bmod 1),\ \ \text{on } L^2[0,1], \tag{1.7}$$

with $\tilde{S}_0 \mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ denote the constant function.

PROPOSITION 1.3. *The operators $\tilde{S}_0,\dots,\tilde{S}_{M-1}$ satisfy the relation*

$$\sum_{i=0}^{M-1} \tilde{S}_i \tilde{S}_i^* = I_{L^2[0,1]}. \tag{1.8}$$

Define $\Omega_{\mathscr{M}}$ to be the set of all words $\omega_1 \dots \omega_n$ with digits in $\{0,\dots,M-1\}$, that do not end in 0, and the empty word $\emptyset$. (We want the word not to end in 0, to eliminate the repetitions coming from the relation $\tilde{S}_{\omega_1}\dots\tilde{S}_{\omega_n} S_0 \mathbf{1} = \tilde{S}_{\omega_1}\dots\tilde{S}_{\omega_n}\mathbf{1}$).

THEOREM 1.4. *The family of functions*

$$\{\tilde{S}_{\omega_1}\dots\tilde{S}_{\omega_n}\mathbf{1} : \omega_1\dots\omega_n \in \Omega_M\} \tag{1.9}$$

is a Parseval frame in $L^2[0,1]$.

We will start section 2 with the proof of our main result. The proof has the advantage that it shows also how the Parseval frame can be dilated to an orthonormal basis and how the operators $\tilde{S}_\omega$ are dilated to Cuntz isometries, as in Theorems 1.1 and 1.2. It has also the advantage that it goes along the more general lines presented in [7,4,5,1]. In Proposition 2.3, we present some more properties of the Parseval frames constructed in Theorem 1.4, with explicit ways of computing these piecewise constant functions by means of tensor products of matrices. In Remark 2.4, we present a more direct proof of Theorem 1.4, without the use of dilation theory. We end the paper with Proposition 2.5, which shows how one can construct examples of matrices satisfying (1.5) and (1.6).

The construction of Parseval frames, using operators that satisfy (1.8), is possible in a more general context, but we defer this to a later paper.

2. Proofs and other results

Proof of Proposition 1.3. We compute $\tilde{S}_l^*$. We have, for $f,g \in L^2[0,1]$,

$$\langle \tilde{S}_l f\,,\,g\rangle = \int_{[0,1]} m_l(x) f(Nx \bmod 1)\overline{g}(x)\,dx = \frac{1}{N}\sum_{b=0}^{N-1}\int_{[0,1]} m_l\left(\frac{x+b}{N}\right) f(x)\overline{g}\left(\frac{x+b}{N}\right)dx.$$

Thus

$$\tilde{S}_l^* g(x) = \frac{1}{N}\sum_{b=0}^{N-1} \overline{m}_l\left(\frac{x+b}{N}\right) g\left(\frac{x+b}{N}\right) = \frac{1}{N}\sum_{b=0}^{N-1} \overline{\alpha}_{l,b} g\left(\frac{x+b}{N}\right). \tag{2.1}$$

So, if $x \in \left[\frac{b'}{N}, \frac{b'+1}{N}\right)$, and $g \in L^2[0,1]$, then

$$\begin{aligned}\sum_{l=0}^{M-1} \tilde{S}_l \tilde{S}_l^* g(x) &= \sum_{l=0}^{M-1} m_l(x)\frac{1}{N}\sum_{b=0}^{N-1}\overline{\alpha}_{l,b} g\left(\frac{(Nx \bmod 1)+b}{N}\right)\\ &= \sum_{b=0}^{N-1} g\left(x+\frac{-b'+b}{N}\right)\frac{1}{N}\sum_{l=0}^{M-1}\alpha_{l,b'}\overline{\alpha}_{l,b} = \sum_{b=0}^{N-1} g\left(x+\frac{-b'+b}{N}\right)\delta_{b,b'} = g(x).\end{aligned}$$

Proof of Theorem 1.4.

First we will dilate the isometry matrix T to a unitary in a special way.

Pick a number N' such that $NN' \geqslant M$. Denote $B = \{0,\ldots,N-1\}$, $B' = \{0,1,\ldots,N'-1\}$, and $L = \{0,\ldots,M-1\}$.

We can identify L with a subset L' of $B\times B'$ by some injective function $\iota : L \to B\times B'$, in such a way that 0 from L corresponds to $(0,0) = \iota(0)$ from $B\times B'$, and let $\alpha_{(b,b'),c} = \alpha_{l,c}$ if $(b,b') = \iota(l)$, $\alpha_{(b,b'),c} = 0$ if $(b,b') \notin \iota(L)$. In other words we add some zero rows to the matrix $T = \frac{1}{\sqrt{N}}(\alpha_{ij})$ to get NN' rows in total.

The next step consists of dilating the Parseval frame of row vectors for the matrix T to an orthonormal basis, in a way that is compatible with the Cartesian product structure of $B\times B'$.

We construct the numbers $a_{(b,b'),(c,c')}$, $(b,b'),(c,c') \in B\times B'$ with the following properties:

1. The matrix
$$\frac{1}{\sqrt{NN'}}\left(a_{(b,b'),(c,c')}\right)_{(b,b'),(c,c')\in B\times B'} \tag{2.2}$$
is unitary and the first row is constant $\frac{1}{\sqrt{NN'}}$ so $a_{(0,0),(c,c')} = 1$ for all $(c,c') \in B\times B'$,

2. For all $(b,b') \in B\times B'$, $c \in B$,
$$\frac{1}{N'}\sum_{c'\in B'} a_{(b,b'),(c,c')} = \alpha_{(b,b'),c}. \tag{2.3}$$

Let $t_{(b,b'),c} = \frac{1}{\sqrt{N}}\alpha_{(b,b'),c}$. Note that the vectors $t_{\cdot,c}$, $c \in B$, in $\mathbb{C}^{NN'}$ are orthonormal. Therefore, we can complete it to an orthonormal basis in $\mathbb{C}^{NN'}$, so we can define some vectors $t_{\cdot,d}$, $d \in \{1,\ldots,NN'-N\}$ such that

$$\{t_{\cdot,c} : c \in B\} \cup \{t_{\cdot,d} : d \in \{1,\ldots,NN'-N\}\}$$

is an orthonormal basis for $\mathbb{C}^{NN'}$.

For $c \in B$, define the vectors in $\mathbb{C}^{NN'}$ by

$$\tilde{e}_c(c_1,c_1') = \frac{1}{\sqrt{N'}}\delta_{cc_1} \quad \left((c_1,c_1') \in B \times B'\right).$$

It is easy to see that these vectors are orthonormal in $\mathbb{C}^{NN'}$, therefore we can complete them to an orthonormal basis for $\mathbb{C}^{NN'}$ with some vectors $\tilde{e}_d$, $d \in \{1,...,NN'-N\}$.

Note that the vectors $\{\tilde{e}_c : c \in B\}$ span the subspace

$$\mathscr{M} = \{\left(X(c,c')\right)_{(c,c')\in B\times B'} : X \text{ does not depend on } c'\}.$$

Define now

$$s_{(b,b')} = \sum_{c\in B} t_{(b,b'),c}\tilde{e}_c + \sum_{d=1}^{NN'-N} t_{(b,b'),d}\tilde{e}_d.$$

Since the matrix with columns $t_{\cdot,c}$ and $t_{\cdot,d}$ has orthonormal columns, it is unitary. So it has orthogonal rows. So the vectors $t_{(b,b'),\cdot}$ are orthonormal, therefore the vectors $s_{(b,b')}$ are orthonormal. Also, since $\alpha_{(0,0),c} = 1$, we have that $t_{(0,0),c} = \frac{1}{\sqrt{N}}$ for all $c \in B$. But then

$$\sum_{c\in B} |t_{(0,0),c}|^2 = 1 = \|t_{(0,0),\cdot}\|^2.$$

So $t_{(0,0),d} = 0$ for $d \in \{1,...,NN'-N\}$. Therefore, for all $(c_1,c_1') \in B\times B'$:

$$s_{(0,0)}(c_1,c_1') = \sum_{c\in B} \frac{1}{\sqrt{N}}\,\frac{1}{\sqrt{N'}}\delta_{cc_1} = \frac{1}{\sqrt{NN'}}.$$

Since the vectors $\{\tilde{e}_c : c \in B\}$ span the subspace

$$\mathscr{M} = \{\left(X(c,c')\right)_{(c,c')\in B\times B'} : X \text{ does not depend on } c'\},$$

the vectors $\{\tilde{e}_d : d \in \{1,...,NN'-N\}\}$ are orthogonal to $\mathscr{M}$. Let $P_{\mathscr{M}}$ be the projection onto $\mathscr{M}$.

Note that, for $X \in \mathbb{C}^{NN'}$, we have

$$\begin{aligned}(P_{\mathscr{M}}X)(c_1,c_1') &= \sum_{c\in B}\langle X, e_c\rangle e_c(c_1,c_1') = \frac{1}{N'}\sum_{c\in B}\left(\sum_{(c_0,c_0')\in B\times B'} X(c_0,c_0')\delta_{cc_0}\right)\delta_{cc_1} \\ &= \frac{1}{N'}\sum_{c_0'\in B'} X(c_1,c_0').\end{aligned}$$

Also, since $P_{\mathscr{M}}\tilde{e}_c = \tilde{e}_c$ for $c \in B$ and $P_{\mathscr{M}}\tilde{e}_d = 0$ for $d = 1,\dots,NN'-N$, we have,

$$\left(P_{\mathscr{M}}s_{(b,b')}\right)(c_1,c_1') = \sum_{c\in B} t_{(b,b'),c}e_c(c_1,c_1') = \sum_{c\in B} t_{(b,b'),c}\frac{1}{\sqrt{N'}}\delta_{cc_1} = \frac{1}{\sqrt{N'}}t_{(b,b'),c_1}.$$

Define now

$$a_{(b,b'),(c,c')} := \sqrt{NN'}\, s_{(b,b')}(c,c').$$

Then we have

$$a_{(0,0),(c,c')} = 1 \text{ for all } (c,c').$$

The matrix

$$\frac{1}{\sqrt{NN'}}\left(a_{(b,b'),(c,c')}\right)_{(b,b'),(c,c')}$$

is the matrix with rows $s_{(b,b')}$. So it is unitary.

$$\begin{aligned}\frac{1}{N'}\sum_{c'\in B'} a_{(b,b'),(c,c')} &= \frac{1}{N'}\sum_{c'\in B'}\sqrt{NN'}\, s_{(b,b')}(c,c') = \sqrt{NN'}\left(P_{\mathscr{M}} s_{(b,b')}\right)(c,c')\\ &= \sqrt{NN'}\cdot\frac{1}{\sqrt{N'}} t_{(b,b'),c} = \alpha_{(b,b'),c}.\end{aligned}$$

Thus, the conditions 1 and 2 for the numbers $a_{(b,b'),(c,c')}$ are satisfied.

Next, using the unitary matrix in (2.2), we construct some Cuntz isometries $S_{(b,b')}$, $(b,b')\in B\times B'$ in the dilation space $L^2([0,1]\times[0,1])$ and with them we construct an orthonormal set, by applying the Cuntz isometries to the constant function $\mathbf{1}$.

Define now the maps $\mathscr{R},\mathscr{R}' : [0,1]\to[0,1]$ by

$$\mathscr{R}x = Nx \bmod 1, \quad \mathscr{R}'x = N'x \bmod 1, \tag{2.4}$$

and define the maps

$$\Upsilon_{(b,b')}(x,x') = \left(N^{-1}(x+b), N'^{-1}(x'+b')\right) \tag{2.5}$$

for $(x,x')\in\mathbb{R}^d\times\mathbb{R}^{d'}$ and $(b,b')\in B\times B'$. Define the functions

$$m_{(b,b')}(x,x') := \sum_{(c,c')\in B\times B'} a_{(b,b'),(c,c')}\chi_{\Upsilon_{(c,c')}([0,1]\times[0,1])}(x,x'),$$

where χ_A denotes the characteristic function of the set A.

With these filters we define the operators $S_{(b,b')}$ on $L^2([0,1]\times[0,1])$ by

$$\left(S_{(b,b')}f\right)(x,x') = m_{(b,b')}(x,x')f(\mathscr{R}x,\mathscr{R}x'). \tag{2.6}$$

LEMMA 2.1. *The operators* $S_{(b,b')}$, $(b,b')\in B\times B'$ *are a representation of the Cuntz algebra* $\mathscr{O}_{NN'}$, *i.e., they satisfy the relations in* (1.1). *The adjoint* $S^*_{(b,b')}$ *is given by the formula*

$$(S^*_{(b,b')}f)(x,x') = \frac{1}{NN'}\sum_{(c,c')\in B\times B'}\overline{m_{(b,b')}}(\Upsilon_{(c,c')}(x,x'))f(\Upsilon_{(c,c')}(x,x')), \tag{2.7}$$

for $f\in L^2([0,1]\times[0,1]), (x,x')\in[0,1]\times[0,1]$.

Proof. First, we compute the adjoint, using the invariance equations for the Lebesgue measure under the maps $\Upsilon_{(c,c')}$, i.e.,

$$\int_{[0,1]^2} f(x,x')\,d(x,x') = \frac{1}{NN'}\sum_{(c,c')}\int_{[0,1]^2} f(\Upsilon_{(c,c')}(x,x'))\,d(x,x').$$

We have:

$$\begin{aligned}\langle S_{(b,b')}f,g\rangle &= \int_{[0,1]^2} m_{(b,b')}(x,x')f(\mathscr{R}x,\mathscr{R}'x')\overline{g}(x,x')\,d(x,x')\\ &= \frac{1}{NN'}\sum_{(c,c')}\int_{[0,1]^2} m_{(b,b')}(\Upsilon_{(c,c')}(x,x'))f(x,x')\overline{g}(\Upsilon_{(c,c')}(x,x'))\;d(x,x'),\end{aligned}$$

and this proves (2.7).

We check the Cuntz relations:

$$\begin{aligned}&S^*_{(i,i')}S_{(j,j')}f(x,x')\\ =&\frac{1}{NN'}\sum_{(c,c')}\overline{m}_{(i,i')}(\Upsilon_{(c,c')}(x,x'))m_{(j,j')}(\Upsilon_{(c,c')}(x,x'))f(\mathscr{R}\times\mathscr{R}'(\Upsilon_{(c,c')}(x,x'))\\ =&\frac{1}{NN'}\sum_{(c,c')}\overline{a}_{(i,i'),(c,c')}a_{(j,j'),(c,c')}f(x,x') = \delta_{(i,i'),(j,j')}f(x,x'),\end{aligned}$$

by (2.2). Therefore

$$S^*_{(i,i'}S_{(j,j')} = \delta_{(i,i'),(j,j')}I.$$

Now take $(x,x') \in \Upsilon_{(c_0,c_0')}[0,1]^2$ so $\mathscr{R}x = Nx - c_0$, $\mathscr{R}'x' = N'x' - c_0'$. Then

$$\begin{aligned}\sum_{(i,i')} S_{(i,i')}S^*_{(i,i')}f(x,x') &= \sum_{(i,i')}\frac{1}{NN'}\sum_{(c,c')}\overline{m}_{(i,i')}(\Upsilon_{(c,c')}(\mathscr{R}x,\mathscr{R}'x'))f(\Upsilon_{(c,c')}(\mathscr{R}x,\mathscr{R}'x'))\\ &= \frac{1}{NN'}\sum_{(c,c')}\sum_{(i,i')}a_{(i,i'),(c_0,c_0')}\overline{a}_{(i,i'),(c,c')}f(\left(x+\frac{c-c_0}{N},x'+\frac{c'-c_0'}{N'}\right)\\ &= \sum_{(c,c')}\delta_{(c_0,c_0'),(c,c')}f\left(x+\frac{c_0-c}{N},x'+\frac{c_0'-c'}{N'}\right) = f(x,x').\end{aligned}$$

Therefore

$$\sum_{(i,i')} S_{(i,i')}S^*_{(i,i')} = I.$$

For a word $\omega = (b_1,b_1')\dots(b_k,b_k{}')$ we compute

$$\begin{aligned}(S_\omega\mathbf{1})(x,x') &= (S_{(b_1,b_1')}\dots S_{(b_k,b_k')}\mathbf{1})(x,x') = S_{(b_1,b_1{}')}\dots S_{(b_{k-1},b_{k-1}')}m_{(b_k,b_k')}(x,x')\\ &= S_{(b_1,b_1')}\dots S_{(b_{k-2},b_{k-2}')}m_{(b_{k-1},b_{k-1}')}(x,x')m_{(b_k,b_k')}(\mathscr{R}x,\mathscr{R}'x') = \dots\\ &= m_{(b_1,b_1')}(x,x')m_{(b_2,b_2')}(\mathscr{R}x,\mathscr{R}'x')\dots m_{(b_{k-1},b_{k-1}')}(\mathscr{R}^{k-1}x,\mathscr{R}'^{k-1}x').\end{aligned}$$

Next we will need to compute the projection $P_V S_\omega \mathbf{1}$, onto the subspace V of functions which depend only on the first component,

$$V = \{f(x,y) = g(x) : g \in L^2[0,1]\}.$$

It is easy to see that the projection onto V is given by the formula

$$(P_V f)(x) = \int_{[0,1]} f(x,x')\,dx', \quad (f \in L^2([0,1]\times[0,1])).$$

Using the invariance equation for the Lebesgue measure under the maps $\tau'_{c'}(x') = (x'+c')/N'$, $c' \in \{0,\dots,N'-1\}$, we have

$$\begin{aligned}(P_V S_\omega \mathbf{1})(x) &= \int_{[0,1]} m_{(b_1,b'_1)}(x,x') \,\dots\, m_{(b_k,b'_k)}(\mathscr{R}^{k-1}x, \mathscr{R}'^{k-1}x')\,dx' \\ &= \frac{1}{N'}\sum_{c'\in B'}\int_{[0,1]} m_{(b_1,b'_1)}(x,\tau'_{c'}x') \,\dots\, m_{(b_k,b'_k)}(\mathscr{R}^{k-1}x, \mathscr{R}'^{k-1}\tau'_{c'}x')dx'.\end{aligned}$$

But, by (2.3),

$$\frac{1}{N'}\sum_{c'\in B'} m_{(b_1,b'_1)}(x,\tau'_{c'}x') = \frac{1}{N'}\sum_{c'\in B'} a_{(b_1,b'_1),(b(x),c')} = \alpha_{(b,b'),b(x)},$$

where $b(x) = b$ if $x \in \left[\frac{b}{N}, \frac{b+1}{N}\right)$. So

$$(P_V S_\omega \mathbf{1})(x) = \alpha_{(b_1,b'_1),b(x)} \int_{[0,1]} m_{(b_2,b'_2)}(\mathscr{R}x,x')\dots m_{(b_k,b'_k)}(\mathscr{R}^{k-1}x, \mathscr{R}'^{k-2}x')\,d\mu'(x').$$

It now follows by induction that

$$(P_V S_\omega \mathbf{1})(x) = \prod_{j=1}^{k} \alpha_{(b_j,b'_j),b(\mathscr{R}^{j-1}x)}.$$

We can compute that, if $\tilde{\omega}$ is a word over $L = \{0,\dots,M-1\}$, then

$$\begin{aligned}\tilde{S}_{\tilde{\omega}}\mathbf{1}(x) &= m_{\tilde{\omega}_1}(x)\, m_{\tilde{\omega}_2}(\mathscr{R}x) \,\dots\, m_{\tilde{\omega}_k}(\mathscr{R}^{k-1}x) \\ &= \alpha_{\tilde{\omega}_1,b(x)}\, \alpha_{\tilde{\omega}_2,b(\mathscr{R}x)} \cdots\, \alpha_{\tilde{\omega}_k,b(\mathscr{R}^{k-1}x)}.\end{aligned} \tag{2.8}$$

So for the word ω over $B\times B'$, $P_V S_\omega \mathbf{1} = \tilde{S}_{\tilde{\omega}}\mathbf{1}$, if all the digits ω are in $\iota(L)$ and $\omega_j = \iota(\tilde{\omega}_j)$, and $P_V S_\omega \mathbf{1} = 0$ if at least one of the digits ω_j is not in $\iota(L)$.

We will prove that

$$\{S_\omega \mathbf{1} : \omega \text{ is a word over } B\times B',\ \text{not ending in } (0,0)\} \tag{2.9}$$

is an orthonormal basis for $L^2([0,1]\times[0,1])$.

It is easy to see that the family is orthonormal: if two words ω and ω' differ on the i-th position, since the Cuntz isometries S_{ω_i} and $S_{\omega'_i}$ have orthogonal ranges, it follows that $S_\omega \mathbf{1}$ and $S_{\omega'}\mathbf{1}$ are orthogonal; if ω and ω' do not differ on any position, then one is a prefix of the other, and by completing with zeros at the end and using the fact that $S_{(0,0)}\mathbf{1} = \mathbf{1}$, again one obtains orthogonality.

It remains to prove the completeness.

Note first that, if

$$e_{(t,t')}(x,x') := e^{2\pi i (t,t')\cdot(x,x')}, \ \ ((t,t') \in \mathbb{R}\times\mathbb{R},\ (x,x') \in [0,1]\times[0,1]), \tag{2.10}$$

then

$$\begin{aligned}
\left(S^*_{(b,b')} e_{(t,t')}\right)(x,x') &= \frac{1}{NN'} \sum_{(c,c')\in B\times B'} \overline{m}_{(b,b')}(\Upsilon_{(c,c')}(x,x'))\ e_{(t,t')}(\Upsilon_{(c,c')}(x,x')) \\
&= \frac{1}{NN'} \sum_{(c,c')\in B\times B'} \overline{a}_{(b,b'),(c,c')}\ e^{2\pi i\left(\frac{t\cdot(x+c)}{N}+\frac{t'\cdot(x'+c')}{N'}\right)} \\
&= \left\{ \frac{1}{NN'} \sum_{(c,c')\in B\times B'} \overline{a}_{(b,b'),(c,c')} e^{2\pi i\left(\frac{t\cdot c}{N}+\frac{t'\cdot c'}{N'}\right)} \right\} e_{\left(\frac{t}{N},\frac{t'}{N'}\right)}(x,x') \\
&= v_{(b,b')}(t,t')\ e_{\left(\frac{t}{N},\frac{t'}{N'}\right)}(x,x'),
\end{aligned}$$

where

$$v_{(b,b')}(t,t') = \frac{1}{NN'} \sum_{(c,c')\in B\times B'} \overline{a}_{(b,b'),(c,c')} e^{2\pi i\left(\frac{t\cdot c}{N}+\frac{t'\cdot c'}{N'}\right)}. \tag{2.11}$$

Let $\mathscr{K}$ be the closed span of the family $\{S_\omega \mathbf{1}\}$ in (2.9), and let $P_{\mathscr{K}}$ be the orthogonal projection onto $\mathscr{K}$. Let Ω be the set of words over $B\times B'$ that do not end in zero, including the empty word. Define, for $(t,t') \in \mathbb{R}\times\mathbb{R}$,

$$h(t,t') = ||P_{\mathscr{K}} e_{(t,t')}||^2. \tag{2.12}$$

We have

$$\begin{aligned}
h(t,t') &= \sum_{\omega\in\Omega} \left|\langle e_{(t,t')}, S_\omega \mathbf{1}\rangle\right|^2 = \sum_{\omega_1\in B\times B'} \sum_{\omega\in\Omega} \left|\langle e_{(t,t')}, S_{\omega_1} S_\omega \mathbf{1}\rangle\right|^2 \\
&= \sum_{\omega_1\in B\times B'} \sum_{\omega\in\Omega} \left|\langle S^*_{\omega_1} e_{(t,t')}, S_\omega \mathbf{1}\rangle\right|^2 = \sum_{\omega_1\in B\times B'} \left|v_{\omega_1}(t,t')\right|^2 \sum_{\omega\in\Omega} \left|\langle e_{\left(\frac{t}{N},\frac{t'}{N'}\right)}, S_\omega \mathbf{1}\rangle\right|^2 \\
&= \sum_{\omega_1\in B\times B'} \left|v_{\omega_1}(t,t')\right|^2\ h\left(\frac{t}{N}, \frac{t'}{N'}\right).
\end{aligned}$$

Now, note that

$$\begin{aligned}
&\sum_{\omega_1\in B\times B'}\left|v_{\omega_1}(t,t')\right|^2\\
=&\sum_{(b,b')}\frac{1}{(NN')^2}\sum_{(c,c')\in B\times B'}\sum_{(d,d')\in B\times B'}\overline{a}_{(b,b'),(c,c')}a_{(b,b'),(d,d')}\,e^{2\pi i(\frac{t}{N},\frac{t'}{N'})\cdot[(c,c')-(d,d')]}\\
=&\frac{1}{(NN')^2}\sum_{(c,c'),(d,d')}e^{2\pi i(\frac{t}{N},\frac{t'}{N'})\cdot[(c,c')-(d,d')]}\sum_{(b,b')}\overline{a}_{(b,b'),(c,c')}a_{(b,b'),(d,d')}\\
=&1,
\end{aligned}$$

which follows from the fact that the matrix $\frac{1}{\sqrt{NN'}}(a_{(b,b'),(c,c')})$ is unitary.

So $h(t,t')=h\left(\frac{t}{N},\frac{t'}{N'}\right)$. Since $h(t,t')=||P_{\mathscr{K}}e_{(t,t')}||^2$, we can easily see that h is continuous on $\mathbb{R}^2$. Also, since $e_{(0,0)}=\mathbf{1}\in\mathscr{K}$, we get that $h(0,0)=1$.

By induction, we have

$$h(t,t')=h\left(\frac{t}{N^n},\frac{t'}{(N')^n}\right)\xrightarrow{n\to\infty}h(0,0)=1.$$

It follows that $h(t,t')$ is the constant 1, which means that $e_{(t,t')}$ is in $\mathscr{K}$ for any (t,t'). By the Stone-Weierstrass theorem, $\mathscr{K}=L^2([0,1]\times[0,1])$. Thus we have a complete orthonormal basis. Hence $P_VS_\omega\mathbf{1}$ is a Parseval frame. Eliminating the zeros, according to the statement after (2.8), we obtain that the functions $\tilde{S}_{\tilde{\omega}}\mathbf{1}$, with $\tilde{\omega}\in\Omega_M$ form a Parseval frame for $L^2[0,1]$.

REMARK 2.2. Our proof shows how the Parseval frame $\{\tilde{S}_{\tilde{\omega}}\mathbf{1}:\tilde{\omega}\in\Omega_M\}$ can be dilated to an orthonormal basis and also how the operators $\tilde{S}_l$, $l\in\{0,\dots,M-1\}$ can be dilated to a representation of the Cuntz algebra as in Theorem 1.2. We describe here, more precisely, what we mean by this.

Note first that we changed the index set $L=\{0,\dots,M-1\}$ to the index set $B\times B'=\{0,\dots,N-1\}\times\{0,\dots,N'-1\}$ and we embedded L into $B\times B'$ by the map ι, with $\iota(0,0)=0$. We also defined $\alpha_{(b,b'),c}=\alpha_{l,c}$ if $(b,b')=\iota(l)$, $c\in B$, and $\alpha_{(b,b'),c}=0$ otherwise.

The operators $(S_{(b,b')})_{(b,b')\in B\times B'}$ form a representation of the Cuntz algebra $\mathscr{O}_{NN'}$ on the Hilbert space $L^2([0,1]\times[0,1])$.

Let

$$V=\{f\in L^2([0,1]\times[0,1]):f(x,y)=g(x),g\in L^2[0,1]\},$$

which can be identified with $L^2[0,1]$. Define the operators $\tilde{S}_{(b,b')}$ on $L^2[0,1]$, for $(b,b')\in B\times B'$,

$$\tilde{S}_{(b,b')}=\begin{cases}\tilde{S}_l, & \text{if } (b,b')=\iota(l),\\ 0, & \text{otherwise.}\end{cases}$$

We prove that

$$S^*_{(b,b')}P_V=\tilde{S}^*_{(b,b')},\quad((b,b')\in B\times B'). \tag{2.13}$$

Using the relation before (2.11) and the relation (2.3), we see that

$$\left(S^*_{(b,b')}e_{(t,0)}\right)(x,x') = \frac{1}{NN'} \sum_{(c,c')\in B\times B'} \overline{a}_{(b,b'),(c,c')} e^{2\pi i \frac{t\cdot c}{N}} = \frac{1}{N}\sum_{c\in B} \overline{\alpha}_{(b,b'),c} e^{2\pi i\frac{t\cdot c}{N}}$$
$$= \left(\tilde{S}^*_{(b,b')}e_t\right)(x).$$

The last equality follows from a similar computation to the one just before (2.11), when $(b,b')\in\iota(L)$, and, if $(b,b')\notin\iota(L)$, $\alpha_{(b,b'),c}=0$ for all $c\in B$.

Since the functions $e_{(t,0)}$, $t\in\mathbb{R}$ are dense in V, we obtain (2.13). Also, since the vector $\mathbf{1}$ is cyclic for the representation (it generates an orthonormal basis for $L^2([0,1]\times[0,1])$ as we have seen before), we get that V is also cyclic for this representation.

Thus, by Theorem 1.4, the operators $(S_{(b,b')})$ form a representation of the Cuntz algebra $\mathscr{O}_{NN'}$ which is the dilation of the operators $(\tilde{S}_{(b,b')})$ which satisfy the relation

$$\sum_{(b,b')\in B\times B'} \tilde{S}_{(b,b')}\tilde{S}^*_{(b,b')} = I_{L^2[0,1]}.$$

The advantage of enlarging the index set from L to $B\times B'$ is that the dilation has a nice structure of a Cartesian product. The disadvantage is that when we project back to the original space we get some extra zeros.

If we want to avoid these zeros, then we can consider the subspace $\tilde{K}$ which is spanned by the vectors

$$\{S_{(b_1,b_1')}\dots S_{(b_k,b_k')}\mathbf{1} : (b_1,b_1')\dots(b_k,b_k')\in\iota(L), k\in\mathbb{N}\}.$$

It is easy to see that the space $\tilde{K}$ is invariant for $S_{\iota(l)}$ and $S^*_{\iota(l)}$, $l\in L$, the relation (2.13) is preserved, when restricted to $\tilde{K}$, and therefore $\tilde{K}$, with the restrictions of the operators $S_{\iota(l)}$ to it, are exactly the dilation of the operators $\tilde{S}_l$ as in Theorem 1.2.

PROPOSITION 2.3. *Let A be the matrix $(\alpha_{l,b})_{l=0,\dots,M-1,b=0,\dots,N-1}$ and $\overrightarrow{\alpha}_l = (\alpha_{l,0},\alpha_{l,1},\dots,\alpha_{l,N-1})$ be the l^{th} row of A.*

1. *Let $l_0,l_1,\dots,l_{k-1}\in\{0,\dots,M-1\}$ and let $b_0,b_1,\dots,b_{k-1}\in\{0,\dots,N-1\}$. Define $l := l_0+Ml_1+\dots+M^{k-1}l_{k-1}$ and $b := b_{k-1}+Nb_{k-2}+\dots+N^{k-1}b_0$. Then, for $x\in\left[\frac{b}{N^k},\frac{b+1}{N^k}\right]$,*

$$\left(\tilde{S}_{l_0}\tilde{S}_{l_1}\dots\tilde{S}_{l_{k-1}}\mathbf{1}\right)(x) = \alpha_{l_0,b_0}\alpha_{l_1,b_1}\dots\alpha_{l_{k-1},b_{k-1}} = (A^{\otimes k})_{l,b}. \tag{2.14}$$

   Here $A^{\otimes k}$ is the tensor product $A\otimes A\otimes\dots\otimes A$, k times.

2. *The l^{th} row of $\frac{1}{\sqrt{N^k}}A^{\otimes k}$ is $\frac{1}{\sqrt{N^k}}(\overrightarrow{\alpha}_{l_0}\otimes\overrightarrow{\alpha}_{l_1}\otimes\dots\otimes\overrightarrow{\alpha}_{l_{k-1}})$ and these rows form a Parseval frame for $\mathbb{C}^{N^k}$. The family*

$$\{\tilde{S}_\omega\mathbf{1} : \omega \text{ is a word over } \{0,\dots,M-1\} \text{ of length } \leqslant k \text{ not ending in } 0\}$$

coincides with the family

$$\{\tilde{S}_\omega \mathbf{1} : \omega \text{ is a word of length } = k\}$$

and it forms a Parseval frame for the subspace $\mathscr{F}_k$ *of* L^2*-functions which are constant on every interval of the form* $\left[\frac{b}{N^k}, \frac{b+1}{N^k}\right]$, *for* $b \in \{0, \dots, N^{k-1}\}$.

3. *If* $\tilde{\omega} \in \Omega_M = \{\text{words not ending in } 0\}$, *and* $\text{length}(\tilde{\omega}) =: |\tilde{\omega}| \geqslant k+1$, *then* $\tilde{S}_{\tilde{\omega}}\mathbf{1}$ *is orthogonal to the subspace* $\mathscr{F}_k$.

Proof. (i) This follows from (2.8), so , if $x \in \left[\frac{b_1+Nb_0}{N^2}, \frac{b_1+Nb_0+1}{N^2}\right]$, then

$$\tilde{S}_{l_0}\tilde{S}_{l_1}\mathbf{1}(x) = \alpha_{l_0,b(x)}\alpha_{l_1,b(\mathscr{R}x)} = \alpha_{l_0,b_0}\alpha_{l_1,b_1},$$

and the fact that

$$(A \otimes A)_{l_0+Ml_1,\ b_0+Mb_1} = \alpha_{l_0,b_0}\alpha_{l_1,b_1},$$

and

$$\left(\overrightarrow{\alpha}_{l_0} \otimes \overrightarrow{\alpha}_{l_1}\right)(b_0+Nb_1) = \overrightarrow{\alpha}_{l_0}(b_0)\overrightarrow{\alpha}_{l_1}(b_1).$$

(ii) Since $\frac{1}{\sqrt{N}}A$ is an isometry, i.e., $\frac{1}{N}A^*A = I_N$, then

$$\frac{1}{N^k}(A^{\otimes k})^*(A^{\otimes k}) = \frac{1}{N^k}\underbrace{(A^*A \otimes \dots \otimes A^*A)}_{k \text{ terms}} = \underbrace{I_{\mathbb{C}^N} \otimes \dots \otimes I_{\mathbb{C}^N}}_{k \text{ terms}} = I_{\mathbb{C}^{N^k}}.$$

So the rows of $\frac{1}{N^k}A^{\otimes k}$ form a Parseval frame for $\mathbb{C}^{N^k}$. The subspace $\mathscr{F}_k$ is isometric to $\mathbb{C}^{N^k}$ by the map $\psi_k : \mathbb{C}^{N^k} \to \mathscr{F}_k$, which is defined by

$$\psi_k(V_0, \dots, V_{N^k-1}) := \sqrt{N^k}\sum_{b=0}^{N^k-1} V_b\, \chi_{\left[\frac{b}{N^k}, \frac{b+1}{N^k}\right]}$$

and

$$\psi_k\left(\frac{1}{\sqrt{N^k}}(\overrightarrow{\alpha}_{l_0} \otimes \overrightarrow{\alpha}_{l_1} \otimes \dots \otimes \overrightarrow{\alpha}_{l_{k-1}})\right) = \tilde{S}_{l_0} \dots \tilde{S}_{l_{k-1}}\mathbf{1}.$$

Therefore, the family

$$\{\tilde{S}_\omega \mathbf{1} : \omega \text{ is a word over } \{0, \dots, M-1\} \text{ of length } k\}$$

forms a Parseval frame for $\mathscr{F}_k$.

Since $\tilde{S}_0\mathbf{1} = \mathbf{1}$, we can see that this family coincides with

$$\{\tilde{S}_\omega \mathbf{1} : \omega \text{ is a word over } \{0, \dots, M-1\} \text{ of length } \leqslant k \text{ not ending in } 0\}.$$

(iii) With $\tilde{\omega}$ given as before (so $|\tilde{\omega}| \geqslant k+1$), we have for $f \in \mathscr{F}_k$, by (ii) above, that

$$||f||^2 = \sum_{\substack{\omega \in \Omega_M, \\ |\omega| \leqslant k}} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2. \tag{2.15}$$

On the other hand, since $\{\tilde{S}_\omega \mathbf{1} : \omega \in \Omega_M\}$ is a Parseval frame, it follows that

$$||f||^2 = \sum_{\omega\in\Omega_M} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2 = \sum_{\substack{\omega\in\Omega_M,\\ |\omega|\leqslant k}} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2 + \sum_{\substack{\omega\in\Omega_M,\\ |\omega|\geqslant k+1}} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2.$$

From this, we get

$$\sum_{\substack{\omega\in\Omega_M,\\ |\omega|\geqslant k+1}} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2 = 0.$$

So $\tilde{S}_{\tilde{\omega}}\mathbf{1}$ is orthogonal to f if $|\tilde{\omega}| \geqslant k+1$.

REMARK 2.4. Now we shall provide an alternative proof for Theorem 1.4, using Part (ii) in the Proposition 2.3 above (which we note that it does not require the proof of Theorem 1.4).

By (ii) in Proposition 2.3, if $|\tilde{\omega}| = m \geqslant k+1$, we have that $\{\tilde{S}_\omega \mathbf{1} : \omega \in \Omega_M, |\omega| \leqslant m\}$ is a Parseval frame for $\mathscr{F}_m$. So it follows that

$$||f||^2 = \sum_{\substack{\omega\in\Omega_M,\\ |\omega|\leqslant m}} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2 = \sum_{\substack{\omega\in\Omega_M,\\ |\omega|\leqslant k}} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2 + \sum_{\substack{\omega\in\Omega_M,\\ m\geqslant|\omega|\geqslant k+1}} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2.$$

From this, we get

$$\sum_{\substack{\omega\in\Omega_M,\\ m\geqslant|\omega|\geqslant k+1}} \left|\langle f, \tilde{S}_\omega \mathbf{1}\rangle\right|^2 = 0.$$

So $\tilde{S}_{\tilde{\omega}}\mathbf{1}$ is orthogonal to f if $|\tilde{\omega}| \geqslant k+1$.

Now take $f \in \mathscr{F}_k$. We have

$$\sum_{\tilde{\omega}\in\Omega_M} |\langle f, \tilde{S}_{\tilde{\omega}}\mathbf{1}\rangle|^2 = \sum_{\tilde{\omega}\in\Omega_M, |\omega|\leqslant k} |\langle f, \tilde{S}_{\tilde{\omega}}\mathbf{1}\rangle|^2 + \sum_{\tilde{\omega}\in\Omega_M, |\omega|\geqslant k+1} |\langle f, \tilde{S}_{\tilde{\omega}}\mathbf{1}\rangle|^2 = \|f\|^2 + 0.$$

But the union of the spaces $\mathscr{F}_k$, $k \in \mathbb{N}$ is dense in $L^2[0,1]$, and therefore $\tilde{S}_{\tilde{\omega}}\mathbf{1}$, $\tilde{\omega} \in \Omega_M$ do form a Parseval frame for $L^2[0,1]$.

In the following, we present a way to construct examples of matrices T satisfying (1.5) and (1.6). As we mentioned in the introduction, this is equivalent to the fact that the rows $\overrightarrow{T}_l = \frac{1}{\sqrt{N}}(\alpha_{l,0}, \dots, \alpha_{l,N-1})$, $l = 0, \dots, M-1$ form a Parseval frame for $\mathbb{C}^N$ and $\overrightarrow{T}_0(i) = \frac{1}{\sqrt{N}}$ for all $i = 0, \dots N-1$.

Let $\langle \overrightarrow{T}_0 \rangle$ be the subspace spanned by $\overrightarrow{T}_0$ and $\langle \overrightarrow{T}_0 \rangle^\perp$ be its orthogonal complement in $\mathbb{C}^N$.

PROPOSITION 2.5. *Let $\overrightarrow{T}_l$, $l = 0, \dots, M-1$ be a set of vectors in $\mathbb{C}^N$ with $\overrightarrow{T}_0(i) = \frac{1}{\sqrt{N}}$ for all $i = 0, \dots, N-1$. The following affirmations are equivalent:*

1. *The vectors $\overrightarrow{T}_l$, $l = 0, \dots, M-1$ form a Parseval frame for $\mathbb{C}^N$.*

2. *$\overrightarrow{T}_l \perp \overrightarrow{T}_0$ for all $l = 1,\dots,M-1$ and the vectors $\overrightarrow{T}_l$, $l = 1,\dots,M-1$, form a Parseval frame for $\langle \overrightarrow{T}_0\rangle^{\perp}$.*

Proof. Note first that $\|\overrightarrow{T}_0\| = 1$.
(i)$\Rightarrow$(ii). Since the vectors $\overrightarrow{T}_l$, $l = 0,\dots,M-1$ form a Parseval frame, we have

$$1 = \|\overrightarrow{T}_0\|^2 = |\left\langle \overrightarrow{T}_0, \overrightarrow{T}_0 \right\rangle|^2 + \sum_{l=1}^{M-1} |\left\langle \overrightarrow{T}_0, \overrightarrow{T}_l \right\rangle|^2 = 1 + \sum_{l=1}^{M-1} |\left\langle \overrightarrow{T}_0, \overrightarrow{T}_l \right\rangle|^2.$$

This implies that $\overrightarrow{T}_l \perp \overrightarrow{T}_0$ for all $l = 1,\dots,M-1$.

Take now a vector f in $\langle \overrightarrow{T}_0\rangle^{\perp}$. We have

$$\|f\|^2 = |\left\langle f, \overrightarrow{T}_0 \right\rangle|^2 + \sum_{l=1}^{M-1} |\left\langle f, \overrightarrow{T}_l \right\rangle|^2 = \sum_{l=1}^{M-1} |\left\langle f, \overrightarrow{T}_l \right\rangle|^2,$$

so $\overrightarrow{T}_l$, $l = 1,\dots,M-1$ form a Parseval frame for $\langle \overrightarrow{T}_0\rangle^{\perp}$.

(ii)$\Rightarrow$(i). Let f be a vector in $\mathbb{C}^N$. We can decompose f as $f = \left\langle f, \overrightarrow{T}_0 \right\rangle \overrightarrow{T}_0 + f_1$, with $f_1 \in \langle \overrightarrow{T}_0\rangle^{\perp}$. We have

$$\|f\|^2 = |\left\langle f, \overrightarrow{T}_0 \right\rangle|^2 + \|f_1\|^2 = |\left\langle f, \overrightarrow{T}_0 \right\rangle|^2 + \sum_{l=1}^{M-1} |\left\langle f_1, \overrightarrow{T}_l \right\rangle|^2$$

$$= |\left\langle f, \overrightarrow{T}_0 \right\rangle|^2 + \sum_{l=1}^{M-1} |\left\langle f, \overrightarrow{T}_l \right\rangle|^2 = \sum_{l=0}^{M-1} |\left\langle f, \overrightarrow{T}_l \right\rangle|^2.$$

With Proposition 2.5, we see that, to construct matrices T as in (1.5) and (1.6), we just have to construct a Parseval frame for $\langle \overrightarrow{T}_0\rangle^{\perp}$. This can be done by picking an isometry Ψ from $\mathbb{C}^{N-1}$ to $\langle \overrightarrow{T}_0\rangle^{\perp}$, a Parseval frame e_l, $l = 1,\dots,M-1$ for $\mathbb{C}^{N-1}$, and letting $\overrightarrow{T}_l = \Psi(e_l)$, $l = 1,\dots,M-1$.

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