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ON THE LOCAL SPECTRAL PROPERTIES OF THE LEFT MULTIPLICATION OPERATORS

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Abstract. Let X and Y be complex Banach spaces, and $B(Y,X)$ (resp. $B(X)$) be the space of all bounded linear operators from Y into X (resp. from X into itself). Fix an operator $T \in B(X)$ and an open subset U of $\mathbb{C}$, and denote by L_T the left multiplication operator on $B(Y,X)$ induced by T. Let

$$\mathscr{X}_T(\mathbb{C}\setminus U) := \{x \in X : (T-\lambda)f(\lambda) = x \text{ has an analytic solution } f \text{ on } U\}$$

denote the glocal spectral subspace of T on $\mathbb{C}\setminus U$. In this paper, we establish an operator valued factorization theorem type when X is a Hilbert space or Y is an ℓ^1-space, and prove that

$$\{Q \in B(Y,X) : Q(Y) \subseteq \mathscr{X}_T(\mathbb{C}\setminus U)\} \subseteq \mathscr{B}(\mathscr{Y},\mathscr{X})_{L_T}(\mathbb{C}\setminus O)$$

for all nonempty relatively compact open subsets O of U. We also prove that if T has the single-valued extension property (SVEP), then

$$\mathscr{B}(\mathscr{Y},\mathscr{X})_{L_T}(\mathbb{C}\setminus U) = \{Q \in B(Y,X) : Q(Y) \subseteq \mathscr{X}_T(\mathbb{C}\setminus U)\}.$$

Furthermore, we characterize the local spectra and the glocal spectral spaces of the left multiplication operators on $B(\mathscr{A})$ where $\mathscr{A}$ stands for certain Banach spaces and algebras. Moreover, we introduce and study some natural extensions of local, surjective and right spectra of any operator $S \in B(X)$, mainly the minimal and maximal local spectra of S at paracomplete subspaces of X.

1. Introduction

The local spectral theory of bounded linear operators on a complex Banach space X has always been an interesting area of research, and a considerable number of excellent papers and books on this theory have been published; see for instance [6, 21, 26, 30] and the references therein. Recently, several authors have been attracted by preserver problems and local spectral theory. These problems demand the characterization of maps on the algebra $B(X)$, of all bounded linear operators on X, that leave a local spectra invariant; see for instance [1, 2, 3, 4, 5, 10, 11, 12, 13]. Other authors explore the connection between the local spectra of operators and their corresponding left multiplication operators; see for instance [9, 23, 24, 30]. These operators are interesting in the abstract theory, mainly they are used to generalize some known notions and concepts of operators in $B(X)$ to elements of Banach algebras; see for example [9] and [30,

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Remark I.1.7 and Theorem II.9.26]. Roughly speaking, we discuss the local spectra and the glocal spectral spaces of the left multiplication operators.

Throughout this paper, let X, $\widehat{X}$ and Y be Banach spaces over the field $\mathbb{C}$ of complex numbers, and $B(Y,X)$ be the space of all bounded linear operators from Y to X. When $Y=X$, we simply write $B(X)$ instead of $B(X,X)$ and denote its identity operator by $\mathbf{1}_X$. The left multiplication operator L_T on $B(Y,X)$ induced by an operator $T\in B(X)$ is defined by $L_T : Q \mapsto T\circ Q$. For every L_T-invariant Banach subspace A of $B(Y,X)$, the restriction of L_T to A, will be denoted also by L_T. If U is an open subset of $\mathbb{C}^p$, we denote by $H(U,X)$ the space of all X-valued analytic functions on U. For any $f\in H(U,X)$, we write f_λ instead of $f(\lambda)$ for all $\lambda\in U$. We also denote by $\mathbb{N}$ the set of all positive integers, and let

$$\Delta_r(\lambda) := \{\mu\in\mathbb{C} \,:\, |\mu-\lambda|<r\}$$

denote the open disc centered at any $\lambda\in\mathbb{C}$ and of a given radius $r>0$.

This paper contains seven sections and is organized as follows. After the foregoing introduction, a preliminary section is given where we collect some basic facts about vector valued analytic functions, local spectral theory, and ℓ^1-spaces and paracomplete spaces. In Section 3, we suppose that U is an open subset of $\mathbb{C}^p$, and $T\in H\left(U,B(X,\widehat{X})\right)$ and $Q\in H\left(U,B(Y,\widehat{X})\right)$, and then discuss when, on a relatively compact open subset O of U, the equation

$$T_\lambda\circ R_\lambda = Q_\lambda,\ (\lambda\in O) \tag{1.1}$$

has a global solution R in $H(O,B(Y,X))$. In our first main result of this paper (Theorem 3.3), we show that if X is a Hilbert space or Y is an ℓ^1-space such that for every $y\in Y$ the equation

$$T_\lambda\cdot f_\lambda = Q_\lambda(y),\ (\lambda\in U)$$

has a solution $f\in H(U,X)$ depending on y, then (1.1) holds for every relatively compact open subset O of U. Furthermore, we also show that if $\{f\in H(U,X) : T_\lambda\cdot f_\lambda = 0\}$ is topologically complemented in $H(U,X)$, then (1.1) has a global solution in $H(U,B(Y,X))$.

In Section 4, we study the glocal spectral subspaces of the left multiplication operators $L_T\in B(B(Y,X))$ induced by operators $T\in B(X)$. When X is a Hilbert space, or Y is an ℓ^1-space, we provide in our second main result (Theorem 4.1) a complete description of such spaces. Furthermore, we show that if $T\in B(X)$ is an operator having the single valued extension property (in short: SVEP), then

$$B(Y,X)_{L_T}(\mathbb{C}\setminus U) = \{Q\in B(Y,X) \,:\, \mathrm{Ran}(Q) := Q(Y)\subseteq X_T(\mathbb{C}\setminus U)\}.$$

This is a significant extension of Corollary 3.6.11 in [26] where T was supposed to have the Dunford's property (C), which is stronger than the SVEP. Such a description is the backbone of the most results of our paper.

In Section 5, we show that for a paracomplete subspace Z of X and an operator $T\in B(X)$, the set

$$\{\sigma_{L_T}(Q) : Q\in B(Y,X),\ Y \text{ is a Banach space and } \mathrm{Ran}(Q)=Z\},$$

partially ordered by the inclusion, possesses a greatest (resp. smallest) element denoted by $\sigma_T(\varepsilon_Z)$ (resp. $\sigma_T(\pi_Z)$). These sets are called the maximal (resp. minimal) local spectrum of T at Z. If they are equal, we simply denoted them by $\sigma_T(Z)$ and say that T has a local spectrum at Z. We also show that if Z is a finite-dimensional space with a Hamel basis $(x_1,...,x_n)$, then T has a local spectrum at Z and $\sigma_T(Z) = \bigcup_{1\leqslant i\leqslant n}\sigma_T(z_i)$. This result shows that the notions of minimal and maximal local spectra of operators at paracomplete subspaces can be seen as extensions of that of the classical local spectrum at a point.

In Section 6, we show that

$$\sigma_T(\pi_Z) = \mathbb{C}\setminus\{\lambda\in\mathbb{C} : \exists r>0,\ Z\subseteq\mathscr{X}_T(\mathbb{C}\setminus\Delta_r(\lambda))\}$$

for all paracomplete subspaces Z of X and all $T\in B(X)$; see Proposition 6.1. This implies, in particular, that $\sigma_T(\pi_X)$ is exactly the surjectivity spectrum $\sigma_{su}(T) := \{\lambda\in\mathbb{C} : T-\lambda\mathbf{1}_X$ is not surjective$\}$ of T. Whence the "minimal local spectrum" is a natural extension of the "surjectivity spectrum". Moreover, we show that if T has the SVEP, then

$$\sigma_{L_T}(Q) = \sigma_T(Z) = \overline{\bigcup_{z\in Z}\sigma_T(z)}$$

for all $Q\in B(Y,X)$ with $\mathrm{Ran}(Q)=Z$. However, we also show that if X is a Hilbert space, then every operator $T\in B(X)$ has a local spectrum at any paracomplete subspace of X. Thus, one can derive from this the well known equality between the right spectrum and the surjectivity spectrum of any given operator on a given Hilbert space, since $\sigma_T(\varepsilon_X)$ is exactly the right spectrum $\sigma_r(T) := \{\lambda\in\mathbb{C} : T-\lambda\mathbf{1}_X$ is not right invertible in $B(X)\}$ of T; see Proposition 6.3.

In the last Section, for $T\in B(X)$ being an operator with the SVEP, Lemma 7.1 gives a complete description of local spectra of $L_T\in B(B(Y,X))$ when restricted to some subspaces of $B(Y,X)$ satisfying certain functional identities. We then deduce in particular, among other things, "Eschmeier, Laursen and Neumann" characterization of the glocal spectral subspace and the local spectrum of the left multiplication operator induced by a multiplier operator ([22, Proposition 16]).

2. Preliminaries

In this section, we gather some basic facts about vector valued analytic functions, local spectral theory, and ℓ^1-spaces and paracomplete spaces. Our basic references are the books [6] by Aiena, [26] by Laursen and Neumann and [30] by Müller.

2.1. Background from local spectral theory

A Fréchet space is a complete locally convex metrizable space. It is known that the closed graph theorem and the open mapping theorem remain valid in Fréchet spaces; see [8, I.17.3] or [34, Chap.2]. Given a nonempty open subset U of $\mathbb{C}^p$, the space $H(U,X)$ is a Fréchet space when endowed with the topology of uniform convergence of sequences on every compact subset of U. For any nonempty subset Λ of U and

$f \in H(U,X)$, we set $\|f\|_\Lambda := \sup_{\lambda\in\Lambda}\|f_\lambda\| \in [0,\infty]$. Now, for $R \in H(U,B(Y,X))$, we define $\|R\|_\Lambda$ analogously.

For a subset Λ of a topological space, we denote, as usual, the interior by Λ° and the closure by $\overline{\Lambda}$.

Given an operator $T \in B(X)$ and a closed set $F \subseteq \mathbb{C}$, the corresponding glocal spectral subspace is the set $\mathscr{X}_T(F)$ of $x \in X$ such that $(T-\lambda\mathbf{1}_X)f_\lambda = x$ has an analytic solution $\lambda \mapsto f_\lambda$ on $\mathbb{C}\backslash F$. It is a hyperinvariant subspace of T but not necessarily closed. By the way, the glocal spectral subspace of $L_T \in B(B(Y,X))$ will be denoted by $\mathscr{B}(\mathscr{Y},\mathscr{X})_{L_T}(F)$. The local resolvent of T at a vector $x \in X$, denoted by $\rho_T(x)$, is the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U_λ of λ and an analytic function $f : U_\lambda \to X$ such that $(T-\mu\mathbf{1}_X)f_\mu = x$, for all $\mu \in U_\lambda$. The local spectrum of T at x is

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x),$$

and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. The local spectral subspace of T on a subset F of $\mathbb{C}$ is

$$X_T(F) := \left\{x \in X \;:\; \sigma_T(x) \subseteq F\right\},$$

and is as well a T-hyperinvariant subspace. If $X_T(F)$ (or $\mathscr{X}_T(F)$) is closed for any closed subset F of $\mathbb{C}$, the operator T is said to have the Dunford's property (C); see [26, Definition 1.2.18; Proposition 3.3.4].

Given an open subset U of $\mathbb{C}$, an operator $T \in B(X)$ is said to have the U-single valued extension property (in short: U-SVEP) if the only analytic function $f : U \to X$ which satisfies the equation $(T-\lambda\mathbf{1}_X)\cdot f_\lambda = 0$ is the constant function $f = 0$. As known, T is said to have the SVEP at $\lambda \in \mathbb{C}$ if T has the U-SVEP for every open connected neighborhood U of λ, and is said to have the SVEP if T has the U-SVEP for every nonempty open subset U of $\mathbb{C}$; see [6, Definition 2.3]. As in [6, pp.77], we denote

$$\Xi_T := \left\{\lambda \in \mathbb{C} \;:\; T \text{ does not have the SVEP at } \lambda\right\}.$$

It is an open subset of $\mathbb{C}$ contained in $\sigma(T)$ and is empty precisely when T has the SVEP. If T has the Dunford's property (C), then T possesses the SVEP. A collection of examples show that the converse is not true in general; see [26, Proposition 1.2.19].

Note that we can apply these notions to $(L_T,Q) \in B(A)\times A$, instead of $(T,x) \in B(X)\times X$, where A is any L_T-invariant Banach subspace of $B(Y,X)$. In this case, if there is a risk of confusion, we will denote $\rho_{L_T}(Q)_A$, $\sigma_{L_T}(Q)_A$, and so on. Note that if T has the U-SVEP (resp. SVEP at some $\lambda \in \mathbb{C}$, resp. SVEP), then so does $L_T \in B(B(Y,X))$.

2.2. On paracomplete subspaces and ℓ^1-spaces

A Banach space X is called an ℓ^1-space if X is isomorphic, for a given nonempty set I, to the Banach space $\ell^1(I) := \{x = (x_i)_{i\in I} \in \mathbb{C}^I \;:\; |x|_1 := \sum_{i\in I}|x_i| < +\infty\}$. The space $\ell^1(\mathbb{N})$ will be denoted by ℓ^1. It is known that every separable Banach space is a quotient of the space ℓ^1 (see [15, Theorem 6.1]) and that every Banach space is a quotient of an ℓ^1-space; see for instance [26, Lemma 3.2.3]. Now, for a Banach space $(X, \| \ \|)$, we will need the Banach space

$$(X)_{\ell^1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \ : \ \|x\|_{\ell^1} := \sum_{n\in\mathbb{N}} \|x_n\| < \infty \right\}.$$

A subspace Z of a Banach space X is called paracomplete if there are a Banach space Y and an operator $Q \in B(Y,X)$ with $\mathrm{Ran}(Q) = Z$; see [30, C.10.4] and [16]. It is known that a subspace Z of X is paracomplete if there is a complete norm on Z which is greater than the original norm induced by X; see [16, Proposition 2.1]. So, every closed subspace of X is paracomplete. It is also known that Z is a paracomplete space if and only if Z is the image of an ℓ^1-space; see [16, pp. 220].

We finish this section by a lemma in which we collect some well known "lifting" properties of ℓ^1-spaces; see [28, pp.107,108] or [33, Chap.5, 11.29] for a survey.

LEMMA 2.1. *The following assertions are equivalent:*

1. *For every pair of Banach spaces E, F, every linear surjection $u \in B(E,F)$ and every $v \in B(X,F)$, there is an operator $w \in B(X,E)$ which lifts v, i.e. $u \circ w = v$.*

2. *For every Banach space E, every surjection $u \in B(E,X)$ is right invertible (i.e there is $v \in B(X,E)$, satisfying $u \circ v = \mathbf{1}_X$).*

3. *For every Banach space E and every surjection $u \in B(E,X)$, $\ker(u)$ is a complemented subspace of E.*

4. *X is an ℓ^1-space.*

3. Factorization of analytic operator-valued functions

In this section, we let U be a nonempty open subset of $\mathbb{C}^p$, and for any $B(X,\widehat{X})$-valued analytic function $T : \lambda \mapsto T_\lambda$ on U, we introduce

$$T_U : H(U,X) \to H(U,\widehat{X}); \ (T_U f)_\lambda := T_\lambda(f(\lambda)), \ (\lambda \in U).$$

Now, if R belongs to $H(U,B(Y,X))$, we define $T_U \circ R \in H(U,B(Y,\widehat{X}))$ by

$$(T_U \circ R)_\lambda := T_\lambda \circ R_\lambda, \ (\lambda \in U).$$

Also, denote $\ker(T_U) := \{f \in H(U,X) \ : \ T_U f = 0\}$, $T_U H(U,X) := \{T_U f \ : \ f \in H(U,X)\}$ and $T_U\big(H(U,B(Y,X))\big) := \{T_U \circ R \ : \ R \in H(U,B(Y,X))\}$. The Fréchet space of all continuous linear mappings from Y to $H(U,\widehat{X})$ will be denoted by $B(Y,H(U,\widehat{X}))$.

We will need the following result which can be seen as a restatement of a, non trivial, classical one.

LEMMA 3.1. *The mapping* $\Phi_U : H(U,B(Y,\widehat{X})) \to B(Y,H(U,\widehat{X}))$ *defined by*

$$\Phi_U(f)_y(\lambda) := f_\lambda(y),\ (f \in H(U,B(Y,\widehat{X}))), \tag{3.1}$$

is a linear isomorphism, and $\sup_{\|y\|_Y=1} \|\Phi_U(f)_y\|_K = \|f\|_K$ *for any compact subset* K *of* U.

Proof. Let $f : U \to B(Y,\widehat{X})$ be a mapping. By [30, A2. Theorem 9], "f is holomorphic" if and only if "the $\widehat{X}$-valued function $\lambda \mapsto f_\lambda(y)$ is holomorphic on U, for all $y \in Y$". Now, for any $g \in B(Y,H(U,\widehat{X}))$, we consider $f : U \to \widehat{X}^Y$, with f_λ defined by $f_\lambda(y) := g_y(\lambda)$, $\lambda \in U, y \in Y$. Clearly, $f \in H(U,B(Y,\widehat{X}))$ and $\Phi_U(f) = g$. Thus, Φ_U is a well defined surjective mapping. Obviously such a mapping is linear and injective, and therefore it is a linear isomorphism. The last property of the lemma is easy. □

In the next lemma we present some technical arguments needed in the proof of Theorem 3.3.

LEMMA 3.2. *Let* $T \in H(U,B(X,\widehat{X}))$ *and* $Q \in H(U,B(Y,\widehat{X}))$. *Suppose that* A *is a Fréchet space such that* $A \subseteq H(U,X)$ *with continuous embedding and* $\mathrm{Ran}(\Phi_U(Q)) \subseteq T_U A$*, where* Φ_U *is defined, in Lemma 3.1, by (3.1). Then:*

1. $E_U(A) := \{(f,y) \in A \times Y \ :\ T_U f = \Phi_U(Q)_y\}$ *is a Fréchet subspace of* $A \times Y$ *and the mapping* $\Pi_2 : E_U(A) \to Y,\ (f,y) \mapsto y$ *is a continuous surjection for which* $\ker(\Pi_2) = \{f \in A \ :\ T_U f = 0\} \times \{0\}$.
2. *If* Π_2 *possesses a continuous right inverse, then* $T_U \circ R = Q$ *for some* $R \in H(U,B(Y,X))$.

Proof. (1) To show that $E_U(A)$ is a Fréchet subspace of $A \times Y$, it suffices to show that $E_U(A)$ is closed in $A \times Y$. To do so, observe that, in view of Lemma 3.1, we can define the following continuous linear mapping

$$\Psi : A \times Y \to H(U,\widehat{X});\ \Psi(f,y) := T_U f - \Phi_U(Q)_y,\ \ ((f,y) \in A \times Y).$$

To check the continuity of Ψ, choose any compact subset K of U. We have for every $(f,y) \in A \times Y$,

$$\|T_U f - \Phi_U(Q)_y\|_K \leqslant \max\left\{\|T_U\|_K, \|Q\|_K\right\} \cdot \left(\|f\|_K + \|y\|\right).$$

Clearly, $E_U(A) = \Psi^{-1}(\{0\})$ and therefore it is closed in $A \times Y$; as desired.

Now, let $y \in Y$ and note that, since $\Phi_U(Q)_y \in T_U A$, there is $f \in A$ such that $\Phi_U(Q)_y = T_U f$. This implies that $(f,y) \in E_U(A)$ and thus Π_2 is surjective. The expression of $\ker(\Pi_2)$ can be obtained easily.

(2) Assume that $\widehat{\Pi}_2 : Y \to E_U(A)$ is a continuous right inverse of Π_2. Since the mapping $\Pi_1 : E_U(A) \to A$ defined by $(f,y) \mapsto f$ is continuous, we note that $S := \Pi_1 \circ \widehat{\Pi}_2$ is also continuous. Hence, $S \in B(Y,H(U,X))$ and $T_U S(y) = \Phi_U(Q)_y$, $(y \in Y)$. Then, by Lemma 3.1, $R := \Phi_U^{-1}(S) \in H(U,B(Y,X))$ and $T_U \circ R = Q$. □

Keep in mind that Φ_U is defined by (3.1), and let us denote:

- $H_Y^{X,\widehat{X}}(U)$ the set of all $T \in H(U,B(X,\widehat{X}))$ for which

$$\Big\{Q \in H(U,B(Y,\widehat{X})) : \mathrm{Ran}(\Phi_U(Q)) \subseteq T_U H(U,X)\Big\} \subseteq T_U\big(H(U,B(Y,X))\big).$$

- $H_Y^{X,\widehat{X}}(U)_0$ the set of all $T \in H(U,B(X,\widehat{X}))$ for which

$$\Big\{Q \in H(U,B(Y,\widehat{X})) : \mathrm{Ran}(\Phi_U(Q)) \subseteq T_U H(U,X)\Big\} \subseteq T_U\big(H(O,B(Y,X))\big),$$

for every relatively compact open subset O of U.

In other words, $T \in H_Y^{X,\widehat{X}}(U)$ $\big($resp. $T \in H_Y^{X,\widehat{X}}(U)_0\big)$ means that if, for every $y \in Y$, the equation $T_\lambda \circ f_\lambda = Q_\lambda(y)$ has an analytic X-valued solution on U, for some $Q \in H(U,B(Y,\widehat{X}))$, then the equation $T_\lambda \circ R_\lambda = Q$ has an analytic $B(Y,X)$-valued solution on U $\big($resp. on every relatively compact open subset O of $U\big)$. Clearly,

$$H_Y^{X,\widehat{X}}(U) \subseteq H_Y^{X,\widehat{X}}(U)_0 \subseteq H(U,B(X,\widehat{X})).$$

Now, we are in a position to state and prove our first main result.

THEOREM 3.3. *The following statements hold.*

1. *If* $T \in H(U,B(X,\widehat{X}))$ *such that* $\ker(T_U)$ *is topologically complemented in* $H(U,X)$, *then* $T \in H_Y^{X,\widehat{X}}(U)$.
2. X *is isomorphic to a Hilbert space, if and only if, for every pair of Banach spaces* $Y,\widehat{X}$, *we have* $H(U,B(X,\widehat{X})) = H_Y^{X,\widehat{X}}(U)_0$.
3. Y *is an* ℓ^1*-space, if and only if, for every pair of Banach spaces* $X,\widehat{X}$, *we have* $H(U,B(X,\widehat{X})) = H_Y^{X,\widehat{X}}(U)_0$.

Proof. (1) Let $Q \in H(U,B(Y,\widehat{X}))$ such that $\mathrm{Ran}(\Phi_U(Q)) \subseteq T_U H(U,X)$. Set $A := H(U,X)$, and note that $\mathrm{Ran}(\Phi_U(Q)) \subseteq T_U A$, where Φ_U is defined by (3.1). Let F be a closed complement of $\ker(T_U)$ in $H(U,X)$. Then, by Lemma 3.2, $\{(f,y) \in E_U(A) : f \in F\}$ is a closed subspace of $E_U(A)$ and it is a linear supplement of $\ker(\Pi_2)$ in $E_U(A)$. So, by [8, Corollary 4, I.17.3], Π_2 is right invertible. Thus by Lemma 3.2, there is $R \in H(U,B(Y,X))$ such that $T_U \circ R = Q$; as desired.

Necessity in (2) *and* (3). Pick an open relatively compact subset O of U. Take the vector-valued Bergman space

$$A(O,X) := H(O,X) \cap L^2(O,X),$$

where $(L^2(O,X), \|\ \|_2)$ denotes the Banach space of all X-valued measurable functions on O which are Bochner square-integrable with respect to the normalized Lebesgue measure μ on O. For the basic background on vector-valued integration theory, we refer the interested reader to [19] and [20]. Using a straightforward extension, to p-variables case, of the arguments given in [26, Chap.2, §3] and identifying $\mathbb{C}^p$ with $\mathbb{R}^{2p}$ and each complex coordinate z_k with $x_k + \sqrt{-1}y_k$, one sees that $A(O,X)$ consists precisely of all $f \in L^2(O,X)$ for which

$$\int_O \frac{1}{2}\left(\frac{\partial g}{\partial x_k} + \sqrt{-1}\frac{\partial g}{\partial y_k}\right) \cdot f \mathrm{d}\mu = 0, \ \ (k = 1, \dots, p),$$

for all indefinitely differentiable functions $g : O \to \mathbb{C}$ with compact support. Therefore, $(A(O,X), \| \ \|_2)$ is a Banach subspace of $L^2(O,X)$. Moreover, if X is a Hilbert space, then so is $(A(O,X), \| \ \|_2)$. Moreover, it is easily seen that $(A(O,X), \| \ \|_2)$ is continuously embedded in $H(O,X)$.

Now, let $Q \in H(U, B(Y, \widehat{X}))$ such that $\mathrm{Ran}(\Phi_U(Q)) \subseteq T_U H(U,X)$. Observe that $H(U,X) \subseteq A(O,X)$, hence we have $\mathrm{Ran}(\Phi_U(Q)) \subseteq T_U A(O,X)$. Assume, first, that X is a Hilbert space. With the notations of Lemma 3.2 we can see that $\{(f,y) \in E_O(A(O,X)) \ : \ f \in \ker(T_O)^{\perp}\}$ is a closed complement of $\ker(\Pi_2)$ in $E_O(A(O,X))$, where $\ker(T_O)^{\perp}$ denotes the orthogonal complement of $\ker(T_O)$ in the Hilbert space $A(O,X)$. Hence, by [8, Corollary 4, I.17.3], Π_2 is right invertible. However, if Y is an ℓ^1-space and X is an arbitrary Banach space, Lemma 2.1 tells us that Π_2 is right invertible. In the both cases, Lemma 3.2 ensures that $Q \in T_O\big(H(O, B(Y,X))\big)$.

The reverse implication in (2). If X is not a Hilbert space, then X possesses an uncomplemented closed subspace Z. Let $Y = \widehat{X} = X/Z$ be the quotient space, and $q : X \to \widehat{X}; x \mapsto x + Z$ be the canonical surjection. Clearly

$$T : \mathbb{C} \to B(X, \widehat{X}); \ \lambda \mapsto q \text{ and } Q : \mathbb{C} \to B(\widehat{X}); \ \lambda \mapsto \mathbf{1}_{\widehat{X}}$$

are analytic, and $\mathrm{Ran}(\Phi_{\mathbb{C}}(Q)) \subseteq H(\mathbb{C}, \widehat{X})$, while the equality $H(\mathbb{C}, \widehat{X}) = T_{\mathbb{C}} H(\mathbb{C}, X)$ is nontrivial and it follows from [26, Proposition 2.1.4]. Therefore, $\mathrm{Ran}(\Phi_{\mathbb{C}}(Q)) \subseteq T_{\mathbb{C}} H(\mathbb{C}, X)$. However, for every nonempty open set $O \subseteq \mathbb{C}$, there is no $R \in H(O, B(\widehat{X}, X))$ satisfying $T_O \circ R = Q$. Indeed, assume for the sake of contradiction that there is a nonempty open set $O \subseteq \mathbb{C}$ and $R \in H(O, B(\widehat{X}, X))$ satisfying $T_O \circ R = Q$. Then $q \circ R_\lambda = \mathbf{1}_{\widehat{X}}$ for all $\lambda \in O$, and q is right invertible and this yields a contradiction since Z assumed uncomplemented closed subspace in X.

The reverse implication in (3). If Y is not an ℓ^1-space, then there is some $v \in B(Y, \widehat{X})$ and a surjective $u \in B(X, \widehat{X})$, for suitable Banach spaces X and $\widehat{X}$, such that $u \circ w \neq v$ for all $w \in B(\widehat{X}, X)$; see Lemma 2.1. Clearly,

$$T : \mathbb{C} \to B(X, \widehat{X}); \ \lambda \mapsto u \text{ and } Q : \mathbb{C} \to B(Y, \widehat{X}); \ \lambda \mapsto v$$

are analytic. It is easy to see that for every nonempty open set $O \subseteq \mathbb{C}$, there is no $R \in H(O, B(\widehat{X}, X))$ satisfying $T_O \circ R = Q$. However, by [26, Proposition 2.1.4], we have $T_{\mathbb{C}} H(\mathbb{C}, X) = H(\mathbb{C}, \widehat{X})$, hence $\mathrm{Ran}(\Phi_{\mathbb{C}}(Q)) \subseteq T_{\mathbb{C}} H(\mathbb{C}, X)$. Thus, $H(U, B(X, \widehat{X})) \neq H_Y^{X,\widehat{X}}(U)_0$ and the reverse implication in (3) is established. □

Several necessary conditions that ensure the existence of an analytic solution of the equation $T_\lambda \cdot f_\lambda = g_\lambda$, for a given $g \in H(U,\widehat{X})$, can be found in the literature; see [6], [7], [27, Proposition 2.1.4 & Theorem 3.2.1], [30, Chap.II, §11, Theorem 12]. Recall that the minimum modulus of $T \in B(X)$ is defined by $\gamma(T) := \inf\{\|T(x)\|/\mathrm{dist}(x,\ker(T)) : x \in X \setminus \ker(T)\}$; see [26, pp.203] or [6, Definition 1.12]. When the mapping $\lambda \to \gamma(T_\lambda)$ is continuous on U and $\mathrm{Ran}(\gamma(T_\lambda))$ closed for all $\lambda \in U$, we say that T is regular on U. Combining Theorem 3.3 and [30, Chap.II, §11, Theorem 12] we get:

COROLLARY 3.4. *Let U be a domain of holomorphy in $\mathbb{C}^p$, O be a relatively compact open subset of U and $T : U \to B(X,\widehat{X})$ be an analytic regular function. If either X is isometric to a Hilbert space or Y is an ℓ^1-space, then for every $Q \in H(U,B(Y,\widehat{X}))$ satisfying $Q_\lambda(y) \in \mathrm{Ran}(T_\lambda)$, for each $(\lambda,y) \in U \times Y$, we have $Q \in T_U(H(O,B(Y,\widehat{X})))$.*

4. Glocal spectral subspaces of a left multiplication operator

In this section we give a description of glocal spectral subspaces of the left multiplication operators induced by a bounded linear operator on a Banach space.

THEOREM 4.1. *Let U be a nonempty open subset of $\mathbb{C}$ and $T \in B(X)$ be an operator. Then the following statements hold.*

1. *If T has the U-SVEP, then so does L_T, and*

$$\mathscr{B}(\mathscr{Y},\mathscr{X})_{L_T}(\mathbb{C}\setminus U) = \big\{Q \in B(Y,X) \;:\; Q(Y) \subseteq \mathscr{X}_T(\mathbb{C}\setminus U)\big\}.$$

2. *If either X is isomorphic to a Hilbert space, or Y is an ℓ^1-space, then for every nonempty relatively compact open subset O of U,*

$$\mathscr{B}(\mathscr{Y},\mathscr{X})_{L_T}(\mathbb{C}\setminus U) \subseteq \big\{Q \in B(Y,X) : Q(Y) \subseteq \mathscr{X}_T(\mathbb{C}\setminus U)\big\} \subseteq \mathscr{B}(\mathscr{Y},\mathscr{X})_{L_T}(\mathbb{C}\setminus O).$$

Proof. (1) Since T has the U-SVEP, then $\{f \in H(U,X) \;:\; T_U f = 0\} = \{0\}$ is topologically complemented in $H(U,X)$, and thus Theorem 3.3-(1) can be applied.

(2) We note that the first inclusion is obvious. We therefore only need to establish the other one. To show this it suffices to apply Theorem 3.3, when $\widehat{X} = X$, $Q \in B(Y,X)$ is regarded as a constant function in $H(U,B(Y,X))$ and using the holomorphic $B(X)$-valued function $U \ni \lambda \mapsto T - \lambda \mathbf{1}_X$ instead of T. In this case the condition $\mathrm{Ran}(\Phi_U(Q)) \subseteq T_U H(U,X)$ is nothing but $\mathrm{Ran}(Q) \subseteq \mathscr{X}_T(\mathbb{C}\setminus U)$. Therefore, Theorem 4.1-(2), is a restatement, in this particular setting, of the direct sense of Theorem 3.3-(2),(3). □

Clearly, from this theorem we see that if $T \in B(X)$ has the SVEP, then so does L_T. Therefore, for every closed subset F of $\mathbb{C}$, we have

$$B(Y,X)_{L_T}(F) = \{Q \in B(Y,X) \;:\; Q(Y) \subseteq X_T(F)\}.$$

We conclude that if T has the Dunford's property (C), then so does L_T. Thus, Theorem 4.1 is a generalization of [26, Corollary 3.6.11, (a)] since Dunford's property (C) is stronger than the SVEP.

Following [29, pp.73], the inner local spectral radius of an operator $T \in B(X)$ at any vector $x \in X$ is defined by

$$\iota_T(x) := \sup\{r \geqslant 0 \,:\, x \in \mathscr{X}_T(\mathbb{C} \setminus \Delta_r(0))\}.$$

Hence for every $\lambda \in \mathbb{C}$ we have

$$\iota_{T-\lambda \mathbf{1}_X}(x) = \sup\{r \geqslant 0 \,:\, x \in \mathscr{X}_T(\mathbb{C} \setminus \Delta_r(\lambda))\}. \tag{4.1}$$

So $\iota_{T-\lambda \mathbf{1}_X}(x) = 0$ means precisely that $\lambda \in \sigma_T(x)$. We have obviously

$$\iota_{L_T}(Q) \leqslant \inf_{y \in Y} \iota_T(Q(y)) \tag{4.2}$$

for all $Q \in B(Y,X)$ and $T \in B(X)$. The following corollary is a consequence of Theorem 4.1, and tells us that the reverse inequality of (4.2) always holds provided that Y is an ℓ^1-space or X is isomorphic to a Hilbert space, or T has the SVEP at 0.

COROLLARY 4.2. *Let Q be in $B(Y,X)$. If either Y is an ℓ^1-space or X is isomorphic to a Hilbert space, or T has the SVEP at 0, then*

$$\iota_{L_T}(Q) = \inf_{y \in Y} \iota_T(Q(y)). \tag{4.3}$$

Proof. In view of (4.2), it is easy to have the equality if $\inf_{y \in Y} \iota_T(Q(y)) = 0$. Thus, we may and shall assume that $\inf_{y \in Y} \iota_T(Q(y)) > 0$ and then we prove the reverse inequality of (4.2). Indeed, for every positive real $r < \inf_{y \in Y} \iota_T(Q(y))$, Theorem 4.1 tells us that $Q \in \mathscr{B}(\mathscr{Y}, \mathscr{X})_{L_T}(\mathbb{C} \setminus \Delta_r(0))$. Thus, $\iota_{L_T}(Q) > r$, and the desired inequality follows. $\square$

5. Maximal and minimal local spectra of an operator

In the following result we define in a natural way the minimal and maximal local spectra of an operator at a paracomplete subspace.

DEFINITION AND PROPOSITION 5.1. Let $T \in B(X)$ and Z be a paracomplete subspace of X. Denote by $\Sigma_T(Z)$ the set

$$\left\{\sigma_{L_T}(Q) : Q \in B(Y,X), Y \text{ is a Banach space, } \operatorname{Ran}(Q) = Z\right\}.$$

Then the following statements hold.

1. $(\Sigma_T(Z), \subseteq)$ possesses a greatest element denoted by $\sigma_T(\varepsilon_Z)$ which will be called the *maximal local spectrum* of T at Z. Moreover

$$\sigma_T(\varepsilon_Z) = \sigma_{L_T}(\varepsilon_Z)_{B(Z,X)}, \tag{5.1}$$

where $\varepsilon_Z : (Z, |||\ |||) \to X$ is the natural embedding and $|||\ |||$ is any Banach norm on Z which is greater than that induced by X.

2. $(\Sigma_T(Z),\subseteq)$ possesses a smallest element denoted by $\sigma_T(\pi_Z)$ which will be called the *minimal local spectrum* of T at Z. Moreover

$$\{\sigma_T(\pi_Z)\} = \{\sigma_{L_T}(Q) : Q \in B(Y,X), Y \text{ is an } \ell^1\text{-space, and } \mathrm{Ran}(Q) = Z\}. \quad (5.2)$$

In the case when the equality $\sigma_T(\varepsilon_Z) = \sigma_T(\pi_Z)$ holds, this last set will be denoted by $\sigma_T(Z)$ and we will say that T *has a local spectrum* at Z.

Proof. Observe first that if Y_1 are Y_2 are two Banach spaces and $Q_i \in B(Y_i,X)$, $i = 1,2$, such that $Q_2 \circ S = Q_1$ for some $S \in B(Y_1,Y_2)$, then for every $R \in H(U,B(Y_2,X))$ with $(T-\lambda \mathbf{1}_X)\circ R_\lambda = Q_2$, the mapping $\widehat{R} : \lambda \mapsto R_\lambda \circ S$ is analytic on U and $(T-\lambda \mathbf{1}_X)\circ \widehat{R}_\lambda = Q_1$. Thus $\sigma_{L_T}(Q_1)_{B(Y_1,X)} \subseteq \sigma_{L_T}(Q_2)_{B(Y_2,X)}$.

(1) If Y is a Banach space and $Q \in B(Y,X)$ such that $\mathrm{Ran}(Q) \subseteq Z$, then $\widehat{Q} : Y \to (Z, |||\ |||)$ defined by $\varepsilon_Z \circ \widehat{Q} = Q$ is bounded. Thus $\sigma_{L_T}(Q)_{B(Y,X)} \subseteq \sigma_{L_T}(\varepsilon_Z)_{B(Z,X)}$ and (1) is proved.

(2) Let $\pi_Z \in B(Z_1,X)$ be a bounded operator on an ℓ^1-space Z_1, such that $\mathrm{Ran}(\pi_Z) = Z$. Note that π_Z exists since Z is assumed paracomplete. If Y is a Banach space and $Q \in B(Y,X)$ satisfies $\mathrm{Ran}(Q) = Z$, then $Q \circ S = \pi_Z$ for some $S \in B(Z_1,Y)$; see Lemma 2.1. Thus, $\sigma_{L_T}(\pi_Z)_{B(Z_1,X)} \subseteq \sigma_{L_T}(Q)_{B(Y,X)}$. Thus $\sigma_{L_T}(\pi_Z)_{B(Z_1,X)}$ is the smallest element of $(\Sigma_T(Z),\subseteq)$ and does not depend on the choice of the ℓ^1-space Z_1. This proves (2). □

Note that if Z is an ℓ^1-space for a norm finer than the norm induced by X, then every operator $T \in B(X)$ has a local spectrum at Z. The next result shows that the spectra defined previously are extensions of the local spectrum of an operator at a point.

PROPOSITION 5.2. *Let T be a bounded operator on X and Z be a finite dimensional subspace of X, with a Hamel basis $(x_1,\ldots,x_n)$, then T has a local spectrum at Z and* $\sigma_T(Z) = \bigcup_{x\in Z} \sigma_T(x) = \sigma_T(x_1)\cup\ldots\cup\sigma_T(x_n)$.

Proof. Since Z is an ℓ^1-space, T has a local spectrum at Z. Now, if $\lambda_0 \in \bigcap_{1\leqslant i\leqslant n} \rho_T(x_i)$, then there is a real $r>0$ and $\xi^i \in H(\Delta_r(\lambda_0),X)$, such that $(T-\lambda\mathbf{1}_X)(\xi^i_\lambda) = x_i$, $\lambda \in \Delta_r(\lambda_0)$, $1\leqslant i\leqslant n$. Then the mapping $R : \Delta_r(\lambda_0) \to B(Z,X)$ defined by $R_\lambda(x_i) = \xi^i_\lambda x_i$ is analytic and satisfies $(T-\lambda\mathbf{1}_X)\circ R_\lambda = \varepsilon_Z$. Thus $\lambda_0 \in \rho_T(Z)$. So, $\sigma_T(Z) \subseteq \bigcup_{1\leqslant i\leqslant n}\sigma_T(x_i)$. The rest is obvious. □

6. Local spectrum of multiplication operators

As a consequence of Theorem 4.1, we derive an interesting description of different spectra of left multiplication operators.

PROPOSITION 6.1. *Let Z be a paracomplete subspace of X and $T \in B(X)$. Then the following assertions hold :*

1. *$\sigma_T(\pi_Z) = \{\lambda \in \mathbb{C} : \inf_{z\in Z} \iota_{T-\lambda \mathbf{1}_X}(z) = 0\}$, i.e.*

$$\rho_T(\pi_Z) = \{\lambda \in \mathbb{C} : \exists r > 0,\, Z \subseteq \mathscr{X}_T(\mathbb{C}\setminus \Delta_r(\lambda))\} \\ = \{\lambda \in \mathbb{C} : \inf_{z\in Z} \iota_{T-\lambda \mathbf{1}_X}(z) > 0\}.$$

2. *Let $Q \in B(Y,X)$ be such that $\operatorname{Ran}(Q) = Z$, then*

$$\overline{\bigcup_{z\in Z} \sigma_T(z)} \subseteq \sigma_T(\pi_Z) \subseteq \sigma_T(Q) \subseteq \sigma_T(\varepsilon_Z) \subseteq \sigma_T(\pi_Z) \cup \Xi_T \subseteq \overline{\bigcup_{z\in Z} \sigma_T(z) \cup \Xi_T}.$$

3. *If X is isomorphic to a Hilbert space, then T has a local spectrum at Z and*

$$\overline{\bigcup_{z\in Z} \sigma_T(z)} \subseteq \sigma_T(Z) = \left\{\lambda \in \mathbb{C} : \inf_{z\in Z} \iota_{T-\lambda \mathbf{1}_X}(z) = 0\right\} \subseteq \overline{\bigcup_{z\in Z} \sigma_T(z) \cup \Xi_T};$$

4. *if T has the SVEP, then T has a local spectrum at Z and*

$$\sigma_T(Z) = \overline{\bigcup_{z\in Z} \sigma_T(z)}. \tag{6.1}$$

Proof. (1) Follows immediately from formula (4.1) and Corollary 4.2.

(2) The inclusion $\overline{\bigcup_{z\in Z} \sigma_T(z)} \subseteq \sigma_T(\pi_Z)$ follows easily from (1). The inclusions $\sigma_T(\pi_Z) \subseteq \sigma_T(Q) \subseteq \sigma_T(\varepsilon_Z)$ are consequences of (5.1) and (5.2).

Now, for every $\lambda \notin \sigma_T(\pi_Z) \cup \Xi_T$, there is an open connected neighborhood U of λ such that $U \subseteq \rho_T(\pi_Z)$. Thus, T has the U-SVEP. By Theorem 4.1-(1) we obtain that $\lambda \in \rho_T(\varepsilon_Z)$. This proves that $\sigma_T(\varepsilon_Z) \subseteq \sigma_T(\pi_Z) \cup \Xi_T$.

For the last inclusion, observe that for every $\lambda \in \mathbb{C} \setminus \left(\overline{\bigcup_{z\in Z} \sigma_T(z) \cup \Xi_T}\right)$, there is a positive real number r such that $\Delta_r(\lambda) \cap \left(\bigcup_{z\in Z} \sigma_T(z) \cup \Xi_T\right) = \emptyset$. Hence, we have $Z \subseteq X_T(\mathbb{C}\setminus \Delta_r(\lambda))$ and T has the SVEP in each point of $\Delta_r(\lambda)$. This means $X_T(\mathbb{C}\setminus \Delta_r(\lambda)) = \mathscr{X}_T(\mathbb{C}\setminus \Delta_r(\lambda))$. Thus, $\lambda \in \rho_T(\pi_Z)$; see Assertion (1). Hence, $\lambda \notin \sigma_T(\pi_Z) \cup \Xi_T$ and $\sigma_T(\pi_Z) \cup \Xi_T \subseteq \overline{\bigcup_{z\in Z} \sigma_T(z) \cup \Xi_T}$.

(3) If X is isomorphic to a Hilbert space, then the equality $\rho_T(\pi_Z) = \rho_T(\varepsilon_Z)$ is a consequence of (4.1) and Corollary 4.2. Thus we can write $\sigma_T(Z)$ and we deduce the rest from statements (1) and (2). For (4), it derives immediately from (2). □

The following corollary shows that the minimal local spectrum is a ”good” extension of the local spectrum.

COROLLARY 6.2. *Let X be a Banach space and $T \in B(X)$. For every paracomplete subspaces Z_1,..., Z_n of X, the space $Z = Z_1 + \ldots + Z_n$ is a paracomplete subspace of X and $\sigma_T(\pi_Z) = \sigma_T(\pi_{Z_1}) \cup \cdots \cup \sigma_T(\pi_{Z_n})$. Furthermore, if the sum $Z = Z_1 \oplus \ldots \oplus Z_n$ is direct, we have also $\sigma_T(\varepsilon_Z) = \sigma_T(\varepsilon_{Z_1}) \cup \cdots \cup \sigma_T(\varepsilon_{Z_n})$.*

Proof. By [16, Proposition 2.2] we have that Z is paracomplete. The first equality is a consequence of Proposition 6.1-(1). The proof of the second equality is similar to the proof of Proposition 5.2. $\square$

The following variant of "[26, Theorem 3.2.1] and [30, Chap.II, §.11, Coro. 14]" tells us that the maximal and the minimal local spectra are, respectively, natural extensions of the surjectivity and the right spectra.

PROPOSITION 6.3. *For every $T \in B(X)$, the following assertions hold.*

1. $\sigma_T(\varepsilon_X) = \sigma_T(\mathbf{1}_X) = \sigma_r(T)$ *and there is* $R \in H(\mathbb{C} \setminus \sigma_T(\varepsilon_X), B(X))$ *such that* $(T - \lambda \mathbf{1}_X) \circ R_\lambda = \mathbf{1}_X$ *for all* $\lambda \in \mathbb{C} \setminus \sigma_T(\varepsilon_X)$.
2. *If* X_1 *is an* ℓ^1*-space and* $\pi_X \in B(X_1, X)$ *is a surjective operator, then* $\sigma_T(\pi_X) = \sigma_{su}(T) = \sigma_{su}(L_T) = \sigma_{L_T}(\pi_{B(X_1,X)})$, *and there exists* $R \in H(\mathbb{C} \setminus \sigma_T(\pi_X), B(X_1, X))$ *such that* $(T - \lambda \mathbf{1}_X) \circ R_\lambda = \pi_X$ *for all* $\lambda \in \mathbb{C} \setminus \sigma_T(\pi_X)$.

Proof. (1) Since $(T - \lambda \mathbf{1}_X)$ is right invertible for every λ in the open set $\mathbb{C} \setminus \sigma_r(T)$, there is $R \in H(\rho_T(\mathbf{1}_X), B(X))$ such that $(T - \lambda \mathbf{1}_X) \circ R_\lambda = \mathbf{1}_X$ for all $\lambda \in \mathbb{C} \setminus \sigma_r(T)$; see [30, Chap.II, §.11, Corollary 14]. Hence, $\sigma_T(\varepsilon_X) \subset \sigma_r(T)$. The other inclusion is obvious.

(2) By [26, Proposition 1.3.2] and Proposition 6.1,

$$\sigma_T(\pi_X) = \sigma_{su}(T) \text{ and } \sigma_{su}(L_T) = \sigma_{L_T}(\pi_{B(X_1,X)}).$$

But $\sigma_{su}(T) = \sigma_{su}(L_T)$ by Lemma 2.1. Thus, by [26, Theorem 3.2.1], there is $R \in H(\mathbb{C} \setminus \sigma_{su}(T), B(X_1, X))$ such that, for all λ in $\mathbb{C} \setminus \sigma_{su}(T)$, one has $(T - \lambda \mathbf{1}_X) \circ R_\lambda = \pi_X$; as desired. $\square$

Observe that, unlike the setting of Proposition 6.3, we may have $\rho_T(Q) \subsetneq \{\lambda \in \mathbb{C} : \exists S \in B(X), (T - \lambda \mathbf{1}_X) \circ S = Q\}^o$. For example, if T is a nonzero bounded operator on X satisfying $T^2 = 0$, then

$$\{\lambda \in \mathbb{C} : \exists S \in B(X), (T - \lambda \mathbf{1}_X) \circ S = T\} = \mathbb{C} \neq \mathbb{C} \setminus \{0\} = \rho_T(\mathrm{Ran}(T)).$$

We finish this section with the following example.

EXAMPLE 6.4. As mentioned in [30, Chap. II, §9, Example 19], we can construct an operator T on a Banach space X for which $\sigma_{su}(T) \subsetneq \sigma_r(T)$. Thus, by Proposition 6.3, $\sigma_T(\pi_X) = \sigma_{su}(T) \subsetneq \sigma_r(T) = \sigma_T(\varepsilon_X)$. Indeed, denote $\ell^\infty := \{a = (a_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|a\|_\infty := \sup_{n \in \mathbb{N}} |a_n| < \infty\}$ and $c_0 := \{(a_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \lim\limits_{n \to \infty} a_n = 0\}$. It is known that c_0 is an uncomplemented Banach subspace of ℓ^∞. Let $T \in B(c_0 \times \ell^\infty)$ be the surjective operator defined by $T(((a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}})) = ((a_{2n-1})_{n \in \mathbb{N}}, (b_n - a_{2n})_{n \in \mathbb{N}})$. We have obviously $\ker(T) = \{(a_n)_{n \in \mathbb{N}} \in c_0 : a_{2n-1} = 0, \ n \in \mathbb{N}\} \times c_0$ is an uncomplemented subspace of $c_0 \times \ell^\infty$.

7. Local spectra of some restrictions of some left multiplication operators

Here, we provide a description of local spectra for the restriction of the left multiplication operator L_T at some invariant subspaces of $B(Y,X)$.

LEMMA 7.1. *Let A and B be two Banach algebras. Assume that $\widehat{X}$ is a Banach (A,B)-bimodule, X is a closed subspace of $\widehat{X}$ and Y is a Banach space. Let I, J_i, $i \in I$, α and β be nonempty sets, and*

$$(\Psi_{i,j},\chi_{i,j},\Phi_{i,j}) : \Lambda_i \subseteq A^{\alpha} \times B^{\beta} \to A \times Y \times B,\ (j \in J_i,\ i \in I)$$

be a family of mappings. If $T \in B(X)$ satisfies

$$\sum_{j\in J_i} T\left(\Psi_{i,j}(a,b)\cdot x \cdot \Phi_{i,j}(a,b)\right) = \sum_{j\in J_i} \Psi_{i,j}(a,b)\cdot T(x)\cdot \Phi_{i,j}(a,b)$$

for all $i \in I$, $(a,b) \in \Lambda_i$ and $x \in X$, then the following statements hold.

1. *The following space*

$$\mathscr{A} = \{Q \in B(Y,X) : \sum_{j\in J_i} \Psi_{i,j}(a,b)\cdot Q(\chi_{i,j}(a,b))\cdot \Phi_{i,j}(a,b) = 0, i \in I, (a,b) \in \Lambda_i\}$$

   is a L_T-invariant Banach subspace of $B(Y,X)$.

2. *For every $Q \in \mathscr{A}$, we have*

$$\overline{\bigcup_{y\in Y} \sigma_T(Q(y))} \subseteq \sigma_T(\pi_{Q(Y)}) \subseteq \sigma_T(Q)_{B(Y,X)} \subseteq \sigma_T(Q)_{\mathscr{A}} \subseteq \overline{\bigcup_{y\in Y} \sigma_T(Q(y)) \cup \Xi_T}.$$

3. *If T has the SVEP, then $\sigma_T(Q)_{\mathscr{A}} = \overline{\bigcup_{y\in Y} \sigma_T(Q(y))}$, $Q \in \mathscr{A}$, and*

$$\mathscr{A}_{L_T}(F) = \left\{S \in \mathscr{A} : S(Y) \subseteq X_T(\mathbb{C}\setminus F)\right\}$$

   for all closed subsets F of $\mathbb{C}$.

Proof. (1) is a folklore result.

(2) The inclusion $\sigma_T(Q)_{B(Y,X)} \subseteq \sigma_T(Q)_{\mathscr{A}}$ is obvious and due to Proposition 6.1, we have $\overline{\bigcup_{y\in Y} \sigma_T(Q(y))} \subseteq \sigma_T(\pi_{Q(Y)}) \subseteq \sigma_T(Q)_{B(Y,X)}$. For $\sigma_T(Q)_{\mathscr{A}} \subseteq \overline{\bigcup_{y\in Y} \sigma_T(Q(y)) \cup \Xi_T}$, pick a $\lambda \in \left(\bigcap_{y\in Y} \rho_T(Q(y))\right)^o \setminus \Xi_T$. By Proposition 6.1, we have $\lambda \in \rho_T(Q)_{B(Y,X)}$. Then there exists $R \in H(\Delta_r(\lambda), B(Y,X))$, for some real $r > 0$, such that $(T-\mu)\circ R_\mu = Q$, $\mu \in \Delta_r(\lambda)$. Hence, for every $\mu \in \Delta_r(\lambda)$, $i \in I$, and $(a,b) \in \Lambda_i$, we have

$$(T-\mu\mathbf{1}_X)\circ \Big[\sum_{j\in J_i} \Psi_{i,j}(a,b)\cdot R_\mu(\chi_{i,j}(a,b))\cdot \Phi_{i,j}(a,b)\Big]$$

$$= \sum_{j\in J_i} \Psi_{i,j}(a,b)\cdot (T-\mu \mathbf{1}_X)\circ R_\mu(\chi_{i,j}(a,b))\cdot \Phi_{i,j}(a,b)$$
$$= \sum_{j\in J_i} \Psi_{i,j}(a,b)\cdot Q(\chi_{i,j}(a,b))\cdot \Phi_{i,j}(a,b) = 0.$$

As T has the SVEP at λ, $\sum_{j\in J_i} \Psi_{i,j}(a,b)\cdot R_\mu(\chi_{i,j}(a,b))\cdot \Phi_{i,j}(a,b) = 0$, and thus $R \in H(\Delta_r(\lambda),\mathscr{A})$. Finally we obtain $\lambda \in \rho_T(Q)_{\mathscr{A}}$.

Since T has the SVEP, (3) follows from (2) and the fact that $\mathscr{A}_{L_T}(F) = \{Q \in \mathscr{A} : \sigma_T(Q)_{\mathscr{A}} \subseteq F\}$, when F is a closed subset of $\mathbb{C}$. □

Now, we collect some consequences of the preceding lemma. In particular, we obtain [22, Proposition 16].

PROPOSITION 7.2. *If one of the following statements (1)-(6) holds, then*

$$\overline{\bigcup_{e\in E} \sigma_T(Q(e))} \subseteq \sigma_{L_T}(Q)_{B(E)} \subseteq \sigma_{L_T}(Q)_{\mathscr{A}} \subseteq \overline{\bigcup_{e\in E} \sigma_T(Q(e)) \cup \Xi_T}.$$

Furthermore, if T has the SVEP, then

$$\mathscr{A}_{L_T}(O) = \{Q \in \mathscr{A} \ : \ Q(E) \subseteq E_T(O)\} \text{ and } \sigma_{L_T}(Q)_{\mathscr{A}} = \overline{\bigcup_{e\in E} \sigma_T(Q(e))}.$$

1. *$\mathscr{A} = \{S \in B(E) : SR = RS,\ R \in A\}$, where A is a nonempty subset of $B(E)$, E is any Banach space, and $(T,Q) \in A \times \mathscr{A}$.*

2. *$\mathscr{A}$ is a unitary von Neumann algebra acting on a Hilbert space E and $T,Q \in \mathscr{A}$.*

3. *$\mathscr{A} = \mathscr{M}(A) := \{M \in B(E) \ : \ \forall (b,c) \in A^2,\ M(b)c = bM(c)\}$ is the multiplier space of a Banach algebra E, and $T,Q \in \mathscr{M}(E)$.*

4. *$\mathscr{A} = \mathscr{M}_l(E) := \{M \in B(E) : \forall (b,c) \in E^2, M(bc) = M(b)c\}$ (resp. $\mathscr{M}_r(E) := \{M \in B(E) : \forall (b,c) \in E^2, M(bc) = bM(c)\}$) is the algebra of left (resp. right) multipliers of a Banach algebra E, and $T,Q \in \mathscr{A}$.*

5. *A and B are two Banach algebras, X and Y are two Banach (A,B)-bimodules and $\mathscr{A}$ is the Banach space of all bounded (A,B)-bimodule morphisms from X into Y and $T,Q \in \mathscr{A}$.*

6. *$E = \mathscr{A} = \mathscr{D}(A,X)$ is the space of bounded derivations from a Banach algebra A into a Banach A-bimodule X, $T \in B(X)$ is a A-bimodule morphism and $Q \in \mathscr{D}(A,X)$.*

Proof. If (1), (3) or (4) holds, then we can apply directly Lemma 7.1.

Suppose that (2) holds. By the double commuting theorem [31, Theorem 4.5.1], we have $\mathscr{A} = \mathscr{A}'' := \{a \in B(H) \ : \ \forall a \in \mathscr{A}',\ ab - ba = 0\}$, where $\mathscr{A}' = \{b \in B(H) : \forall a \in \mathscr{A},\ ba = ab\}$. Then the case (1) applies.

Now, if (5) holds, then $\mathscr{A}$ is the Banach space of all $S \in B(Y,X)$ for which $S(a \cdot y) = a \cdot S(y)$ and $S(y \cdot b) = S(y) \cdot b$, $\forall (a,b,y) \in A \times B \times Y$. Thus we can apply Lemma 7.1.

Finally, assume that (6) holds and note that, by definition,

$$\mathscr{D}(A,X) = \{d \in B(A,X) \; : \; \forall (a,b) \in A^2, \; d(a \cdot b) - d(a) \cdot b - a \cdot d(b) = 0\}.$$

Therefore Lemma 7.1 applies in this case as well. □

Note that the spaces presented in Proposition 7.2 are very popular (see for instance [17]). Finally, one can observe that more applications of Lemma 7.1 can be given in the Banach context with the appropriate examples illustrated in [14].

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