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TOPOLOGICAL ORBIT DIMENSION OF MF C*–ALGEBRAS AND NFD ALGEBRAS

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Abstract. This paper is a continuation of our work on D. Voiculescu's topological free entropy dimension $\delta_{\text{top}}(x_1,\ldots,x_n)$ for a family $\{x_1,\ldots,x_n\}$ of elements in a unital C*-algebra. We first give a relation between the topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ and the modified free orbit dimension $\mathfrak{K}_2^{(2)}$ by using MF-traces. We also introduce a new invariant $\mathfrak{K}_{top}^{(3)}$ which is a modification of the topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ when $\mathfrak{K}_{top}^{(2)}$ is defined. As an application of $\mathfrak{K}_{top}^{(3)}$, we prove that $\mathfrak{K}_{top}^{(3)}(\mathscr{A})=0$ if the separable C*-algebra $\mathscr{A}$ has property c*-Γ and has no non-zero finite-dimensional representations. We also introduce the property MF-c*-Γ. We then show that $\mathfrak{K}_{top}^{(3)}(\mathscr{A})=0$ if the finitely generated C*-algebra $\mathscr{A}$ has property MF-c*-Γ and has no non-zero finite-dimensional representations.

1. Introduction

This paper is a continuation of the work in [5], [8], [10] on D. Voiculescu's topological free entropy dimension $\delta_{\text{top}}(x_1,\cdots,x_n)$ for the family $\{x_1,\cdots,x_n\}$ of elements in a unital C*-algebra.

Here we first give a relation between the topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ and the modified free orbit dimension $\mathfrak{K}_2^{(2)}$ by using MF-traces (Theorem 4). This result allows us to give a new proof of our main result in [10], which gave an estimation of the upper bound of topological free entropy dimension for MF-nuclear algebras. Then we introduce a new invariant $\mathfrak{K}_{top}^{(3)}$ which is a modification of the topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ when $\mathfrak{K}_{top}^{(2)}$ is defined. The idea for defining $\mathfrak{K}_{top}^{(3)}$ arises from the concept $\mathfrak{K}_3$ in [6]. We then extend the domain of $\mathfrak{K}_{top}^{(3)}$ to all MF algebras and prove that $\mathfrak{K}_{top}^{(3)}$ is a C*-algebra invariant. We also modify the notion $\mathfrak{K}_3$ in [6] by using the modified free orbit dimension $\mathfrak{K}_2^{(2)}$ and denote it by $\mathfrak{K}_3^{(3)}$. We give a relation between $\mathfrak{K}_{top}^{(3)}$ and $\mathfrak{K}_3^{(3)}$ for countably generated MF algebras by using the relation between the topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ and the modified free orbit dimension $\mathfrak{K}_2^{(2)}$. Several properties of $\mathfrak{K}_{top}^{(3)}$ are given as follows:

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1. $\mathfrak{K}_{top}^{(3)}(\mathcal{N}_1) = \mathfrak{K}_{top}^{(3)}(\mathcal{N}_2)$ if $C^*(\mathcal{N}_1) = C^*(\mathcal{N}_2)$.

2. If $\mathscr{A}$ is finitely generated, then $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$ if $\mathfrak{K}_{top}^{(2)}(\mathscr{A}) = 0$.

3. If $\mathcal{N}_1 \cap \mathcal{N}_2$ is finitely generated and has no non-zero finite-dimensional representations (i.e. NFD), then

$$\mathfrak{K}_{top}^{(3)}(C^*(\mathcal{N}_1 \cup \mathcal{N}_2)) \leqslant \mathfrak{K}_{top}^{(3)}(\mathcal{N}_1) + \mathfrak{K}_{top}^{(3)}(\mathcal{N}_2).$$

4. Suppose $\mathcal{N}$ is an MF C*-algebra and $\mathscr{D} \subseteq \mathscr{A} \subseteq \mathcal{N}$ are C*-subalgebras where $\mathscr{D}$ is NFD and has finitely many generators. If there is a unitary $u \in \mathcal{N}$ such that $u\mathscr{D}u^* \subseteq \mathscr{A}$, then

$$\mathfrak{K}_{top}^{(3)}(C^*(\mathscr{A} \cup \{u\})) \leqslant \mathfrak{K}_{top}^{(3)}(\mathscr{A}).$$

As an application, we prove that $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$ if the separable C*-algebra $\mathscr{A}$ has property c*-Γ and has no finite-dimensional representations. We also introduce the property MF-c*-Γ. We then conclude that, for a finitely generated C*-algebra $\mathscr{A}$, if $\mathscr{A}$ has property MF-c*-Γ and has no non-zero finite-dimensional representations, then $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$.

The organization of the paper is as follows. In Section 2, we recall the definition of topological free entropy dimension $\delta_{top}(x_1, \cdots, x_n)$ and topological orbit dimension $\mathfrak{K}_{top}^{(2)}(x_1, \cdots, x_n)$ of n-tuple $(x_1, \cdots, x_n)$ of elements in a unital C*-algebra. In Section 3, we first give a relation between $\mathfrak{K}_{top}^{(2)}$ and $\mathfrak{K}_2^{(2)}$, then we give a new proof of our main result in [10]. In Section 4, we introduce topological orbit dimension $\mathfrak{K}_{top}^{(3)}$ for general MF C*-algebras. Several properties of $\mathfrak{K}_{top}^{(3)}$ are discussed there. Section 5 focuses on applications of $\mathfrak{K}_{top}^{(3)}$ to central sequence algebras. We prove that $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$ if the separable C*-algebra $\mathscr{A}$ has property c*-Γ and has no finite-dimensional representations. We also introduce the property MF-c*-Γ. We then conclude that, for a finitely generated C*-algebra $\mathscr{A}$, if $\mathscr{A}$ has property MF-c*-Γ and has no non-zero finite-dimensional representations, then $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$.

2. Definitions and preliminaries

In this section, we are going to recall Voiculescu's definition of the topological free entropy dimension [12] and topological orbit dimension in a unital C*-algebra [8].

2.1. A covering of a set in a metric space

Suppose (X, d) is a metric space and K is a subset of X. A family of balls in X is called a *covering of* K if the union of these balls covers K and the centers of these balls lie in K.

2.2. Covering numbers in complex matrix algebra $(\mathcal{M}_k(\mathbb{C}))^n$

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}Tr$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{s.a}(\mathbb{C})$ be the subalgebra of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $\left(\mathcal{M}_k^{s.a}(\mathbb{C})\right)^n$ be the direct sum (or orthogonal sum) of n copies of $\mathcal{M}_k^{s.a}(\mathbb{C})$. Let $\|\cdot\|$ be an operator norm on $\mathcal{M}_k(\mathbb{C})^n$ defined by

$$\|(A_1,\cdots,A_n)\| = \max\{\|A_1\|,\cdots,\|A_n\|\}$$

for all $(A_1,\cdots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_2$ denote the norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1,\cdots,A_n)\|_2 = \sqrt{\tau_k(A_1^*A_1)+\cdots+\tau_k(A_n^*A_n)}$$

for all $(A_1,\cdots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|$-*ball* $Ball(B_1,\cdots,B_n;\omega,\|\cdot\|)$ *centered at* $(B_1,\cdots,B_n)$ *in* $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1,\cdots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1,\cdots,A_n)-(B_1,\cdots,B_n)\| < \omega.$$

DEFINITION 1. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_\infty(\Sigma,\ \omega)$ to be the minimal number of ω-$\|\cdot\|$-balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|_2$-*ball* $Ball(B_1,\cdots,B_n;\omega,\|\cdot\|_2)$ *centered at* $(B_1,\cdots,B_n)$ *in* $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1,\cdots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1,\cdots,A_n)-(B_1,\cdots,B_n)\|_2 < \omega.$$

DEFINITION 2. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_2(\Sigma,\omega)$ to be the minimal number of ω-$\|\cdot\|_2$-balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

2.3. Unitary orbits of balls in $\mathcal{M}_k(\mathbb{C})^n$

For every $\omega > 0$, we define the ω-*orbit*-$\|\cdot\|$-*ball* $\mathcal{U}(B_1,\cdots,B_n;\omega)$ *centered at* $(B_1,\cdots,B_n)$ *in* $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1,\cdots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1,\cdots,A_n)-(WB_1W^*,\cdots,WB_nW^*)\| < \omega.$$

DEFINITION 3. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $o_\infty(\Sigma,\omega)$ to be the minimal number of ω-*orbit*-$\|\cdot\|$-balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-*orbit*-$\|\cdot\|_2$-*ball* $\mathcal{U}(B_1,\cdots B_n;\omega,\|\cdot\|_2)$ centered at $(B_1,\cdots,B_n)$ *in* $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1,\cdots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1,\cdots,A_n)-(WB_1W^*,\cdots,WB_nW^*)\|_2 < \omega.$$

DEFINITION 4. Suppose that Σ is a subset of $\mathscr{M}_k(\mathbb{C})^n$. We define $o_2(\Sigma,\omega)$ to be the minimal number of ω-*orbit*-$\|\cdot\|_2$-balls that constitute a covering of Σ in $\mathscr{M}_k(\mathbb{C})^n$.

2.4. Noncommutative polynomials

In this article, we always assume that $\mathscr{A}$ is a unital C*-algebra. Let $x_1,\cdots,x_n,y_1,\cdots,y_m$ be self-adjoint elements in $\mathscr{A}$. Let $\mathbb{C}\langle X_1,\cdots,X_n,Y_1,\cdots,Y_m\rangle$ be the set of all noncommutative polynomials in the indeterminates $X_1,\cdots,X_n,Y_1,\cdots,Y_m$. Let $\mathbb{C}_{\mathbb{Q}}=\mathbb{Q}+i\mathbb{Q}$ denote the complex-rational numbers, i.e., the numbers whose real and imaginary parts are rational. The set $\mathbb{C}_{\mathbb{Q}}\langle X_1,\cdots,X_n,Y_1,\cdots,Y_m\rangle$ of noncommutative polynomials with complex-rational coefficients is countable. For notational convenience, throughout this paper we write

$$\mathbb{C}_{\mathbb{Q}}\langle X_1,\cdots,X_n,Y_1,\cdots,Y_m\rangle=\{P_r:r\in\mathbb{N}\}\text{ and }\mathbb{C}_{\mathbb{Q}}\langle X_1,\cdots,X_n\rangle=\{Q_r:r\in\mathbb{N}\}$$

and

$$\mathbb{C}_{\mathbb{Q}}\langle X_1,X_2\cdots\rangle=\cup_{m=1}^{\infty}\mathbb{C}_{\mathbb{Q}}\langle X_1,\cdots,X_m\rangle.$$

REMARK 1. We always assume that $P_1=1$ and $Q_1=1$.

2.5. Voiculescu's norm-microstates space

For all integers $r,k\geqslant 1$, real numbers $R,\varepsilon>0$ and noncommutative polynomials $P_1,\cdots,P_r$, we define

$$\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n,y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r)$$

to be the subset of $\left(\mathscr{M}_k^{s.a}(\mathbb{C})\right)^{n+m}$ consisting of all these

$$(A_1,\cdots,A_n,B_1,\cdots,B_m)\in\left(\mathscr{M}_k^{s.a}(\mathbb{C})\right)^{n+m}$$

satisfying

$$\max\{\|A_1\|,\cdots,\|A_n\|,\|B_1\|,\cdots\|B_m\|\}\leqslant R$$

and

$$\left|\|P_j(A_1,\cdots,A_n,B_1,\cdots,B_m)\|-\|P_j(x_1,\cdots,x_n,y_1,\cdots,y_m)\|\right|\leqslant\varepsilon,\forall 1\leqslant j\leqslant r.$$

REMARK 2. In the original definition of norm-microstates space in [12], the parameter R was not introduced. Note the following observation: Let

$$R>\max\{\|x_1\|,\|x_2\|,\cdots,\|x_n\|,\|y_1\|,\cdots,\|y_m\|\}.$$

When R is large enough and ε is small enough,

$$\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n,y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r)=\Gamma_\infty^{(\mathrm{top})}(x_1,\cdots,x_n,y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r)$$

for all $k\geqslant 1$. This definition agrees with the one in [12] for large R,r and small ε.

Define the norm-microstates space of $x_1,\cdots,x_n$ in the presence of $y_1,\cdots,y_m$, denoted by

$$\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n : y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r),$$

as the projection of $\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n,y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r)$ onto the space $(\mathscr{M}_k^{s.a}(\mathbb{C}))^n$ via the mapping

$$(A_1,\cdots,A_n,B_1,\cdots,B_m) \to (A_1,\cdots,A_n).$$

2.6. Voiculescu's topological free entropy dimension

Define

$$\nu_\infty\left(\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n : y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r),\omega\right)$$

to be the covering number of the set $\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n : y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r)$ by ω-$\|\cdot\|$-balls in the metric space $(\mathscr{M}_k^{s.a}(\mathbb{C}))^n$ equipped with operator norm.

DEFINITION 5. Define

$$\delta_{\mathrm{top}}(x_1,\cdots,x_n;\omega) = \sup_{R>0}\inf_{\varepsilon>0,r\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log\left(\nu_\infty\left(\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n;k,\varepsilon,Q_1,\cdots,Q_r),\omega\right)\right)}{-k^2\log\omega}.$$

The topological free entropy dimension of $x_1,\cdots,x_n$ is defined by

$$\delta_{top}(x_1,\cdots,x_n) = \limsup_{\omega\to0^+}\delta_{top}(x_1,\cdots,x_n;\omega).$$

Similarly, define

$$\begin{aligned}&\delta_{\mathrm{top}}(x_1,\cdots,x_n : y_1,\cdots,y_m;\omega)\\ =&\sup_{R>0}\inf_{\varepsilon>0,r\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log\left(\nu_\infty\left(\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n : y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r),\omega\right)\right)}{-k^2\log\omega}.\end{aligned}$$

The topological free entropy dimension of $x_1,\cdots,x_n$ in the presence of $y_1,\cdots,y_m$ is defined by

$$\delta_{top}(x_1,\cdots,x_n : y_1,\cdots,y_m) = \limsup_{\omega\to0^+}\delta_{top}(x_1,\cdots,x_n : y_1,\cdots,y_m;\omega).$$

REMARK 3. Let $R > \max\{\|x_1\|,\cdots,\|x_n\|,\|y_1\|,\cdots,\|y_m\|\}$. By Remark 2, we know the supremum over $R>0$ is unnecessary, i.e.,

$$\begin{aligned}&\delta_{top}(x_1,\cdots,x_n : y_1,\cdots,y_m)\\ =&\limsup_{\omega\to0^+}\inf_{\varepsilon>0,r\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log\left(\nu_\infty\left(\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n : y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r),\omega\right)\right)}{-k^2\log\omega}.\end{aligned}$$

2.7. Topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ and modified free entropy dimension $\mathfrak{K}_2^{(2)}$

In this subsection, first we are going to recall the C*-algebra invariant "topological orbit dimension $\mathfrak{K}_{top}^{(2)}$" and its basic properties.

DEFINITION 6. ([5]) Define

$$\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n;\omega)=\sup_{R>0}\inf_{\varepsilon>0,r\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R^{(\mathrm{top})}(x_1,\cdots,x_n;k,\varepsilon,Q_1,\cdots,Q_r),\omega\right)\right)}{k^2}.$$

The topological orbit dimension of $x_1,\cdots,x_n$ is defined by

$$\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n)=\sup_{\omega>0}\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n;\omega)=\lim_{\omega\to0^+}\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n;\omega).$$

Similarly, define

$$\begin{aligned}&\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:y_1,\cdots,y_m;\omega)\\ =&\sup_{R>0}\inf_{\varepsilon>0,r\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R^{(top)}(x_1,\cdots,x_n:y_1,\cdots,y_m;k,\varepsilon,P_1,\cdots,P_r),\omega\right)\right)}{k^2}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:y_1,\cdots,y_m)&=\sup_{\omega>0}\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:y_1,\cdots,y_m;\omega)\\&=\lim_{\omega\to0^+}\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:y_1,\cdots,y_m;\omega).\end{aligned}$$

After slightly modifying Lemma 3.1.3 in [5], we can quickly get the following lemma.

LEMMA 1. *Let $x_1,\cdots,x_n,v_1,\cdots,v_p$ be self-adjoint elements in a unital C*-algebra $\mathscr{A}$. If $d_1,\cdots,d_r$ are in the C*-subalgebra generated by $x_1,\cdots,x_n$ in $\mathscr{A}$, then for every $\omega>0$*

$$\begin{aligned}&\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:v_1,\cdots,v_p;4\omega)\\ \leqslant&\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:d_1,\cdots d_r,v_1,\cdots,v_p;2\omega)\leqslant\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:v_1,\cdots,v_p;\omega).\end{aligned}$$

THEOREM 1. *([5]) Suppose that $\mathscr{A}$ is a unital C*-algebra and $\{x_1,\cdots,x_n\}$, $\{y_1,\cdots,y_p\}$ are two families of self-adjoint generators of $\mathscr{A}$. Then*

$$\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n)=\mathfrak{K}_{top}^{(2)}(y_1,\cdots,y_p).$$

The topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ is in fact a C*-algebra invariant. In view of this result, we use $\mathfrak{K}_{top}^{(2)}(\mathscr{A})$ to denote $\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n)$ for an arbitrary generating set $\{x_1,\cdots,x_n\}$ for $\mathscr{A}$. By slightly modifying the proof of Theorem 1, we can get the following theorem.

THEOREM 2. *Suppose that $\mathscr{A}$ is a unital C*-algebra and*

$$x_1,\cdots,x_n,y_1,\cdots,y_p,w_1,\cdots,w_t$$

are self-adjoint elements in $\mathscr{A}$. If $C^*(x_1,\cdots,x_n)=C^*(y_1,\cdots,y_p)$, then

$$\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:w_1,\cdots,w_t)=\mathfrak{K}_{top}^{(2)}(y_1,\cdots,y_p:w_1,\cdots,w_t).$$

REMARK 4. From the definition, it is clear that:

1. $\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:y_1,\cdots,y_p)\geqslant\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:y_1,\cdots y_p,y_{p+1})$;
2. If $\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n:x_1,\cdots,x_{n+j})=0\ \ (j\geqslant 0)$,*then*

$$\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_{n-1}:x_1,\cdots,x_{n+j})=0.$$

Now let's recall the modified free orbit-dimension $\mathfrak{K}_2^{(2)}$. Let $\mathscr{M}$ be a von Neumann algebra with a tracial state τ, and let $x_1,\cdots,x_n$ be self-adjoint elements in $\mathscr{M}$. For any positive numbers R and ε, and any $m,k\in\mathbb{N}$, let $\Gamma_R(x_1,\cdots,x_n;m,k,\varepsilon;\tau)$ be the subset of $\left(\mathscr{M}_k^{s.a.}(\mathbb{C})\right)^n$ consisting of all $(A_1,\cdots,A_n)$ in $\left(\mathscr{M}_k^{s.a.}(\mathbb{C})\right)^n$ such that

$$\max_{1\leqslant j\leqslant n}\left\|A_j\right\|\leqslant R \text{ and } \left|\tau_k\left(A_{i_1}\cdots A_{i_q}\right)-\tau\left(x_{i_1}\cdots x_{i_q}\right)\right|<\varepsilon,$$

for all $1\leqslant i_1,\cdots,i_q\leqslant n$ and $1\leqslant q\leqslant m$. Now we define, successively,

$$\mathfrak{K}_2^{(2)}(x_1,\cdots,x_n;\omega;\tau)=\sup_{R>0}\inf_{\varepsilon>0,m\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R(x_1,\cdots,x_n;m,k,\varepsilon;\tau),\omega\right)\right)}{k^2}$$

$$\mathfrak{K}_2^{(2)}(x_1,\cdots,x_n;\tau)=\limsup_{\omega\to 0^+}\mathfrak{K}_2^{(2)}(x_1,\cdots,x_n;\omega;\tau)$$

where $\mathfrak{K}_2^{(2)}(x_1,\cdots,x_n;\tau)$ is called the *modified free orbit-dimension of* $x_1,\cdots,x_n$ with respect to the tracial state τ [5].

REMARK 5. ([5]) Suppose $x_1,\cdots,x_n$ is a family of self-adjoint elements in a von Neumann algebra with a tracial state τ. Let $\mathfrak{K}_2(x_1,\cdots,x_n;\tau)$ be the upper orbit dimension of $x_1,\cdots,x_n$ defined in Definition 1 of [7]. Then $\mathfrak{K}_2^{(2)}(x_1,\cdots,x_n;\tau)=0$ if $\mathfrak{K}_2(x_1,\cdots,x_n;\tau)=0$.

2.8. MF-Traces and MF-nuclear algebras

We note that the definition of $\delta_{top}(x_1,\cdots,x_n)$ makes sense if and only if, for every $\varepsilon>0$ and every $r,k_0\in\mathbb{N}$, there is a $k\geqslant k_o$ such that

$$\Gamma^{(top)}(x_1,\cdots,x_n;k,\varepsilon,Q_1,\cdots,Q_r)\neq\varnothing.$$

In [8], it has shown that this is equivalent to $C^*(x_1,\cdots,x_n)$ being an MF C*-algebra in the sense of Blackadar and Kirchberg [1]. A C*-algebra $\mathscr{A}$ is an *MF-algebra* if $\mathscr{A}$ can be embedded into $\Pi_{1\leqslant k<\infty}\mathscr{M}_{m_k}(\mathbb{C})/\sum_{1\leqslant k<\infty}\mathscr{M}_{m_k}(\mathbb{C})$ for some increasing sequence $\{m_k\}$ of positive integers. In particular C* $(x_1,\cdots,x_n)$ is an MF-algebra if there is a sequence $\{m_k\}$ of positive integers and sequences $\{A_{1k}\},\cdots,\{A_{nk}\}$ with $A_{1k},\cdots,A_{nk}\in\mathscr{M}_{m_k}(\mathbb{C})$ such that

$$\lim_{k\to\infty}\|Q(A_{1k},\cdots,A_{nk})\|=\|Q(x_1,\cdots,x_n)\|$$

for every *-polynomial $Q(X_1,\cdots,X_n)$.

Throughout the rest of this paper, we always assume that a C*-algebra is MF.

DEFINITION 7. ([10]) Suppose $\mathscr{A}=C^*(x_1,\cdots,x_n)$ is an MF C*-algebra. A tracial state τ on $\mathscr{A}$ is an MF-trace if there is a sequence $\{m_k\}$ of positive integers and sequences $\{A_{1k}\},\cdots,\{A_{nk}\}$ with $A_{1k},\cdots,A_{nk}\in\mathscr{M}_{m_k}(\mathbb{C})$ such that, for every $*$-polynomial Q:

1. $\lim_{k\to\infty}\|Q(A_{1k},\cdots,A_{nk})\|=\|Q(x_1,\cdots,x_n)\|$, and
2. $\lim_{k\to\infty}\tau_{m_k}(Q(A_{1k},\cdots,A_{nk}))=\tau(Q(x_1,\cdots,x_n))$.

We let $\mathscr{TS}(\mathscr{A})$ denote the set of all tracial states on $\mathscr{A}$ and $\mathscr{T}_{MF}(\mathscr{A})$ denote the set of all MF-traces on $\mathscr{A}$.

DEFINITION 8. ([10]) A C*-algebra $\mathscr{A}=C^*(x_1,\cdots,x_n)$ is MF-nuclear if $\pi_\tau(\mathscr{A})''$ is hyperfinite for every $\tau\in\mathscr{T}_{MF}(\mathscr{A})$ where π_τ is the GNS representation of $\mathscr{A}$ with respect to τ.

DEFINITION 9. ([10]) A tracial state τ on a unital C*-algebra $\mathscr{A}$ is called finite-dimensional state if there is a finite dimensional C*-algebra $\mathscr{B}$ with a tracial state ρ and a unital $*$-homomorphism $\pi:\mathscr{A}\to\mathscr{B}$ such that $\tau=\rho\circ\pi$.

PROPOSITION 1. *([10]) Suppose $\mathscr{A}=C^*(x_1,\cdots,x_n)$ is an MF-algebra. Then:*

1. *$\mathscr{T}_{MF}(\mathscr{A})$ is a nonempty weak*-compact convex set.*
2. *Every finite-dimensional state on $\mathscr{A}$ is in $\mathscr{T}_{MF}(\mathscr{A})$.*
3. *If π is a unital $*$-homomorphism on $\mathscr{A}$ and $\pi(\mathscr{A})$ is an MF-algebra, then*

$$\{\varphi\circ\pi:\varphi\in\mathscr{T}_{MF}(\pi(\mathscr{A}))\}\subseteq\mathscr{T}_{MF}(\mathscr{A}).$$

3. Relation between $\mathfrak{K}_{top}^{(2)}$ and $\mathfrak{K}_2^{(2)}$

DEFINITION 10. ([5]) Let $\mathscr{A}$ be a unital C*-algebra and $\mathscr{T}\mathscr{S}(\mathscr{A})$ be the set of all tracial states of $\mathscr{A}$. Suppose that $x_1,\cdots,x_n$ is a family of self-adjoint elements in $\mathscr{A}$. Define

$$\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1,\cdots,x_n) = \sup_{\tau\in\mathscr{T}\mathscr{S}(\mathscr{A})} \mathfrak{K}_2^{(2)}(x_1,\cdots,x_n;\tau).$$

THEOREM 3. *([5]) Suppose that $\mathscr{A}$ is a unital C*-algebra and $x_1,\cdots,x_n$ is a family of self-adjoint generating elements in $\mathscr{A}$. Then*

$$\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n) \leqslant \mathfrak{K}\mathfrak{K}_2^{(2)}(x_1,\cdots,x_n).$$

We can generalize the preceding theorem as follows.

THEOREM 4. *Let $\mathscr{A}$ be a unital C*-algebra and $\{x_1,\cdots,x_n\}$ be a family of self-adjoint generating elements in $\mathscr{A}$. Then*

$$\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n) \leqslant \sup_{\tau\in\mathscr{T}_{MF}(\mathscr{A})} \mathfrak{K}_2^{(2)}(x_1,\cdots,x_n;\tau)$$

where $\mathscr{T}_{MF}(\mathscr{A})$ is the set of all MF-tracial states on $\mathscr{A}$.

The proof of the above theorem is similar to the proof of Theorem 4.3.1 in [5]. We also need to note that τ in the original proof of Theorem 4.3.1 in [5] is an MF-tracial state by the equality

$$|\tau(Q(a_1,\cdots,a_n))| = \lim_{s\to\infty} \left|\tau_{\eta(q_s)}(Q(a_1,\cdots,a_n))\right|$$

for every noncommutative polynomial Q and $\tau_{\eta(q_s)} = \frac{Tr_{k_{q_s}}\circ\Psi_{\eta(q_s)}}{k_{q_s}}$. Hence we omit the proof of Theorem 4 here.

Now we are ready to simplify the proof of the following theorem.

THEOREM 5. *([10]) Suppose $\mathscr{A}$ is an MF-nuclear C*-algebra with a family of self-adjoint generators $x_1,\cdots,x_n$. Then*

$$\delta_{top}(x_1,\cdots,x_n) \leqslant 1.$$

Proof. It is known that the GNS representation of an MF-nuclear C*-algebra with respect to an MF-tracial state yields an injective von Neumann algebra. From [7] and Remark 5

$$\mathfrak{K}_2^{(2)}(x_1,\cdots,x_n;\tau) = 0 \text{ for any } \tau\in\mathscr{T}_{MF}(\mathscr{A}).$$

So, from Theorem 4, we know that $\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n) = 0$. Hence, by Theorem 3.1.2 in [5],

$$\delta_{top}(x_1,\cdots,x_n) \leqslant 1.$$

4. Definition and properties of $\mathfrak{K}_{top}^{(3)}$

Suppose $\mathscr{A}$ is a unital C*-algebra. We agree $\infty\cdot 0 = 0$. For any subset $\mathscr{G} \subseteq \mathscr{A}$, we define

$$\begin{aligned}&\mathfrak{K}_{top}^{(3)}(x_1,\cdots,x_n : \mathscr{G})\\ =&\inf\left\{\infty\cdot\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n : y_1,\cdots,y_t) : \{y_1,\cdots,y\} \text{ is a finite subset of } \mathscr{G}\right\},\end{aligned}$$

$$\mathfrak{K}_{top}^{(3)}(\mathscr{G}) = \sup_{\substack{E\subseteq\mathscr{G}\\ E \text{ is finite}}} \inf_{\substack{F\subseteq\mathscr{G}\\ F \text{ is finite}}} \mathfrak{K}_{top}^{(3)}(E : F).$$

REMARK 6. When $\mathscr{G}$ is finite, it is not difficult to see that

$$\mathfrak{K}_{top}^{(3)}(x_1,\cdots,x_n : \mathscr{G}) = \infty\cdot\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n : \mathscr{G})$$

and

$$\mathfrak{K}_{top}^{(3)}(\mathscr{G}) = \infty\cdot\mathfrak{K}_{top}^{(2)}(\mathscr{G}).$$

The proof of the following theorem is similar to the Theorem 3.3 in [6], so we omit it.

THEOREM 6. *If $\mathscr{A}$ is an MF-algebra, then the following are equivalent:*

1. $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$;
2. *if $x_1,\cdots,x_n \in \mathscr{A}$, then there exist $y_1,\cdots,y_t \in \mathscr{A}$ such that $\mathfrak{K}_{top}^{(2)}(x_1,\cdots x_n : y_1,\cdots,y_t)$ $= 0$;*
3. *for any generating set $\mathscr{G}$ of $\mathscr{A}$, $\mathfrak{K}_{top}^{(3)}(\mathscr{G}) = 0$;*
4. *there exists a generating set $\mathscr{G}$ of $\mathscr{A}$ such that $\mathfrak{K}_{top}^{(3)}(\mathscr{G}) = 0$;*
5. *if $\mathscr{G}$ is a generating set of $\mathscr{A}$, and A_0 is a finite subset of $\mathscr{G}$, then, for any finite subset A with $A_0 \subseteq A \subseteq \mathscr{G}$, there exists a finite subset B of $\mathscr{G}$ so that $\mathfrak{K}_{top}^{(3)}(A : B) = 0$;*
6. *there is an increasing directed family $\{\mathscr{A}_i : i \in \Lambda\}$ of C*-subalgebras of $\mathscr{A}$ such that:*
   - *(a) each $\mathscr{A}_i$ is countably generated;*
   - *(b) $\mathfrak{K}_{top}^{(3)}(\mathscr{A}_i) = 0$;*
   - *(c) $\mathscr{A} = \cup_{i\in\Lambda}\mathscr{A}_i$;*
7. *if A is a countable subset of $\mathscr{M}$, then there exists a countably generated subalgebra $\mathscr{B}$ of $\mathscr{M}$ such that $A \subseteq \mathscr{B}$ and $\mathfrak{K}_{top}^{(3)}(\mathscr{B}) = 0$.*

REMARK 7. If $\mathscr{A}$ is finitely generated, then $\mathfrak{K}_{top}^{(3)}(\mathscr{A})=0$ if $\mathfrak{K}_{top}^{(2)}(\mathscr{A})=0$.

COROLLARY 1. *Suppose $\mathscr{A}$ is a C*-algebra, $\mathscr{G}$ is a generating set of $\mathscr{A}$. Then $\mathfrak{K}_{top}^{(3)}(\mathscr{A})=\mathfrak{K}_{top}^{(3)}(\mathscr{G})$.*

COROLLARY 2. *Suppose $\{\mathscr{A}_l\}_{l\in\Lambda}$ is an increasingly directed family of C*-algebras. Then*

$$\mathfrak{K}_{top}^{(3)}(\cup\mathscr{A}_l)\leqslant\liminf_{l}\mathfrak{K}_{top}^{(3)}(\mathscr{A}_l).$$

Proof. If $\liminf_l \mathfrak{K}_{top}^{(3)}(\mathscr{A}_l)=\infty$, the inequality holds clearly. Suppose that $\liminf_l \mathfrak{K}_{top}^{(3)}(\mathscr{A}_l)=0$. Let $x_1,\cdots,x_n\in\cup\mathscr{A}_l$. Then we can find an $l\in\Lambda$ such that $x_1,\cdots,x_n\in\mathscr{A}_l$ and $\mathfrak{K}_{top}^{(3)}(\mathscr{A}_l)=0$. Therefore, we can find $y_1,\cdots,y_p\in\mathscr{A}_l$ with

$$\mathfrak{K}_{top}^{(3)}(x_1,\cdots,x_n:y_1,\cdots,y_p)=0.$$

It follows that $\mathfrak{K}_{top}^{(3)}(\cup\mathscr{A}_l)=0$ by Theorem 6 (6).

We modify the $\mathfrak{K}_3$ in [6] by using the modified free orbit dimension $\mathfrak{K}_2^{(2)}$.

DEFINITION 11. Let $\mathscr{M}$ be a von Neumann algebra with a tracial state τ, and let $x_1,\cdots,x_n$ be self-adjoint elements in $\mathscr{M}$. Define

$$\begin{aligned}&\mathfrak{K}_3^{(3)}((x_1,\cdots,x_n:\mathscr{G});\tau)\\ =&\inf\left\{\infty\cdot\mathfrak{K}_2^{(2)}((x_1,\cdots,x_n:y_1,\cdots,y_t);\tau):\{y_1,\cdots,y_t\}\text{ is a finite subset of }\mathscr{G}\right\}\end{aligned}$$

and

$$\mathfrak{K}_3^{(3)}(\mathscr{G};\tau)=\sup_{\substack{E\subseteq\mathscr{G}\\ E\text{ is finite}}}\ \inf_{\substack{F\subseteq\mathscr{G}\\ F\text{ is finite}}}\mathfrak{K}_3^{(3)}((E:F);\tau).$$

REMARK 8. Note that if $\mathfrak{K}_3(\mathscr{G};\tau)=0$, then $\mathfrak{K}_3^{(3)}(\mathscr{G};\tau)=0$ by Remark 5.

Now we are ready to get the relationship between $\mathfrak{K}_{top}^{(3)}(\mathscr{G})$ and $\mathfrak{K}_3^{(3)}(\mathscr{G};\tau)$.

THEOREM 7. *Let $\mathscr{A}$ be an MF-algebra and $\mathscr{G}$ be a countably generating set of $\mathscr{A}$. Then*

$$\mathfrak{K}_{top}^{(3)}(\mathscr{G})\leqslant\sup_{\tau\in\mathscr{T}\mathscr{S}(\mathscr{A})}\mathfrak{K}_3^{(3)}(\mathscr{G};\tau).$$

Proof. If $\mathfrak{K}_{top}^{(3)}(\mathscr{G})=0$, the inequality holds clearly. Otherwise, $\mathfrak{K}_{top}^{(3)}(\mathscr{G})=\infty$. It follows from the definition of $\mathfrak{K}_{top}^{(3)}$ that, there is a finite subset $E\subseteq\mathscr{G}$ such that for every finite subset $F\subseteq\mathscr{G}$, $\mathfrak{K}_{top}^{(3)}(E:F)=\infty$. It implies that $0<\mathfrak{K}_{top}^{(2)}(E:F)$ for every finite subset $F \subseteq \mathscr{G}$. Hence for any finite subset F of $\mathscr{G}$ we can find a sequence $\{F_i\}_{i=1}^{\infty}$ of finite subsets of $\mathscr{G}$ with

$$F = F_0 \subseteq F_1 \subseteq \cdots$$

and $\cup_i F_i = \mathscr{G}$. Therefore

$$C^*(E \cup F_0) \subseteq C^*(E \cup F_1) \subseteq \cdots$$

and

$$\overline{\cup_i C^*(E \cup F_i)}^{\|\cdot\|} = \mathscr{A}.$$

Since for each i, $\mathfrak{K}_{top}^{(2)}(E : F_i) > 0$, we can find a tracial state τ_i on $C^*(E \cup F_i)$ such that

$$\mathfrak{K}_2^{(2)}(E : F_i; \tau_i) \geqslant \mathfrak{K}_{top}^{(2)}(E : F_i) - \varepsilon_i > 0 \tag{4.1}$$

for some small $\varepsilon_i > 0$ by Theorem 4. We may regard $C^*(E \cup F_0)$ as a subalgebra of $C^*(E \cup F_i)$ for $i \geqslant 1$, then

$$\mathfrak{K}_2^{(2)}(E : F_0 : \tau_i) \geqslant \mathfrak{K}_2^{(2)}(E : F_i; \tau_i) > 0 \text{ for every } i. \tag{4.2}$$

Let

$$\pi : \overline{\cup_i C^*(E \cup F_i)}^{\|\cdot\|} \longrightarrow \Pi_i C^*(E \cup F_i) / \Sigma_i C^*(E \cup F_i)$$

be the embedding defined by

$$\pi(A) = \underbrace{(0, \cdots, 0,}_{\text{the first } i \text{ positions}} \quad A, A, \cdots) \text{ for every } A \in C^*(E \cup F_i)$$

and $\widetilde{\tau}$ be the tracial state on

$$\Pi_{i \in \mathbb{N}} C^*(E \cup F_i) / \Sigma_{i \in \mathbb{N}} C^*(E \cup F_i)$$

define by $\widetilde{\tau}([\{A_i\}]_\alpha) = \lim_{i \to \alpha} \tau_i(A_i)$ where α is a free ultrafilter over $\mathbb{N}$ and

$$[\{A_i\}]_\alpha \in \Pi_{i \in \mathbb{N}} C^*(E \cup F_i) / \Sigma_{i \in \mathbb{N}} C^*(E \cup F_i).$$

Define $\tau = \widetilde{\tau} \circ \pi$, then τ is a tracial state on $\mathscr{A}$. Note for any finite subset G of $\mathscr{G}$, we can always find a suitable index i such that

$$G \subseteq F_i \subseteq F_{i+1} \subseteq \cdots.$$

Therefore $\widetilde{\tau}$ is irrelevant to the selection of finite subset $F = F_0$, so is τ. Let $\{\varepsilon_t\}$ be a decreasing sequence of positive numbers with $\lim_{t \to \infty} \varepsilon_t = 0$ and $\{m_t\}_{t=1}^{\infty}$ be an increasing sequence of integers with $\lim_{t \to \infty} m_t = \infty$. Then, for large $R > 0$, we can find a subsequence $\{i_t\}_{t=1}^{\infty}$ of integers such that when ε is small enough and m is big enough,

$$\Gamma_R(E : F_0; k, \varepsilon, m; \tau_{i_t}) \subseteq \Gamma_R(E : F_0; k, \varepsilon_t, m_t; \tau) \text{ for every } k.$$

It implies that, for any $\omega > 0$,

$$\inf_{\varepsilon,m}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon,m;\tau_{i_t}\right),\omega\right)\right)}{k^2} \leqslant \limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon_t,m_t;\tau\right),\omega\right)\right)}{k^2} \text{ for every } i_t,$$

it follows that

$$\limsup_{t\to\infty}\inf_{\varepsilon,m}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon,m;\tau_{i_t}\right),\omega\right)\right)}{k^2} \leqslant \limsup_{t\to\infty}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon_t,m_t;\tau\right),\omega\right)\right)}{k^2}.$$

Therefore we can find an index t_0 such that

$$\begin{aligned}\inf_{\varepsilon,m}\limsup_{k\to\infty}&\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon,m;\tau_{i_{t_0}}\right),\omega\right)\right)}{k^2}\\ &\leqslant \limsup_{t\to\infty}\inf_{\varepsilon,m}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon,m;\tau_{i_t}\right),\omega\right)\right)}{k^2}\\ &\leqslant \limsup_{t\to\infty}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon_t,m_t;\tau\right),\omega\right)\right)}{k^2}.\end{aligned}$$

So by (4.1) and (4.2),

$$\begin{aligned}0 < \mathfrak{K}_2^{(2)}((E:F_0);\tau_{i_{t_0}}) &= \sup_{0<\omega<1}\sup_{R>0}\inf_{\varepsilon,m}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon,m;\tau_{i_{t_0}}\right),\omega\right)\right)}{k^2}\\ &\leqslant \sup_{0<\omega<1}\sup_{R>0}\limsup_{t\to\infty}\inf_{\varepsilon,m}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon,m;\tau_{i_t}\right),\omega\right)\right)}{k^2}\\ &\leqslant \sup_{0<\omega<1}\sup_{R>0}\limsup_{t\to\infty}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon_t,m_t;\tau\right),\omega\right)\right)}{k^2}\\ &= \sup_{0<\omega<1}\sup_{R>0}\inf_{\varepsilon_t,m_t}\limsup_{k\to\infty}\frac{\log\left(o_2\left(\Gamma_R\left(E:F_0;k,\varepsilon_t,m_t;\tau\right),\omega\right)\right)}{k^2}\\ &= \mathfrak{K}_2^{(2)}((E:F_0);\tau).\end{aligned}$$

Note that $F_0 = F$ is an arbitrary subset of $\mathscr{G}$. Then

$$\mathfrak{K}_3^{(3)}(\mathscr{G};\tau) = \sup_{\substack{E\subseteq\mathscr{G}\\ E \text{ is finite}}}\inf_{\substack{F\subseteq\mathscr{G}\\ F \text{ is finite}}}\mathfrak{K}_3^{(3)}((E:F);\tau) = \sup_{\substack{E\subseteq\mathscr{G}\\ E \text{ is finite}}}\inf_{\substack{F\subseteq\mathscr{G}\\ F \text{ is finite}}}\left\{\infty\cdot\mathfrak{K}_2^{(2)}((E:F);\tau)\right\} = \infty.$$

It follows that

$$\mathfrak{K}_{top}^{(3)}(\mathscr{G}) \leqslant \sup_{\tau\in\mathscr{TS}(\mathscr{A})}\mathfrak{K}_3^{(3)}(\mathscr{G};\tau).$$

In the rest of this section, we are going to give the analogs of Theorem 3.14 and Theorem 3.17 in [6]. The following lemma can be found in [6]:

LEMMA 2. *([6]) Suppose A is a normal element in a von Neumann algebra $\mathcal{M}$ with tracial state τ such that A has no eigenvalues. Then there is a positive element Y with the uniform distribution on $[0,1]$ such that $W^*(A) = W^*(Y)$.*

REMARK 9. ([10]) It is well known that every self-adjoint element in a finite von Neumann algebra $\mathcal{M}$ has an eigenvalue if and only if $\mathcal{M}$ has a finite-dimensional invariant subspace.

To prove our next lemma, we need to recall the concepts of a Haar unitary matrix in $\mathcal{M}_k(\mathbb{C})$ and a Haar unitary element in an infinite-dimensional von Neumann algebra $\mathcal{M}$.

DEFINITION 12. A unitary matrix A in $\mathcal{M}_k(\mathbb{C})$ is called a Haar unitary matrix if the eigenvalues of A are the k-th roots of unity. Equivalently, if $\tau_k(A^i) = 0$ for $1 \leqslant i < k$ and $\tau_k(A^k) = 1$.

DEFINITION 13. Suppose $\mathcal{M}$ is an infinite-dimensional von Neumann algebra with a tracial state τ. Then a unitary u in $\mathcal{M}$ is called a Haar unitary if $\tau(u^m) = 0$ when $m \neq 0$. In addition, $\mathcal{M}$ is called diffuse if $\mathcal{M}$ contains a Haar unitary.

DEFINITION 14. A C*-algebra $\mathcal{A}$ is called NFD if it has no non-zero finite-dimensional representations.

Now we are ready to show the following lemma.

LEMMA 3. *Suppose $\mathcal{A}$ is a unital MF C*-algebra generated by $x_1,\cdots,x_n$. Then the following statements are equivalent:*

1. *$\mathcal{A}$ is an NFD algebra;*
2. *For any $\delta > 0$, there are ε_0, N_0 and k_0 such that for any $k > k_0, \varepsilon < \varepsilon_0$, $N > N_0$, and any*
$$(A_1,\cdots,A_n) \in \Gamma^{(top)}(x_1,\cdots,x_n;k,\varepsilon,Q_1,\cdots,Q_N)$$
there is a noncommutative polynomial $\widetilde{Q}(X_1,\cdots,X_n)$ such that
$$\left\| U_k - \widetilde{Q}(A_1,\cdots,A_n) \right\|_2 < \delta$$
for a Haar unitary $U_k \in \mathcal{M}_k(\mathbb{C})$.

Proof. First, suppose (1) holds and there are δ_0 and sequences $\{\varepsilon_i\}$ with $\varepsilon_i \to 0$, $\{N_i\}$ with $N_i \to \infty$, $\{k_i\}$ with $k_i \to \infty$ as well as
$$\left(A_1^{(i)},\cdots,A_n^{(i)}\right) \in \Gamma^{(top)}(x_1,\cdots,x_n;k_i,\varepsilon_i,Q_1,\cdots,Q_{N_i}) \text{ for each } i$$
such that
$$\left\| U_{k_i} - Q\left(A_1^{(i)},\cdots,A_n^{(i)}\right) \right\|_2 > \delta_0$$

for any noncommutative polynomial $Q(X_1,\cdots,X_n)$ and any Haar unitary $U_{k_i} \in \mathcal{M}_{k_i}(\mathbb{C})$. Note that any subsequence of $\left\{\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\}$ has the same property, so we may assume that

$$\lim_i \left\|Q\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\| = \|Q(x_1,\cdots,x_n)\|$$

and

$$\lim_i \tau_{k_i}\left(Q\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right) = \lim_i \tau(Q(x_1,\cdots,x_n))$$

for any noncommutative polynomial Q where τ on $\mathcal{A}$ is the MF-tracial state defined by $\{\tau_{k_i}\}$. Let α be a free ultrafilter over $\mathbb{N}$, $(\mathcal{N},\rho)$ be the tracial ultraproduct $\overset{\alpha}{\Pi}\left(\mathcal{M}_{k_i}(\mathbb{C}),\tau_{k_i}\right)$ and $y_j = \left[\left\{A_j^{(i)}\right\}\right]_\alpha \in \mathcal{N}$ for $1 \leqslant j \leqslant n$. Note $(\mathcal{N},\rho)$ is a II_1 factor, then for any noncommutative polynomial $Q(X_1,\cdots,X_n)$

$$\|Q(y_1,\cdots,y_n)\| \leqslant \lim_i \left\|Q\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\| = \|Q(x_1,\cdots,x_n)\|.$$

Hence $\pi : \mathcal{A} \to \mathcal{N}$ defined by $\pi(x_i) = y_i$ is a unital $*$-homomorphism and then $\rho \circ \pi = \tau$. Since $\mathcal{A}$ has no non-zero finite-dimensional representations, we know that $\pi(\mathcal{A})''$ has no finite-dimensional invariant subspace. It implies that $\pi(\mathcal{A})''$ is diffused by Lemma 2, Remark 9 and Definition 13. Therefore we can find a Haar unitary $U = \left[\{U_{k_i}\}\right]_\alpha \in \pi(\mathcal{A})''$ where we may assume U_{k_i} is a Haar unitary in $\mathcal{M}_{k_i}(\mathbb{C})$ without loss of generality. It implies that for any $\delta_0 > 0$ there is a noncommutative polynomial $\widetilde{Q}(X_1,\cdots,X_n)$ satisfying

$$\left\|U - \widetilde{Q}(y_1,\cdots,y_n)\right\|_2 = \lim_\alpha \left\|U_{k_i} - \widetilde{Q}\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\|_2 < \frac{\delta_0}{2}.$$

So we can find an integer i such that

$$\left\|U_{k_i} - \widetilde{Q}\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\|_2 < \delta_0.$$

This contradicts the assumption, hence statement (1) implies statement (2).

On the other hand, suppose (2) holds and $\mathcal{A}$ has a non-zero finite-dimensional representation. Then by Proposition 1 and Definition 7, there is a finite-dimensional MF-tracial state τ and sequence $\left\{\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\}_i$ in which $A_k^{(i)} \in \mathcal{M}_{k_i}(\mathbb{C})$ such that

$$\lim_i \left\|Q\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\| = \|Q(x_1,\cdots,x_n)\|$$

and

$$\lim_i \tau_{k_i}\left(Q\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right) = \lim_i \tau(Q(x_1,\cdots,x_n))$$

for any noncommutative polynomial Q. As in the argument above, we let $(\mathcal{N},\rho)$ be the tracial ultraproduct $\overset{\alpha}{\Pi}\left(\mathcal{M}_{k_i}(\mathbb{C}),\tau_{k_i}\right)$ and $y_j = \left[\left\{A_j^{(i)}\right\}\right]_\alpha \in \mathcal{N}$ for $1 \leqslant j \leqslant n$. Then $(\mathcal{N},\rho)$ is a II_1 factor and there is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{N}$ with $\pi(x_i) =$

y_i and $\rho \circ \pi = \tau$. Since τ is finite-dimensional and ρ is faithful, we have $\pi(\mathscr{A})''$ is finite-dimensional. Thus there is no Haar unitary $U = \{(U_{k_i})\}$ in $\mathscr{N}$ which lies in $\pi(\mathscr{A})''$. It implies that

$$\left\| U_{k_i} - Q\left(A_1^{(i)}, \cdots, A_n^{(i)}\right) \right\|_2 > \delta_0$$

for any noncommutative polynomial $Q(X_1, \cdots, X_n)$ and any Haar unitary $U_{k_i} \in \mathscr{M}_{k_i}(\mathbb{C})$. This contradicts the statement (2). Hence (2) implies (1).

LEMMA 4. *Let $\mathscr{A}$ be a unital MF C*-algebra generated by $x_1, \cdots, x_n$. If $\mathscr{A}$ is NFD, then for any $\delta > 0$, there are ε_0, N_0 and k_0 such that for any $k > k_0, \varepsilon < \varepsilon_0$, $N > N_0$, and any*

$$(A_1, \cdots, A_n) \in \Gamma^{(top)}(x_1, \cdots, x_n; k, \varepsilon, Q_1, \cdots, Q_N),$$

there is a self-adjoint noncommutative polynomial $\widetilde{Q}(X_1, \cdots, X_n)$ satisfying

$$\left\| W_k^* \widetilde{Q}(A_1, \cdots, A_n) W_k - \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k \end{pmatrix} \right\|_2 < \delta$$

for a unitary W_k in $\mathscr{M}_k(\mathbb{C})$ where $\{\lambda_1, \cdots, \lambda_k\} \subseteq [0,1]$ with $\lambda_i = \frac{i-1}{k}$ as $1 \leqslant i \leqslant k$.

Proof. Assume to the contrary that there are δ_0 and sequences $\{\varepsilon_i\}_{i\in\mathbb{N}}$ with $\varepsilon_i \to 0$, $\{N_i\}_{i\in\mathbb{N}}$ with $N_i \to \infty$, $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to \infty$ as well as

$$\left(A_1^{(i)}, \cdots, A_n^{(i)}\right) \in \Gamma^{(top)}(x_1, \cdots, x_n; k_i, \varepsilon_i, Q_1, \cdots, Q_{N_i})$$

such that

$$\left\| W_k^* Q\left(A_1^{(i)}, \cdots, A_n^{(i)}\right) W_k - \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k \end{pmatrix} \right\|_2 > \delta_0$$

for any non-commutative polynomial $Q(X_1, \cdots, X_n)$ and any unitary W_k in $\mathscr{M}_k(\mathbb{C})$. Then by Lemma 3, without loss of generality, we may assume that for $\{\varepsilon_i\}_{i\in\mathbb{N}}, \{N_i\}_{i\in\mathbb{N}}$ and $\{k_i\}_{i\in\mathbb{N}}$ there is a sequence $\{Q_i(X_1, \cdots, X_n)\}_{i\in\mathbb{N}}$ of noncommutative polynomials such that

$$\left\| U_{k_i} - Q_i(A_1, \cdots, A_n) \right\|_2 < \frac{1}{i} \tag{4.3}$$

for a Haar unitary $U_{k_i} \in \mathscr{M}_{k_i}(\mathbb{C})$ and any

$$(A_1, \cdots, A_n) \in \Gamma^{(top)}(x_1, \cdots, x_n; k_i, \varepsilon_i, Q_1, \cdots, Q_{N_i}).$$

So by using the same argument in the proof of Lemma 3, we may assume that

$$\lim_i \left\| Q\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\| = \|Q(x_1,\cdots,x_n)\|$$

and

$$\lim_i \tau_{k_i}\left(Q\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right) = \tau(Q(x_1,\cdots,x_n))$$

for any noncommutative polynomial $Q(X_1,\cdots,X_n)$ where τ is the MF-tracial state on $\mathscr{A}$ defined by $\{\tau_{k_i}\}$. Therefore we can get a unital $*$-homomorphism $\pi:\mathscr{A}\to\mathscr{N}$ defined by $\pi(x_l)=y_l$ for $1\leqslant l\leqslant n$ where $(\mathscr{N},\rho)$ is the tracial ultraproduct $\overset{\alpha}{\Pi}\left(\mathscr{M}_{k_i}(\mathbb{C}),\tau_{k_i}\right)$ and $y_l=\left[\left\{A_l^{(i)}\right\}\right]_\alpha\in\mathscr{N}$ for $1\leqslant l\leqslant n$ and $\rho\circ\pi=\tau$. Hence, by (4.3),

$$\lim_{j\to\infty}\lim_{i\to\alpha}\left\|U_{k_i}-Q_j\left(A_1^{(i)},\cdots,A_n^{(i)}\right)\right\|_2=0. \tag{4.4}$$

Let $u=\left[\left\{U_{k_i}\right\}\right]_\alpha\in\mathscr{N}$. Then by (4.4),

$$\lim_j \left\|u-Q_j(y_1,\cdots,y_n)\right\|_2=0.$$

It implies that the Haar unitary $u=\left[\left\{U_{k_i}\right\}\right]_\alpha$ is in $\pi(\mathscr{A})''$. Therefore we can find a positive element $y\in\pi(\mathscr{A})''$ such that $u=e^{2\pi iy}$. Assume $y=[\{A_{k_i}\}]_\alpha$ in which A_{k_i} is self-adjoint in $\mathscr{M}_{k_i}(\mathbb{C})$. Hence there is a self-adjoint element $x\in\pi(\mathscr{A})$ such that

$$\|y-x\|_2<\frac{\delta_0}{2}. \tag{4.5}$$

Without loss of generality, we may also assume $x=\widetilde{Q}(y_1,\cdots,y_n)$ for some noncommutative polynomial $\widetilde{Q}(X_1,\cdots,X_n)$. Since $u=e^{2\pi iy}$, we may assume that $e^{2\pi iA_{k_i}}=U_{k_i}$. We can therefore get a unitary W_{k_i} such that

$$W_{k_i}A_{k_i}W_{k_i}^*=\begin{pmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \lambda_{k_i}\end{pmatrix}$$

where $\left\{\lambda_1,\cdots,\lambda_{k_i}\right\}\subseteq[0,1]$ with $\lambda_j=\frac{j-1}{k_i}$ as $1\leqslant j\leqslant k_i$. It follows that

$$\left\|W_{k_i}^*\widetilde{Q}\left(A_1^{(i)},\cdots,A_n^{(i)}\right)W_{k_i}-A_{k_i}\right\|_2<\delta_0$$

for some i and unitary $W_{k_i}\in\mathscr{M}_{k_i}$ by (4.5). This is a contradiction. Thus, for any $\delta>0$, there are ε_0, N_0 and k_0 such that, for any $k>k_0,\varepsilon<\varepsilon_0$, $N>N_0$, and any

$$(A_1,\cdots,A_n)\in\Gamma^{(top)}(x_1,\cdots,x_n;k,\varepsilon,Q_1,\cdots,Q_N),$$

there is a noncommutative polynomial $\widetilde{Q}(X_1,\cdots,X_n)$ satisfying

$$\left\| W_k^* \widetilde{Q}(A_1,\cdots,A_n) W_k - \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k \end{pmatrix} \right\|_2 < \delta$$

for a unitary W_k in $\mathscr{M}_k(\mathbb{C})$. So the proof will be completed by taking self-adjoint noncommutative polynomial $\frac{\widetilde{Q}(X_1,\cdots,X_n)+\left(\widetilde{Q}(X_1,\cdots,X_n)\right)^*}{2}$.

LEMMA 5. *Let $\mathscr{A}$ be a unital MF C*-algebra generated by $x_1,\cdots,x_n$. If $\mathscr{A}$ is NFD, then for any $\delta > 0$, there is a self-adjoint element $x \in \mathscr{A}$ and ε_1, N_1 as well as k_1 such that for any $k > k_1, \varepsilon < \varepsilon_1$, $N > N_1$ and any*

$$A \in \Gamma^{(top)}(x;k,\varepsilon,Q_1,\cdots,Q_N),$$

the following inequality holds

$$\left\| W_k^* A W_k - \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k \end{pmatrix} \right\|_2 < \delta$$

for a unitary W_k in $\mathscr{M}_k(\mathbb{C})$ where $\{\lambda_1,\cdots,\lambda_k\} \subseteq [0,1]$ with $\lambda_i = \frac{i-1}{k}$ as $1 \leqslant i \leqslant k$.

Proof. By Lemma 4, for any δ, there are ε_0, N_0 and k_0 such that for any $k > k_0, \varepsilon < \varepsilon_0$, $N > N_0$, and any

$$(A_1,\cdots,A_n) \in \Gamma^{(top)}(x_1,\cdots,x_n;k,\varepsilon,Q_1,\cdots,Q_N),$$

there is a noncommutative polynomial $\widetilde{Q}(X_1,\cdots,X_n)$ satisfying

$$\left\| W_k^* \widetilde{Q}(A_1,\cdots,A_n) W_k - \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k \end{pmatrix} \right\|_2 < \frac{\delta}{2} \tag{4.6}$$

and $\widetilde{Q}(A_1,\cdots,A_n)$ is a self-adjoint element in $\mathscr{M}_k(\mathbb{C})$ for a unitary W_k in $\mathscr{M}_k(\mathbb{C})$ where $\{\lambda_1,\cdots,\lambda_k\} \subseteq [0,1]$ with $\lambda_i = \frac{i-1}{k}$ as $1 \leqslant i \leqslant k$.

Let $x = \widetilde{Q}(x_1,\cdots,x_n)$. Then x is self-adjoint. For $\frac{\delta}{2}$, there are ε_1, N_1 and k_1 with $k_1 > k_0$ such that, for any $k > k_1, \varepsilon < \varepsilon_1$, $N > N_1$ and any $A \in \Gamma^{(top)}(x;k,\varepsilon,Q_1,\cdots,Q_N)$, we have

$$\left\| \widetilde{Q}\left(A_1^{(k)},\cdots,A_n^{(k)}\right) - A \right\| < \frac{\delta}{2} \tag{4.7}$$

for some

$$\left(A_1^{(k)},\cdots,A_n^{(k)}\right)\in\Gamma^{(top)}\left(x_1,\cdots,x_n;k,\varepsilon',Q_1,\cdots,Q_{N'}\right)$$

when ε' is small enough and N' is large enough. By (4.6) and (4.7), the proof can be completed.

LEMMA 6. *([7]) Let V_1,V_2 be two Haar unitary matrices in $\mathscr{M}_k(\mathbb{C})$. For every $\delta>0$, let*

$$\Omega(V_1,V_2;\delta)=\{U\in\mathscr{U}(k)\,|\,\|UV_1-V_2U\|_2\leqslant\delta\}.$$

Then for every $0<\delta<r$ there exists a set $\left\{Ball\left(U_\lambda;\frac{4\delta}{r}\right)\right\}_{\lambda\in\Lambda}$ of $\frac{4\delta}{r}$-balls in $\mathscr{U}(k)$ that cover $\Omega(V_1,V_2;\delta)$ with the cardinality of Λ satisfying $|\Lambda|\leqslant\left(\frac{3r}{2\delta}\right)^{4rk^2}$.

LEMMA 7. *Let $\{\lambda_1,\cdots,\lambda_k\}\subseteq[0,1]$ with $\lambda_i=\frac{i-1}{k}$ as $1\leqslant i\leqslant k$ where $k\geqslant 4$. Assume D_1 and D_2 are diagonal matrices in $\mathscr{M}_k(\mathbb{C})$ such that diagonal entries are all from $\{\lambda_1,\cdots,\lambda_k\}$ without repetition. For every $\delta>0$, let*

$$\Omega(D_1,D_2;\delta)=\{U\in\mathscr{U}(k)\,|\,\|UD_1-D_2U\|_2\leqslant\delta\}.$$

Then, for every $0<\delta<r$, there exists a set $\left\{Ball\left(U_\lambda;\frac{4\delta}{r}\right)\right\}_{\lambda\in\Lambda}$ of $\frac{4\delta}{r}$-balls in $\mathscr{U}(k)$ that cover $\Omega(D_1,D_2;\delta)$ with the cardinality of Λ satisfying $|\Lambda|\leqslant\left(\frac{3r}{2\delta}\right)^{8\pi rk^2}$.

Proof. Let $D=diag(\lambda_1,\cdots,\lambda_k)$. Then there exist $W_1,W_2\in\mathscr{U}(k)$ such that $D_1=W_1DW_1^*$ and $D_2=W_2DW_2^*$. Let

$$\widetilde{\Omega}(\delta)=\{U\in\mathscr{U}(k)\,|\,\|UD-DU\|_2\leqslant\delta\}.$$

Clearly

$$\Omega(D_1,D_2;\delta)=\left\{W_2^*UW_1|U\in\widetilde{\Omega}(\delta)\right\},$$

hence $\widetilde{\Omega}(\delta)$ and $\Omega(D_1,D_2;\delta)$ have the same covering numbers.

Let $\{e_{st}\}_{s,t=1}^k$ be the canonical system of matrix units of $\mathscr{M}_k(\mathbb{C})$. For every $U=\sum_{s,t=1}^k x_{st}e_{st}$ in $\widetilde{\Omega}(\delta)$, with $x_{st}\in\mathbb{C}$, we have

$$\begin{aligned}\left\|Ue^{2\pi iD}-e^{2\pi iD}U\right\|_2^2&=\sum_{s,t=1}^k\left|\left(e^{2\pi i\lambda_s}-e^{2\pi i\lambda_t}\right)x_{st}\right|^2\leqslant\sum_{s,t=1}^k4\pi\left|(\lambda_s-\lambda_t)x_{st}\right|^2\\&=4\pi\|UD-DU\|_2^2\leqslant(2\pi\delta)^2.\end{aligned}$$

Hence

$$\left\|Ue^{2\pi iD}-e^{2\pi iD}U\right\|_2^2\leqslant 2\pi\delta$$

for $U\in\widetilde{\Omega}(\delta)$. So the result can be obtained by Lemma 6.

The following lemma is analogous to Lemma 3.13 in [6].

LEMMA 8. *Let* $x_1,\cdots,x_n,y_1,\cdots,y_p,v_1,\cdots,v_s,w_1,\cdots,w_t$ *be elements in an MF* C^**-algebra* $\mathscr{A}$. *If* $C^*(x_1,\cdots,x_n)\cap C^*(y_1,\cdots,y_p)$ *is NFD, then*

$$\begin{aligned}&\mathfrak{K}_{top}^{(3)}(x_1,\cdots,x_n,y_1,\cdots,y_p:v_1,\cdots,v_s,w_1,\cdots,w_t)\\ \leqslant&\mathfrak{K}_{top}^{(3)}(x_1,\cdots,x_n:v_1,\cdots,v_s)+\mathfrak{K}_{top}^{(3)}(y_1,\cdots,y_p:w_1,\cdots,w_t).\end{aligned}$$

Proof. Without loss of generality, we may assume that $\|x_i\|\leqslant 1$ and $\|y_j\|\leqslant 1$ for each $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant p$. If one of $\mathfrak{K}_{top}^{(3)}(x_1,\cdots,x_n:v_1,\cdots,v_s)$ and $\mathfrak{K}_{top}^{(3)}(y_1,\cdots,y_p:w_1,\cdots,w_t)$ is infinite, the proof is clear. So we can assume that

$$\mathfrak{K}_{top}^{(3)}(x_1,\cdots,x_n:v_1,\cdots,v_s)=\mathfrak{K}_{top}^{(3)}(y_1,\cdots,y_p:w_1,\cdots,w_t)=0. \tag{4.8}$$

By Lemma 5, for $r>0,\ \omega>0$ we can find a self-adjoint element

$$d_{r,\omega}\in C^*(x_1,\cdots,x_n)\cap C^*(y_1,\cdots,y_p), \tag{4.9}$$

$\varepsilon_1>0$, $N_1\in\mathbb{N}$ and $k_1\in\mathbb{N}$ such that for any $k>k_1,\varepsilon<\varepsilon_1$, $N>N_1$ and any

$$D\in\Gamma^{(top)}(d_{r,\omega};k,\varepsilon,Q_1,\cdots,Q_N),$$

the following inequality holds

$$\left\|W_k^*DW_k-\begin{pmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\ddots&0\\0&0&0&\lambda_k\end{pmatrix}\right\|_2<\frac{r\omega}{48}$$

for a unitary W_k in $\mathscr{M}_k(\mathbb{C})$ where $\{\lambda_1,\cdots,\lambda_k\}\subseteq[0,1]$ with $\lambda_i=\frac{i-1}{k}$ as $1\leqslant i\leqslant k$. By (4.8), (4.9) and Theorem 2,

$$\mathfrak{K}_{top}^{(2)}(x_1,\cdots,x_n,d_{r,\omega}:v_1,\cdots,v_s)=\mathfrak{K}_{top}^{(2)}(y_1,\cdots,y_p,d_{r,\omega}:w_1,\cdots,w_t)=0.$$

If

$$\begin{aligned}&(A_1,\cdots,A_n,B_1,\cdots,B_p,D)\\ &\in\Gamma^{(top)}(x_1,\cdots,x_n,y_1,\cdots,y_p,d_{r,\omega}:v_1,\cdots,v_s,w_1,\cdots,w_t;P_1,\cdots,P_m,k,\varepsilon),\end{aligned}$$

then

$$(A_1,\cdots,A_n,D)\in\Gamma^{(top)}(x_1,\cdots,x_n,d_{r,\omega}:v_1,\cdots,v_s,w_1,\cdots,w_t;P_1',\cdots,P_{m_1}',k,\varepsilon)$$

and

$$(B_1,\cdots,B_p,D)\in\Gamma^{(top)}\left(y_1,\cdots,y_p,d_{r,\omega}:v_1,\cdots,v_s,w_1,\cdots,w_t;P_1'',\cdots,P_{m_2}'',k,\varepsilon\right)$$

where

$$P_1',\cdots,P_{m_1}' \in \mathbb{C}\langle X_1,\cdots,X_{n+1},V_1,\cdots,V_s,W_1,\cdots,W_t\rangle$$

and

$$P_1'',\cdots,P_{m_2}'' \in \mathbb{C}\langle Y_1,\cdots,Y_{p+1},V_1,\cdots,V_s,W_1,\cdots,W_t\rangle$$

respectively. Let $\left\{\mathscr{U}(A_1^\lambda,\cdots,A_n^\lambda,D^\lambda);\frac{r\omega}{48}\right\}_{\lambda\in\Lambda_k}$ be a set of $\frac{r\omega}{48}$-orbit-balls that cover

$$\Gamma^{(top)}(x_1,\cdots,x_n,d_{r,\omega}:v_1,\cdots,v_s,w_1,\cdots,w_t;k,\varepsilon,P_1',\cdots,P_{m_1}')$$

with the cardinality of Λ_k satisfying

$$|\Lambda_k| = o_2\left(\Gamma^{(top)}(x_1,\cdots,x_n,d_{r,\omega}:v_1,\cdots,v_s,w_1,\cdots,w_t;k,\varepsilon,P_1',\cdots,P_{m_1}');\frac{r\omega}{48}\right).$$

Also let $\left\{\mathscr{U}(B_1^\sigma,\cdots,B_n^\sigma,D^\sigma);\frac{r\omega}{48}\right\}_{\sigma\in\Sigma_k}$ be a set of $\frac{r\omega}{48}$-orbit-balls that cover

$$\Gamma_R\left(y_1,\cdots,y_p,d_{r,\omega}:v_1,\cdots,v_s,w_1,\cdots,w_t;k,\varepsilon,P_1'',\cdots,P_{m_2}''\right)$$

with the cardinality of Σ_k satisfying

$$|\Sigma_k| = o_2\left(\Gamma_R\left(y_1,\cdots,y_p,d_{r,\omega}:v_1,\cdots,v_s,w_1,\cdots,w_t;k,\varepsilon,P_1'',\cdots,P_{m_2}''\right);\frac{r\omega}{48}\right).$$

When m and k are large enough and ε is small enough, we let D^σ,D^λ to be diagonal matrices in Lemma 7. For any

$$\begin{aligned}&(A_1,\cdots,A_n,B_1,\cdots,B_p,D)\\ &\in \Gamma_R(x_1,\cdots,x_n,y_1,\cdots,y_p,d_{r,\omega}:v_1,\cdots,v_s,w_1,\cdots,w_t;P_1,\cdots,P_m,k,\varepsilon),\end{aligned}$$

there exist some $\lambda\in\Lambda_k,\sigma\in\Sigma_k$ and $W_1,W_2\in\mathscr{U}(k)$ such that

$$\left\|(A_1,\cdots,A_n,D)-W_1\left(A_1^\lambda,\cdots,A_n^\lambda,D^\lambda\right)W_1^*\right\|_2 \leqslant \frac{r\omega}{24},$$

$$\left\|(B_1,\cdots,B_p,D)-W_2\left(B_1^\sigma,\cdots,B_p^\sigma,D^\sigma\right)W_2^*\right\|_2 \leqslant \frac{r\omega}{24}.$$

Therefore

$$\left\|W_1D^\lambda W_1^*-W_2D^\sigma W_2^*\right\|_2 = \left\|W_2^*W_1D^\lambda - D^\sigma W_2^*W_1\right\|_2 \leqslant \frac{r\omega}{12}.$$

From Lemma 7, there exists a set $\left\{Ball\left(U_{\lambda,\sigma,\gamma},\frac{\omega}{3}\right)\right\}_{\gamma\in\Delta_k}$ in $\mathscr{U}(k)$ which cover $\Omega\left(D^\lambda,D^\sigma;\frac{r\omega}{12}\right)$ with cardinality $|\Delta_k|\leqslant\left(\frac{18}{\omega}\right)^{8\pi rk^2}$. This implies that

$$\begin{aligned}&\Big\|(A_1,\cdots,A_n,B_1,\cdots,B_p,D)\\ &-\left(W_2U_{\lambda,\sigma,\gamma}A_1^\lambda U_{\lambda,\sigma,\gamma}^*W_2^*,\cdots,W_2U_{\lambda,\sigma,\gamma}A_n^\lambda U_{\lambda,\sigma,\gamma}^*W_2^*,W_2B_1^\sigma W_2^*,\cdots,W_2B_p^\sigma W_2^*,W_2D^\sigma W_2^*\right)\Big\|_2\\ &< n\omega.\end{aligned}$$

Therefore

$$\Big\| (A_1,\cdots,A_n,B_1,\cdots,B_p) - \Big(W_2U_{\lambda,\sigma,\gamma}A_1^{\lambda}U_{\lambda,\sigma,\gamma}^*W_2^*,\cdots,W_2U_{\lambda,\sigma,\gamma}A_n^{\lambda}U_{\lambda,\sigma,\gamma}^*W_2^*,W_2B_1^{\sigma}W_2^*,\cdots,W_2B_p^{\sigma}W_2^*\Big)\Big\|_2 < n\omega.$$

Then we get

$$\begin{aligned}&\mathfrak{K}_2^{(top)}(x_1,\cdots,x_n,y_1,\cdots,y_p,:d_{r,\omega},v_1,\cdots,v_s,w_1,\cdots,w_t;2n\omega)\\ &\leqslant \inf_{m\in\mathbb{N},\varepsilon>0}\limsup_{k\longrightarrow\infty}\frac{\log(|\Lambda_k||\Sigma_k||\Delta_k|)}{k^2}\leqslant 8\pi r(\log(18)-\log\omega).\end{aligned}\tag{4.10}$$

Now by Lemma 1 and the fact that $d_{r,\omega}\in C^*(x_1,\cdots,x_n)\cap C^*(y_1,\cdots,y_p)$, we have

$$\begin{aligned}&\mathfrak{K}_2^{(top)}(x_1,\cdots,x_n,y_1,\cdots,y_p,:v_1,\cdots,v_s,w_1,\cdots,w_t;4n\omega)\\ \leqslant&\mathfrak{K}_2^{(top)}(x_1,\cdots,x_n,y_1,\cdots,y_p,:d_{r,\omega},v_1,\cdots,v_s,w_1,\cdots,w_t;2n\omega).\end{aligned}$$

Hence by (4.10) and the fact that r is arbitrary, we can conclude that

$$\mathfrak{K}_3^{(top)}(x_1,\cdots,x_n,y_1,\cdots,y_p:v_1,\cdots,v_s,w_1,\cdots,w_t)=0.$$

This completes the proof.

THEOREM 8. *Suppose $\mathscr{A}$ is a separable C*-algebra, $\mathscr{N}_1$ and $\mathscr{N}_2$ are C*-subalgebras of $\mathscr{A}$. If $\mathscr{N}_1\cap\mathscr{N}_2$ is NFD and finitely generated, then*

$$\mathfrak{K}_{top}^{(3)}(C^*(\mathscr{N}_1\cup\mathscr{N}_2))\leqslant\mathfrak{K}_{top}^{(3)}(\mathscr{N}_1)+\mathfrak{K}_{top}^{(3)}(\mathscr{N}_2).$$

Proof. If $\mathfrak{K}_3^{(top)}(\mathscr{N}_1)=\infty$ or $\mathfrak{K}_3^{(top)}(\mathscr{N}_2)=\infty$, the inequality holds automatically. Now suppose that

$$\mathfrak{K}_{top}^{(3)}(\mathscr{N}_1)=\mathfrak{K}_{top}^{(3)}(\mathscr{N}_2)=0.\tag{4.11}$$

By assumption, we let $\mathscr{N}_1\cap\mathscr{N}_2=C^*(d_1,\cdots,d_l)$ and $\mathscr{G}=\mathscr{N}_1\cup\mathscr{N}_2$. Then $\mathscr{G}$ is a generating set of $C^*(\mathscr{N}_1\cup\mathscr{N}_2)$. Suppose A is any finite subset of $\mathscr{G}$ with $\{d_1,\cdots,d_l\}\subseteq A$. So we may assume that

$$A=\{x_1,\cdots,x_n,d_1,\cdots,d_l,y_1,\cdots,y_m\}$$

where $\{x_1,\cdots,x_n\}\subseteq\mathscr{N}_1$ and $\{y_1,\cdots,y_m\}\subseteq\mathscr{N}_2$. By (4.11), there exist

$$v_1,\cdots,v_s\in\mathscr{N}_1,w_1,\cdots,w_t\in\mathscr{N}_2$$

such that

$$\mathfrak{K}_{top}^{(3)}(x_1,\cdots,x_n,d_1,\cdots,d_l:v_1,\cdots,v_s)=\mathfrak{K}_{top}^{(3)}(y_1,\cdots,y_t,d_1,\cdots,d_l:w_1,\cdots,w_t)=0.$$

Then from Lemma 8, we know that

$$\mathfrak{K}_3^{(top)}(A : v_1,\cdots,v_s,w_1,\cdots,w_t) = 0.$$

Therefore, by Theorem 6 (5), $\mathfrak{K}_3^{(top)}(C^*(\mathscr{N}_1 \cup \mathscr{N}_2)) = 0$. This completes the proof.

THEOREM 9. *Let $\mathscr{N}$ be an MF C*-algebra, and $\mathscr{D}$, $\mathscr{A}$ be C*-subalgebras of $\mathscr{N}$ with $\mathscr{D} \subseteq \mathscr{A} \subseteq \mathscr{N}$ where $\mathscr{D}$ is finitely generated and NFD. If there is a unitary $u \in \mathscr{N}$ such that $u^*\mathscr{D}u \subseteq \mathscr{A}$, then*

$$\mathfrak{K}_3^{(top)}(C^*(\mathscr{A} \cup \{u\})) \leqslant \mathfrak{K}_3^{(top)}(\mathscr{A}).$$

Proof. If $\mathfrak{K}_3^{(top)}(\mathscr{A}) = \infty$, the proof is clear. Now suppose that $\mathfrak{K}_3^{(top)}(\mathscr{A}) = 0$ and $\mathscr{D} = C^*(d_1,\cdots,d_l)$. By Lemma 5, for $r > 0$, $\omega > 0$ we can find a self-adjoint element $d_{r,\omega} \in \mathscr{D}$, $\varepsilon_1 > 0$, $N_1 \in \mathbb{N}$ and $k_1 \in \mathbb{N}$ such that for any $k > k_1, \varepsilon < \varepsilon_1$, $N > N_1$ and any

$$D \in \Gamma^{(top)}(d_{r,\omega}; k, \varepsilon, Q_1,\cdots,Q_N),$$

the following inequality holds

$$\left\| W_k^* D W_k - \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k \end{pmatrix} \right\|_2 < \frac{r\omega}{192}$$

for a unitary W_k in $\mathscr{M}_k(\mathbb{C})$ where $\{\lambda_1,\cdots,\lambda_k\} \subseteq [0,1]$ with $\lambda_i = \frac{i-1}{k}$ as $1 \leqslant i \leqslant k$. Let $x_1,\cdots,x_n$ be elements in $\mathscr{A}$. Then there exist $y_1,\cdots,y_p$ in $\mathscr{A}$ such that

$$\mathfrak{K}_2^{(top)}(x_1,\cdots,x_n,d_1,\cdots d_l,d_{r,\omega},u^*d_{r,\omega}u : y_1,\cdots,y_p) = 0.$$

Suppose

$$\left\{ \mathscr{U}\left(T_1^\lambda,\cdots,T_{n+l}^\lambda,A^\lambda,B^\lambda;\frac{r\omega}{192}\right) \right\}_{\lambda \in \Lambda_k}$$

is a set of $\frac{r\omega}{192}$-orbit-balls in $\mathscr{M}_k(\mathbb{C})^{n+l+2}$ that cover

$$\Gamma^{(top)}(x_1,\cdots,x_n,d_1,\cdots d_l,d_{r,\omega},u^*d_{r,\omega}u : y_1,\cdots,y_p : k,\varepsilon,P_1,\cdots,P_m)$$

where $P_1,\cdots,P_m \in \mathbb{C}\langle X_1,\cdots,X_{n+l+2},Y_1,\cdots,Y_p\rangle$ with the cardinality of Λ_k satisfying

$$|\Lambda_k| = o_2\left(\Gamma^{(top)}(x_1,\cdots,x_n,d_1,\cdots d_l,d_{r,\omega},u^*d_{r,\omega}u : y_1,\cdots,y_p;k,\varepsilon,P_1,\cdots,P_m),\frac{r\omega}{192}\right).$$

When m,k are sufficiently large and ε is sufficiently small, we can let A^λ be a diagonal matrix in Lemma 7 and then $B^\lambda = U^*A^\lambda U$ for some unitary matrix U. It implies that for such A^λ and B^λ,

$$\left\|(T_1,\cdots,T_{n+l},A,B) - V\left(T_1^\lambda,\cdots,T_{n+l}^\lambda,A^\lambda,B^\lambda\right)V^*\right\|_2 \leqslant \frac{r\omega}{64}.$$

For sufficiently large m' and sufficiently small $\varepsilon < \frac{r\omega}{64}$, when

$$(T_1,\cdots,T_{n+l},A,B,C,D)$$
$$\in \Gamma^{(top)}\left(x_1,\cdots,x_n,d_1,\cdots d_l,d_{r,\omega},u^*d_{r,\omega}u,\frac{u+u^*}{2},\frac{u-u^*}{2i}:y_1,\cdots,y_p;P_1',\cdots,P_{m'}',k,\varepsilon\right)$$

where $P_1',\cdots,P_{m'}' \in \mathbb{C}\langle X_1,\cdots,X_{n+l+2},Y_1,\cdots,Y_p\rangle$, we may assume that

$$\|C+iD-W\| < \frac{r\omega}{64} \tag{4.12}$$

for some unitary $W \in \mathscr{M}_k(\mathbb{C})$ and

$$\|A(C+iD)-(C+iD)B\| \leqslant \varepsilon < \frac{r\omega}{64} \tag{4.13}$$

as well as

$$(T_1,\cdots,T_{n+l},A,B) \in \Gamma^{(top)}(x_1,\cdots,x_n,d_1,\cdots d_l,d_{r,\omega},u^*d_{r,\omega}u:y_1,\cdots,y_p;k,\varepsilon,P_1,\cdots,P_m).$$

It implies that there exist some $\lambda \in \Lambda_k$ and $V \in \mathscr{U}(k)$ such that

$$\left\|(T_1,\cdots,T_{n+l},A,B)-\left(VT_1^\lambda V^*,\cdots,VT_{n+l}^\lambda V^*,VA^\lambda V^*,VB^\lambda V^*\right)\right\|_2 \leqslant \frac{r\omega}{64}. \tag{4.14}$$

So, by (4.13) and (4.14), we have

$$\left\|V^*(C+iD)VA^\lambda-B^\lambda V^*(C+iD)V\right\|_2=\left\|(C+iD)VA^\lambda V^*-VB^\lambda V^*(C+iD)\right\|_2 \leqslant 3\cdot\frac{r\omega}{64},$$

therefore, by (4.12),

$$\left\|V^*WVA^\lambda-B^\lambda V^*WV\right\|_2 \leqslant \frac{r\omega}{8}.$$

By Lemma 7, $\left\{Ball\left(U^\sigma;\frac{\omega}{2}\right)\right\}_{\sigma\in\Sigma_k}$ is a set of $\frac{\omega}{2}$-balls in $\mathscr{U}(k)$ that cover $\Omega\left(A^\lambda,B^\lambda,\frac{r\omega}{8}\right)$ with the cardinality of Σ_k satisfying $|\Sigma_k| \leqslant \left(\frac{12}{\omega}\right)^{8\pi rk^2}$. Hence $\|V^*WV-U_\sigma\|_2 \leqslant \frac{\omega}{2}$ for some $\sigma \in \Sigma_k$. Following (4.12),

$$\|V^*(C+iD)V-U^\sigma\|_2 < \omega.$$

It implies that

$$\left\|(T_1,\cdots,T_{n+l},C,D)-\left(VT_1^\lambda V^*,\cdots,VT_{n+l}^\lambda V^*,V\frac{U^\sigma+U^{\sigma*}}{2}V^*,V\frac{U^\sigma-U^{\sigma*}}{2i}V^*\right)\right\|_2 \leqslant 3\omega.$$

Therefore,

$$o_2\left(\Gamma^{(top)}\left(x_1,\cdots,x_n,d_1,\cdots,d_l,\frac{u+u^*}{2},\frac{u-u^*}{2i}:d_{r,\omega},u^*d_{r,\omega}u,y_1,\cdots,y_p;k,\varepsilon,P_1,\cdots,P_m\right),3\omega\right)$$

$\leqslant |\Lambda_k|\,|\Sigma_k|\,.$

Hence, by Lemma 1, we get

$$\begin{aligned}
0 &\leqslant \mathfrak{K}_2^{(top)}\left(x_1,\cdots,x_n,d_1,\cdots d_l,\frac{u+u^*}{2},\frac{u-u^*}{2i}:y_1\cdots,y_p,6\omega\right)\\
&\leqslant \mathfrak{K}_2^{(top)}\left(x_1,\cdots,x_n,d_1,\cdots d_l,\frac{u+u^*}{2},\frac{u-u^*}{2i}:d_{r,\omega},u^*d_{r,\omega}u,y_1\cdots,y_p,3\omega\right)\\
&\leqslant \inf_{m\in\mathbb{N},\varepsilon>0}\limsup_{k\longrightarrow\infty}\frac{\log\left(|\Lambda_k|\,|\Sigma_k|\right)}{k^2}\\
&\leqslant \inf_{m\in\mathbb{N},\varepsilon>0}\limsup_{k\longrightarrow\infty}\left(\frac{\log\left(|\Lambda_k|\right)}{k^2}+8\pi r(\log 12-\log\omega)\right)\\
&= 8\pi r(\log 12-\log\omega).
\end{aligned}$$

Since r is an arbitrarily small positive number, we have

$$\mathfrak{K}_2^{(top)}\left(x_1,\cdots,x_n,d_1,\cdots d_l,\frac{u+u^*}{2},\frac{u-u^*}{2i}:y_1\cdots,y_p,6\omega\right)=0.$$

Therefore

$$\mathfrak{K}_3^{(top)}\left(x_1,\cdots,x_n,d_1,\cdots d_l\frac{u+u^*}{2},\frac{u-u^*}{2i}:y_1\cdots,y_p\right)=0.$$

Then, by Theorem 6 (5),

$$\mathfrak{K}_3^{(top)}\left(C^*\left(\mathscr{A}\cup\{u\}\right)\right)=0.$$

COROLLARY 3. *Let $\mathscr{A}=C^*\left(x_1,\cdots,x_n\right)$ and $\mathscr{B}$ be unital MF C*-algebras. Suppose G is a countable group of actions $\left\{\alpha_g\right\}_{g\in G}$ on $\mathscr{A}$ and $\mathscr{D}=\mathscr{A}\rtimes_\alpha G$ is either a full or reduced crossed product of $\mathscr{A}$ by the actions of G. If $\mathscr{A}$ is NFD and there is an onto *-homomorphism $\pi:\mathscr{A}\rtimes_\alpha G\longrightarrow\mathscr{B}$, then*

$$\mathfrak{K}_3^{(top)}\left(\mathscr{B}\right)\leqslant\mathfrak{K}_3^{(top)}\left(\mathscr{A}\right).$$

Proof. If $\mathfrak{K}_3^{(top)}\left(\mathscr{A}\right)=\infty$, the assertion is clear. Now suppose $\mathfrak{K}_3^{(top)}\left(\mathscr{A}\right)=0$. Note that $\pi\left(g^{-1}\right)\pi\left(\mathscr{A}\right)\pi\left(g\right)\subseteq\pi\left(\mathscr{A}\right)$. Then, by Theorem 9,

$$\mathfrak{K}_3^{(top)}\left(\pi\left(\mathscr{A}\right)\cup\left\{\pi\left(g\right)\right\}\right)=0.$$

From Theorem 8, we know that

$$\mathfrak{K}_3^{(top)}\left(\pi\left(\mathscr{A}\right)\cup\left\{\pi\left(g_1\right)\right\}\cup\left\{\pi\left(g_2\right)\right\}\right)=0.$$

Let

$$\mathscr{B}_n=C^*\left(\pi\left(\mathscr{A}\right)\cup\left\{\pi\left(g_1\right)\right\}\cup\cdots\cup\left\{\pi\left(g_n\right)\right\}\right).$$

Then $\mathfrak{K}_3^{(top)}\left(\mathscr{B}_n\right)=0$ by successively using Theorem 8. Therefore

$$\mathfrak{K}_3^{(top)}\left(\mathscr{B}\right)=\liminf_n\mathfrak{K}_3^{(top)}\left(\mathscr{B}_n\right)=0$$

by Corollary 1, Corollary 2 and the fact that $\mathscr{B}=\overline{\cup\mathscr{B}_n}^{\|\cdot\|}$.

5. Applications

Let $\mathscr{A}$ be a unital C*-algebra and ω be a free ultrafilter on $\mathbb{N}$. Let $c_\omega(\mathscr{A})$ denote the closed two-sided ideal of the C*-algebra $l^\infty(\mathscr{A})$ given by

$$c_\omega(\mathscr{A}) = \left\{(a_n)_{n\geqslant 1} \in l^\infty(\mathscr{A}) \mid \lim_{n\to\omega} \|a_n\| = 0\right\}.$$

The *C*-ultrapower* $\mathscr{A}_\omega$ is defined to be the quotient C*-algebra $l^\infty(\mathscr{A})/c_\omega(\mathscr{A})$, and we use π_ω to denote the quotient mapping $l^\infty(\mathscr{A}) \to \mathscr{A}_\omega$. Let $l: \mathscr{A} \to l^\infty(\mathscr{A})$ denote the "diagonal" inclusion mapping $l(a) = (a,a,\cdots) \in l^\infty(\mathscr{A})$, $a \in \mathscr{A}$; and put $l_\omega = \pi_\omega \circ l : \mathscr{A} \to \mathscr{A}_\omega$. Both mappings l and l_ω are injective. Therefore, we can view $\mathscr{A}$ as a subalgebra of $\mathscr{A}_\omega$. The relative commutant defined by $\mathscr{A}_\omega \cap \mathscr{A}'$ is called the *central sequence algebra* of $\mathscr{A}$.

Suppose $\mathscr{N}$ is a von Neumann algebra with a tracial state τ. Consider the associated norm, $\lim_\omega \|a\|_{2,\tau} = (\tau(a^*a))^{\frac{1}{2}}$, for any $a \in \mathscr{N}$. Let $\mathscr{N}^\omega$ denote the von Neumann algebra $l^\infty(\mathscr{N})/c_{\tau,\omega}(\mathscr{N})$ where $c_{\tau,\omega}(\mathscr{N})$ consists of the bounded sequences $(a_1,a_2,\cdots)$ with $\lim_\omega \|a_n\|_{2,\tau} = 0$.

If $\mathscr{M}$ is a II_1 factor, then $\mathscr{M}$ has *property* Γ if and only if $\mathscr{M}^\omega \cap \mathscr{M}'$ has a representing sequence $(U_1,U_2,\cdots)$ such that each U_n is a Haar unitary in $\mathscr{M}$. If $\mathscr{M}$ is a II_1 von Neumann algebra with a separable predual, then $\mathscr{M}$ is defined in [11] to have *property* Γ if and only if each II_1 factor in the central decomposition of $\mathscr{M}$ has property Γ. It follows from direct integral theory that if $\mathscr{M}$ has property Γ, then $\mathscr{M}^\omega \cap \mathscr{M}'$ contains a representing sequence of Haar unitaries. The following theorem is due to Dixmier [3] and Connes [2].

THEOREM 10. *Let $\mathscr{M}$ be a separable II_1 factor. The following conditions are equivalent:*

1. *$\mathscr{M}$ has property Γ;*
2. *$\mathscr{M}^\omega \cap \mathscr{M}' \neq \mathbb{C}I$;*
3. *$\mathscr{M}^\omega \cap \mathscr{M}'$ is a diffuse von Neumann algebra.*

Let $\pi_\tau(\mathscr{A})''$ be the weak closure of $\mathscr{A}$ under the GNS representation of $\mathscr{A}$ with respect to the state τ.

In [11], a separable unital C*-algebra is said to have *property c*-*Γ if, for every tracial state τ on $\mathscr{A}$ such that $\pi_\tau(\mathscr{A})''$ is a II_1 factor and $\pi_\tau(\mathscr{A})''$ has property Γ, which is equivalent to $\pi_\tau(\mathscr{A})''$ having property Γ whenever $\pi_\tau(\mathscr{A})''$ is a II_1 von Neumann algebra. If $\mathscr{A}$ is NFD, then $\pi_\tau(\mathscr{A})''$ has no finite-dimensional representations. Therefore $\pi_\tau(\mathscr{A})''$ is II_1 for every tracial state τ on $\mathscr{A}$. So if $\mathscr{A}$ is NFD and has property c*-Γ, then $\pi_\tau(\mathscr{A})''$ has property Γ for every tracial state τ on $\mathscr{A}$. Actually we can say more in this case.

LEMMA 9. *Let $\mathscr{A}$ be a separable unital C*-algebra. If $\mathscr{A}$ is NFD, then $\mathscr{A}$ has property c*-Γ if and only if for every tracial state τ on $\mathscr{A}$, the central sequence algebra of $\pi_\tau(\mathscr{A})''$ has no non-zero finite-dimensional representations.*

Proof. If $\mathscr{A}$ has property c*-Γ and is NFD, then $\pi_\tau(\mathscr{A})''$ has property Γ for every tracial state τ on $\mathscr{A}$. So the central sequence algebra of each II_1 factor in the central decomposition of $\pi_\tau(\mathscr{A})''$ has no finite-dimensional representations by Theorem 10.

On the other hand, if the central sequence algebra of $\pi_\tau(\mathscr{A})''$ has no finite-dimensional representations for every tracial state, then the central sequence algebra of each II_1 factor in the central decomposition of $\pi_\tau(\mathscr{A})''$ has no finite-dimensional representations. So by Theorem 10 , $\mathscr{A}$ has property c*-Γ.

The following amazing result is due to Kirchberg and Rørdam [9].

LEMMA 10. *([9])Let $\mathscr{A}$ be a separable unital C*-algebra, let τ be a faithful tracial state on $\mathscr{A}$, let $\mathscr{N}$ be the weak closure of $\mathscr{A}$ under the GNS representation of $\mathscr{A}$ with respect to the state τ, and let ω be a free ultrafilter on $\mathbb{N}$. It follows that the natural *-homomorphisms*

$$\mathscr{A}_\omega \longrightarrow \mathscr{N}^\omega, \qquad \mathscr{A}_\omega \cap \mathscr{A}' \longrightarrow \mathscr{N}^\omega \cap \mathscr{N}'$$

are surjective.

We say that an MF algebra $\mathscr{A}$ with no non-zero finite-dimensional representations has *property MF-c*-Γ* if, for every MF-trace τ on $\mathscr{A}$, the central sequence algebra $\left(\pi_\tau(\mathscr{A})''\right)^\omega \cap \pi_\tau(\mathscr{A})'$ has no non-zero finite-dimensional representations, i.e., $\pi_\tau(\mathscr{A})''$ has property Γ.

THEOREM 11. *([6]) If $\mathscr{M}$ is a von Neumann algebra with a central net of Haar unitaries, then $\mathfrak{K}_3(\mathscr{M}) = 0$.*

THEOREM 12. *Let $\mathscr{A}$ be a unital separable MF C*-algebra. Suppose $\mathscr{A}$ is NFD and has property c*-Γ. Then $\mathfrak{K}_3^{(top)}(\mathscr{A}) = 0$.*

Proof. Let $\mathscr{N}_\tau = \pi_\tau(\mathscr{A})''$ be the weak closure of $\mathscr{A}$ under the GNS representation of $\mathscr{A}$ with respect to the tracial state τ. Since $\mathscr{A}$ is NFD and has property c*-Γ, there is a central sequence $\{u_n\}$ of Haar unitaries in $\mathscr{N}_\tau$ such that $[\{u_n\}] = u \in (\mathscr{N}_\tau)^\omega \cap (\mathscr{N}_\tau)'$. If follows that $\mathfrak{K}_3(\mathscr{A};\tau) = 0$ by Theorem 11 for every tracial state τ. Hence $\mathfrak{K}_3^{(3)}(\mathscr{A};\tau) = 0$ by Remark 8. Since

$$\mathfrak{K}_{top}^{(3)}(\mathscr{A}) \leqslant \sup_{\tau \in TS(\mathscr{A})} \mathfrak{K}_3^{(3)}(\mathscr{A};\tau) = 0,$$

by Theorem 7, $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$.

COROLLARY 4. *Let $\mathscr{A}$ be a unital MF C*-algebra. Suppose each tracial state on $\mathscr{A}$ is faithful and $\mathscr{A}_\omega \cap \mathscr{A}'$ has no non-zero finite-dimensional representations. Then $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$.*

Proof. Since $\mathscr{A}_\omega \cap \mathscr{A}'$ has no non-zero finite-dimensional representations, we know $\mathscr{A}$ is NFD. Let $\mathscr{N}_\tau$ be the weak closure of $\mathscr{A}$ under the GNS representation of $\mathscr{A}$ with respect to the tracial state τ. Since τ is faithful, the natural *-homomorphisms

$$\mathscr{A}_\omega \cap \mathscr{A}' \longrightarrow (\mathscr{N}_\tau)^\omega \cap (\mathscr{N}_\tau)'$$

is surjective by Lemma 10. It follows that $(\mathscr{N}_\tau)^\omega \cap (\mathscr{N}_\tau)'$ has no finite-dimensional representations, hence $\mathscr{A}$ has property c*-Γ by Lemma 9. Therefore $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$ by Theorem 12.

COROLLARY 5. *Suppose $\mathscr{A}$ is a unital, finitely generated MF C*-algebra. If $\mathscr{A}$ is NFD and has property MF-c*-Γ, then $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$.*

Proof. Let $\mathscr{N}_\tau = \pi_\tau(\mathscr{A})''$ be the weak closure of $\mathscr{A}$ under the GNS representation of $\mathscr{A}$ with respect to the tracial state τ. Since $\mathscr{A}$ has property MF-c*-Γ, $(\mathscr{N}_\tau)^\omega \cap (\mathscr{N}_\tau)'$ has no finite-dimensional representation. Then there is a central sequence $\{u_n\}$ of Haar unitaries in $\mathscr{N}_\tau$ such that $[\{u_n\}] = u \in (\mathscr{N}_\tau)^\omega \cap (\mathscr{N}_\tau)'$. It follows that $\mathfrak{K}_3(\mathscr{A};\tau) = 0$ by Theorem 11. Hence $\mathfrak{K}_3^{(3)}(\mathscr{A};\tau) = 0$ by Remark 8. It implies that $\mathfrak{K}_2^{(2)}(\mathscr{A};\tau) = 0$ for every MF-tracial state τ. Note that by Theorem 4

$$\mathfrak{K}_{top}^{(2)}(\mathscr{A}) \leqslant \sup_{\tau \in \mathscr{T}_{MF}(\mathscr{A})} \mathfrak{K}_2^{(2)}(\mathscr{A};\tau) = 0.$$

Hence $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$ by Remark 7.

REMARK 10. We don't know whether the property MF-c*-Γ is equivalent to $\mathfrak{K}_{top}^{(3)}(\mathscr{A}) = 0$ in which $\mathscr{A}$ is finitely generated and NFD. But it is well known that $C_r^*(\mathbb{F}_2)$ is simple, hence $C_r^*(\mathbb{F}_2)$ is NFD. Note $C_r^*(\mathbb{F}_2)$ is an MF C*-algebra and has a unique tracial state. So by the facts that the set of MF-tracial states is not empty and $L(\mathbb{F}_2)$ has no property Γ, we know $C_r^*(\mathbb{F}_2)$ has no property MF-c*-Γ. So we may hope $\mathfrak{K}_{top}^{(3)}(C_r^*(\mathbb{F}_2)) = \infty$, i.e., $\mathfrak{K}_{top}^{(2)}(C_r^*(\mathbb{F}_2)) \neq 0$. Actually, Voiculescu [12] proved that $\delta_{top}(S_1, S_2) = 2$, where S_1 and S_2 are free semicircle elements. Therefore $\mathfrak{K}_{top}^{(2)}(C_r^*(\mathbb{F}_2)) \neq 0$ by Theorem 3.1.2 in [5].

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