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QUANTIZATION OF $A_0(K)$–SPACES

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Abstract. Let $(V,\{\|.\|_n\},\{M_n(V)^+\})$ be a C^*-ordered operator space and $Q_n(V)$ be the quasi-state space of $M_n(V)$. We show that every C^*-ordered operator space V is complete isometrically, completely isomorphic to $\{A_0(Q_n(V),M_n(V^*))\}$. Motivated by this result we study matricial convexity property. We introduce a notion of an L^1-matrix convex set $\{K_n\}$ in a $*$-locally convex space X. We show that every quantized function space $\{A_0(K_n,M_n(X))\}$ is a C^*-ordered operator space. Further, we generalize the notion of regular embedding of a compact convex set to L^1-regular embedding of an L^1-matrix convex set. We show that if a L^1-matricial convex set is L^1-regular embed and L^1-matricial cap, then $A_0(K_n,M_n(V))$ is an abstract operator system.

1. Introduction

Convexity is an important aspect of Functional Analysis. In particular, compact convex sets are building blocks for many important branches. For example, Kadison's affine functional representation of the self-adjoint part of operator systems and Choquet theory primarily based on the properties of compact convex sets. For references, see [1, 8].

Let K be a compact and convex set in a locally convex space X and let $A(K)$ denote the space of all real valued continuous affine functions on K. Then $A(K)$ is an order unit space. In 1951, Kadison observed that the self-adjoint part of an operator system R in a C^*-algebra $\mathscr{A}$ is order isomorphic to $A(R(\mathscr{A}))$, where $R(\mathscr{A})$ is the state space of R defined by $R(\mathscr{A}) := \{f \in R^* : f(a) \geqslant 0 \ \forall a \in R^+, f(1) = 1\}$. [8]. A non-unital version of an order unit space is known as approximate order unit space. In 1968, Asimov [2] introduced the notion of a 'universal cap' K in a locally convex space X and showed the following: Let $(V, V^* \|.\|)$ be an ordered Banach space. Then V is an approximate order unit space if and only if there exists a universal cap K in a locally convex space X such that V is order isomorphic to $A_0(K)$ (see [16, Theorem 9.15]. We note that if K is universal cap, then $A_0(K)$ is an order unit space if and only if K is regularly embedded in a locally convex space X (see Section 4 for the definition of regular embedding). In this article we develop a quantized version of this theory.

Operator spaces may be seen as a quantization of Banach spaces. (See [7] for more details and references therein.) A quantization of $A(K)$ appeared in the work of

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Webster and Winkler [13] where they showed a quantized version of the Krein-Milman theorem for matrix convex set. Our aim in this paper is to present a quantization of $A_0(K)$ to give affine representation for non unital matrix ordered operator spaces. For this purpose, we introduce a notion of an L^1-matrix convex set. This notion is more general than that of a universal cap.

In 2010, the second named author characterized a set of order theoretic conditions on a matrix ordered operator space which are both necessary as well as sufficient to order embed the space (completely isometrically, completely order isomorphic) into a suitable C^*-algebra. These spaces were introduced as C^*-ordered operator spaces (cf. [9]). It is worth to note that every abstract operator system is a C^*-ordered operator space. Thus C^*-ordered operator space are non unital analogue of operator systems.

The plan of this paper as follows: In the second section, we show that a C^*-ordered operator space $(V, \{M_n(V)^+\}, \{\|\cdot\|_n\})$ has a quantized affine functional representation on $\{Q_n(V)\}$ (see Theorem 2.6). Here $Q_n(V)$ is the quasi-state space of $M_n(V)$ in the matrix duality. Following this result, we deduce some important order theoretic properties of $\{Q_n(V)\}$. In the third section, we introduce a new notion a lead point of a compact convex set. Note that all extreme points of a compact convex sets are the lead points. Using the concept of lead points, we introduce a notion of an L^1-matrix convex set in a $*$-locally convex space X. We show that if $(V, \{M_n(V)^+\}, \{\|.\|_n\})$ is a C^*-ordered operator space, then $\{Q_n(V)\}$ is an L^1-matrix convex set (see Remark 3.4). As a main result in this section, we show that if $\{K_n\}$ is an L^1-matrix convex set in a $*$-locally convex space X, then $\{A_0(K_n, M_n(X))\}$ is a C^*-ordered operator space (see Theorem 3.5). In the last section, we discuss certain extra conditions on L^1-matrix convex set under which $\{A_0(K_n, M_n(X))\}$ is an abstract operator system. We generalize the notion of regular embedding of a compact convex set to L^1-regular embedding of an L^1-matrix convex set. We introduce the notion of L^1-matricial extending the notion of universal cap (see [2]). We show that if $\{K_n\}$ is a L^1-regular embedding and L^1-matricial cap in X, then $\{A_0(K_n, M_n(X))\}$ is an abstract operator system (see Theorem 4.4).

2. Preliminaries and correlation result for C^*-ordered operator spaces

Let V be a $*$-vector space. Then $M_n(V)$ is also a $*$-vector space with $[v_{i,j}]^* = [v_{j,i}^*]$. Now, V is called a *matrix ordered space* if there is a sequence $\{M_n(V)^+\}$ of cones with $M_n(V)^+ \subseteq M_n(V)_{sa}$ such that for each $\gamma \in \mathbb{M}_{m,n}$, we have

$$\gamma^* M_m(V)^+ \gamma \subseteq M_n(V)^+.$$

In 1977, Choi and Effros [3] observed the following: if $(V, \{M_n(V)^+\})$ is a matrix ordered space, then its matrix dual $(V^*, M_n(V^*)^+)$ is also a matrix ordered space, where

$$M_n(V^*)^+ = \{f \in M_n(V^*)_{sa} : f(v) \geqslant 0 \ \ \forall v \in M_n(V)^+\}. \tag{1}$$

An *abstract operator system* $(V, \{M_n(V)^+\}, e)$ is a triple, where $(V, \{M_n(V)^+\})$ is a matrix ordered space and $e \in V^+$ is an order unit in V^+ such that $V^+ \cap -(V^+) = \{0\}$

and e^n is Archimedean in $M_n(V)^+$ for each n. An element $e \in V^+$ is called an *order unit* if $v \in V_{sa}$, there exists a $\lambda \in \mathbb{R}^+$ such that $\lambda e \pm v \geqslant 0$ in V.

THEOREM 2.1. ([3]) *Let $(V, \{M_n(V^*)^+\}, e)$ be an abstract operator system. Then there exists complete order isomorphism φ from V into $\mathscr{B}(H)$ for some Hilbert space H.*

An L^∞*-matricially normed space* $(V, \{\|\cdot\|_n\})$ is a complex vector space V together with a sequence $\{\|\cdot\|_n\}$ of matrix norms, where $\|\cdot\|_n$ is a norm on $M_n(V)$ for each n such that:

1. $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$;
2. $\|\alpha v \beta\|_n \leqslant \|\alpha\|\|v\|_n\|\beta\|$.

THEOREM 2.2. [12] *Let $(V, \{\|.\|_n\})$ be an L^∞-matricially normed space. Then there exists complete isometry $\varphi : V \to \mathscr{B}(H)$ for some Hilbert space H.*

Note that a closed subspace of $\mathscr{B}(H)$ is called as a *concrete operator space* and an L^∞-matricially normed space is called as an *abstract operator space*. An L^1*-matricially normed space* is a pair $(V, \{\|\cdot\|_n\})$, where V is a complex vector space, and $\|\cdot\|_n$ is a norm on $M_n(V)$ for each n such that:

1. $\|v \oplus w\|_{n+m} = \|v\|_n + \|w\|_m$;
2. $\|\alpha v \beta\|_n \leqslant \|\alpha\|\|v\|_n\|\beta\|$.

In 1988, Ruan showed that if V is an L^∞-matricially normed space, then its matricial dual V^* is an L^1-matricially normed space under the scalar pairing

$$\langle [v_{i,j}], [f_{i,j}] \rangle (= [f_{i,j}]([v_{i,j}])) = \sum_{i,j=1}^{n} f_{i,j}(v_{i,j}). \tag{2}$$

We refer [12] to see more details. Celebrated Ruan representation Theorem 2.2 paved a new way to visualize the subspaces of $\mathscr{B}(H)$ in the more abstract setting in which underlying Hilbert is not necessary.

It is well known that completely positive maps between C*-algebras (abstract operator systems) play crucial role to study non-commutative dynamics and quantum probability (quantum information theorem). Since every C*-algebras (operator systems) is matrix ordered space, thus completely positive maps are defined between C*-algebras (operator systems).

After Ruan's representation of abstract operator spaces (L$^\infty$-matricially norm spaces) as subspaces of C*-algebras, it was natural to study matrix ordered space and operator space together. We refer [14, 15] to see matrix ordered operator space. To study order embedding properties of matrix ordered operator space in C*-algebras, the second named author obtained certain extra conditions on matrix ordered operator space (see [9] for details and references therein). Such matrix ordered operator space is known as C*-ordered operator space.

DEFINITION 2.3. [9] A C^**-ordered operator space* is a triple $(V,\{M_n(V)^+\},\{\|\cdot\|_n\})$, where $(V,\{M_n(V)^+\})$ is a matrix ordered space and $(V,\{\|\cdot\|_n\})$ is an L^∞-matricially normed space, and V^+ is proper such that for each $n\in\mathbb{N}$ the following conditions hold:

1. $*$ is an isometry on $M_n(V)$;
2. $M_n(V)^+$ is closed;
3. $\|f\|_n\leqslant\max\{\|g\|_n,\|h\|_n\}$, whenever $f\leqslant g\leqslant h$ with $f,g,h\in M_n(V)_{sa}$.

Every operator system is a C^*-ordered space. Now, we describe the concrete representation theorem for C^*-ordered operator space.

THEOREM 2.4. ([9]) *Let $(V,\{\|.\|_n\})$ be a self-adjoint abstract operator space. Then there exist a completely isometrical, completely order isomorphism φ form V into $\mathscr{A}$ for some C^*-algebra $\mathscr{A}$ if and only if V is a C^*-ordered operator space.*

One may treat above theorem as a non unital analogue of Choi and Effros's theorem (see [3] for more details). Let V be a C^*-ordered operator space. Then its matricial dual V^* is a matrix ordered space such that L^1-matricially normed space with a complete isometric involution. We put

$$Q_n(V)=\{f\in M_n(V^*):f\geqslant 0,\|f\|_n\leqslant 1\}.$$

We call $Q_n(V)$ as the *quasi state* of $M_n(V)$. It is easy to see that $Q_n(V)$ is a compact convex set with respect to w^*-topology. Now we prove our first result.

LEMMA 2.5. $M_n(V^*)_{sa}\cap M_n(V^*)_1=\text{co}(Q_n(V)\cup -Q_n(V))$.

Proof. Let $f\in M_n(V^*)_{sa}$. Then by [10, Theorem 2.2], there are $g,h\in M_n(V^*)^+$ such that $f=g-h$ and $\|f\|_n=\|g\|_n+\|h\|_n$. Put $g_1=\left(\frac{\|f\|}{\|g\|}\right)g$ and $h_1=\left(\frac{\|f\|}{\|h\|}\right)h$ (excluding the trivial cases, when either $f=0$ or $g=0$). Then $\|f\|=\|g_1\|=\|h_1\|$ and f is a convex combination of g_1 and $-h_1$. Therefore $M_n(V^*)_{sa}\cap M_n(V)_1\subseteq\text{co}(Q_n(V)\cup -Q_n(V))$. Since $\pm Q_n(V)\subseteq M_n(V^*)_{sa}\cap M_n(V)_1$ and $M_n(V^*)_{sa}\cap M_n(V)_1$ is convex, we have $\text{co}(Q_n(V)\cup -Q_n(V))\subseteq M_n(V^*)_{sa}\cap M_n(V)_1$.

In next theorem, we show that every C^*-ordered operator space has a matricial affine representation.

THEOREM 2.6. *Let $(V,\{M_n(V)^+\},\{\|\cdot\|_n\})$ be a C^*-ordered operator space. For $v\in V$, define $\check{v}:V^*\mapsto\mathbb{C}$ given by $\check{v}(f)=f(v)$ $(f\in V^*)$ and set $\check{v}|_{Q(V)}=\hat{v}$. Then $\hat{v}:Q_1(V)\to\mathbb{C}$ is an affine, w^*-continuous map with $\hat{v}(0)=0$ such that $\check{v}$ is the unique extension of $\hat{v}$ to V^* as a w^*-continuous linear functional. We write, $A_0(Q_1(V),V^*)$ for the space of all w^*-continuous affine mappings from $Q_1(V)\mapsto\mathbb{C}$ vanishing at 0 and having a unique w^*-continuous linear extension to V^*. Thus $v\mapsto\hat{v}$ determines a surjective $*$-isomorphism $\Gamma:V\to A_0(Q_1(V),V^*)$. We can transport a matrix order and a matrix norm on it so that it becomes a C^*-ordered operator space and Γ turns out to be a complete isometric, complete order isomorphism.*

Proof. Note that $\check{v}$ is the unique extension of $\hat{v}$ on V^* as a w^*-continuous linear functional for $V^{*+} = \cup_{k\in\mathbb{N}} kQ_1(V)$ and V^{*+} spans V^*. Further, we note that $v \mapsto \hat{v}$ determines a linear $*$-isomorphism from $\Gamma : V \to A_0(Q_1(V), V^*)$. Also, as w^*-dual of V^* is identified with V, we may conclude that Γ is surjective. For $v \in V$, set $(\check{v})^* = (\check{v^*})$ so that

$$(\check{v})^*(f) = f(v^*) = \overline{f^*(v)} = \overline{\check{v}(f^*)}$$

for all $f \in V^*$. In particular for $v \in V_{sa}$ and $f \in V^*_{sa}$, $(\check{v})^*(f) = \check{v}(f) \in \mathbb{R}$. Similarly, if $v \in V^+$ and $f \in V^{*+}$, then $\check{v}(f) \geqslant 0$. In fact, as $v \in V^+$ if and only if $f(v) \geqslant 0$ for every $f \in Q(V)$, we may conclude that

$$\begin{aligned}\Gamma(V^+) &= \{\phi \in A_0(Q_1(V), V^*)_{sa} : \phi(f) \geqslant 0\ \forall f \in Q_1(V)\}\\ &:= A_0(Q_1(V), V^*)^+.\end{aligned}$$

In other words, Γ is an order isomorphism. Now using matrix duality, we may further conclude that

$$\Gamma_n : M_n(V) \mapsto A_0(Q_n(V), M_n(V^*))$$

given by

$$\Gamma_n([v_{i,j}]) = [\hat{v_{i,j}}],\ \ [v_{i,j}] \in M_n(V)$$

is a surjective order isomorphism for each $n \in \mathbb{N}$. Now, if we identify $A_0(Q_n(V), M_n(V^*))$ with $M_n(A_0(Q_1(V), V^*))$ for each $n \in \mathbb{N}$, then $\Gamma : V \mapsto A_0(Q_1(V), V^*)$ is a surjective order isomorphism.

Next, we describe a norm on $A_0(Q_n(V), M_n(V^*))$. Let $F \in A_0(Q_n(V), M_n(V^*))$. Then there is a unique $v \in M_n(V)$ such that $F = \Gamma_n(v) = \widehat{v}$. We define

$$\|F\|_{\infty,n} = \sup\left\{\left|\begin{bmatrix}0 & \widehat{v}\\ \widehat{v^*} & 0\end{bmatrix}(f)\right| : f \in Q_{2n}(V)\right\}. \tag{3}$$

As $v \in M_n(V)$, we have $\begin{bmatrix}0 & v\\ v^* & 0\end{bmatrix} \in M_{2n}(V)_{sa}$. Since $*$ is isometry in V, using Lemma 2.5, we have

$$\begin{aligned}\left\|\begin{bmatrix}0 & v\\ v^* & 0\end{bmatrix}\right\|_n &= \sup\left\{\left|\begin{bmatrix}0 & \widehat{v}\\ \widehat{v^*} & 0\end{bmatrix}(f)\right| : f \in M_{2n}(V^*)_{sa} \cap \mathbb{M}_n(V^*)_1\right\}\\ &= \sup\left\{\left|\begin{bmatrix}0 & \widehat{v}\\ \widehat{v^*} & 0\end{bmatrix}(f)\right| : f \in Q_n(V)\right\}.\end{aligned}$$

Also as $*$ is isometry and $\{\|\cdot\|_n\}$ is an L^∞-matrix norm, we have $\|v\|_n = \left\|\begin{bmatrix}0 & v\\ v^* & 0\end{bmatrix}\right\|_{2n}$ so that $\|v\|_n = \|\widehat{v}\|_{\infty,n}$.

We notice that affine representation of a C*-ordered operator space V depends on the quasi state space $Q_n(V)$. In what follows, we isolate some properties of $Q_n(V)$ so as to prove a converse of Theorem 2.6. Let W be a complex vector space. Then a collection $\{K_n\}$ and $K_n \subseteq M_n(W)$ is called a *matrix convex set* if $\sum_{i=1}^k \gamma_i^* w_i \gamma_i \in K_m$ whenever $w_i \in M_{n_i}(W)$ and $\gamma_i \in M_{n_i,m} (1 \leqslant i \leqslant k)$ satisfy $\sum_{i=1}^k \gamma_i^* \gamma_i = I_m$ (cf. [13]).

It is well know that if V is an abstract operator space, then $\{M_n(V)_1\}$ is a matrix convex set. Here $M_n(V)_1 := \{v \in M_n(V) : \|v\|_n \leqslant 1\}$. Further, if V is a C*-ordered operator space, then $\{M_n(V)_1^+\}$ is a matrix convex set. However, $\{Q_n(V)\}$ is not a matrix convex set. To see this, let $f \in Q(V)$ with $\|f\| = 1$. Then $\|f \oplus f\|_2 = 2$ so that $f \oplus f \notin Q_2(V)$. Since in a matrix convex set $\{K_n\}$, we have $K_1 \oplus K_1 \subseteq K_2$. Thus $\{Q_n(V)\}$ is not a matrix convex set. Nevertheless, it has some interesting properties which we illustrate in the following result. Put $S_n(V) = \{f \in Q_n(V) : \|f\|_n = 1\}$.

PROPOSITION 2.7. *Let V be a* C**-ordered operator space and let $f \in Q_{m+n}(V)$ so that $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$ for some $f_{11} \in M_m(V^*)^+$, $f_{22} \in M_n(V^*)^+$ and $f_{12} \in M_{m,n}(V^*)$. Then we have the following:*

1. *$f_{11} \in Q_m(V)$ and $f_{22} \in Q_m(V)$.*
2. *$\begin{bmatrix} f_{11} & e^{i\theta} f_{12} \\ e^{-i\theta} f_{12}^* & f_{22} \end{bmatrix} \in Q_{m+n}(V)$ for any $\theta \in \mathbb{R}$.*
3. *$\left\| \begin{bmatrix} 0 & f_{12} \\ f_{12}^* & 0 \end{bmatrix} \right\|_{m+n} \leqslant \left\| \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix} \right\|_{m+n}, \left\| \begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix} \right\|_{m+n} \leqslant \left\| \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix} \right\|_{m+n}$.*
4. *If $m = n$, then $f_{12} + f_{12}^* \in \mathrm{co}(Q_n(V) \cup -Q_n(V))$.*
5. *Let $f \in Q_n(V)$ and let $\gamma_i \in \mathbb{M}_{n,n_i}$ such that $\sum_{i=1}^k \gamma_i \gamma_i^* \leqslant I_n$. Then $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in Q_{\sum_{i=1}^k n_i}(V)$.*
6. *Let $f \in Q_{m+n}(V)$ with $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$ so that $f_{11} \in Q_m(V), f_{22} \in Q_n(V)$ and $f_{12} \in M_n(V)$ and let $f_{11} = \alpha_1 \widehat{f_{11}}, f_{22} = \alpha_{22} \widehat{f_{22}}$ with $\widehat{f_{11}} \in S_m(V), \widehat{f_{22}} \in S_n(V)$. Then $\alpha_1 + \alpha_2 \leqslant 1$.*

Proof. We know that if $\alpha \in \mathbb{M}_{m,n}$ and $f \in M_n(V^*)$, then $\|\alpha f \alpha^*\|_m \leqslant \|\alpha\|^2 \|f\|_n$ [12]. Also, if $f \in M_n(V^*)^+$, then $\alpha f \alpha^* \in M_m(V^*)^+$ [3, Lemma 4.2]. Using these argument, we can prove (1) and (2) if we choose $\alpha = \begin{bmatrix} I_m & 0_{m,n} \end{bmatrix}$, $\alpha = \begin{bmatrix} 0_{n,m} & I_n \end{bmatrix}$ and $\alpha = \begin{bmatrix} e^{i\theta} I_m & 0 \\ 0 & I_n \end{bmatrix}$ respectively. In particular, $\left\| \begin{bmatrix} f_{11} & \pm f_{12} \\ \pm f_{12}^* & f_{22} \end{bmatrix} \right\|_{m+n} \leqslant 1$. Thus as

$$2 \begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix} + \begin{bmatrix} f_{11} & -f_{12} \\ -f_{12}^* & f_{22} \end{bmatrix}$$

and

$$2 \begin{bmatrix} 0 & f_{12} \\ f_{12}^* & 0 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix} - \begin{bmatrix} f_{11} & -f_{12} \\ -f_{12}^* & f_{22} \end{bmatrix},$$

(3) follows from the triangle inequality of a norm. Next, as

$$\left\| \begin{bmatrix} f_{12}^* & 0 \\ 0 & f_{12} \end{bmatrix} \right\|_{2n} = \left\| \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} 0 & f_{12} \\ f_{12}^* & 0 \end{bmatrix} \right\|_{2n} \leqslant \left\| \begin{bmatrix} 0 & f_{12} \\ f_{12}^* & 0 \end{bmatrix} \right\|_{2n} \leqslant 1,$$

we have

$$\|f_{12}+f_{12}^*\|_n \leqslant \|f_{12}^*\|_n + \|f_{12}\|_n \leqslant \left\| \begin{bmatrix} f_{12}^* & 0 \\ 0 & f_{12} \end{bmatrix} \right\|_{2n} \leqslant 1.$$

Since $f_{12}+f_{12}^* \in M_n(V^*)_{sa}$, by Lemma 2.5, we may conclude that $f_{12}+f_{12}^* \in \mathrm{co}(Q_n(V) \cup -Q_n(V))$.

(4) As $f \in Q_n(V) \subset M_n(V^*)^+$, and $\gamma_i \in M_{n,n_i}$, we have $\gamma_i^* f \gamma_i \in M_{n_i}(V^*)^+$ for $1 \leqslant i \leqslant k$. Thus $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in M_{\sum_{i=1}^k n_i}(V^*)^+$. We show that $\| \oplus_{i=1}^k \gamma_i^* f \gamma_i\| \leqslant 1$. Let $v \in (M_{\sum_{i=1}^k n_i}(V)_{sa})_1$, say $v=[v_{i,j}]$ where $v_{i,j} \in M_{n_i,n_j}(V)$ and $v_{i,j}^* = v_{j,i}, 1 \leqslant i,j \leqslant k$. Then

$$\begin{aligned} |\langle \oplus_{i=1}^k \gamma_i^* f \gamma_i, v\rangle| &= |\sum_{i=1}^n \langle \gamma_i^* f \gamma_i, v_{ii}\rangle| \\ &= |\sum_{i=1}^n \langle f, \gamma_i^{*t} v_{i,i} \gamma_i^* \rangle| \\ &\leqslant \|\sum_{i=1}^k \gamma_i^{*t} v_{i,i} \gamma_i^t\| \text{ for } f \in Q_n(V). \end{aligned}$$

Since $\sum_{i=1}^k \gamma_i \gamma_i^* \leqslant I_n$, we have

$$\left\| \sum_{i=1}^k \gamma_i^{*t} \gamma_i^t \right\| = \left\| \left(\sum_{i=1}^k \gamma_i \gamma_i^* \right)^t \right\| = \left\| \sum_{i=1}^k \gamma_i \gamma_i^* \right\| \leqslant 1.$$

Thus $\sum_i \gamma_i^{*t} \gamma_i^t \leqslant I_n$. Since $\|v_{i,i}\|_{n_i} \leqslant \|v\|_{\sum_{i=1}^k n_i} \leqslant 1$ for $1 \leqslant i \leqslant k$ and since $\{(M_n(V)_{sa})_1\}$ is a matrix convex set, we find that $\|\sum_{i=1}^k \gamma_i^{*t} v_{i,i} \gamma_i^t\| \leqslant 1$. Thus $\| \oplus_{i=1}^k \gamma_i^* f \gamma_i\| \leqslant 1$ so that $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in Q_{\sum_{i=1}^k n_i}(V)$.

(5) Let $f = \begin{bmatrix} f_{11} & f_{12} \\ F_{12} & f_{22} \end{bmatrix} \in Q_{m+n}(V)$. Then by (3), $f_{11} \in M_m(V^*)^+$ and $f_{22} \in M_n(V^*)^+$ and we have $\|f_{11}\|_m + \|f_{22}\|_n \leqslant 1$. Find $\widehat{f_{11}} \in S_m(V)$, $\widehat{f_{22}} \in S_n(V)$ such that $f_{11} = \|f_{11}\|_m \widehat{f_{11}}$ and $f_{22} = \|f_{22}\|_n \widehat{f_{22}}$. This completes the proof.

3. L^1-matrix convex sets and quantized $A_0(K)$ space

3.1. L^1-matrix convex sets

Now, we introduce a new kind of matricial convexity.

DEFINITION 3.1. Let K be a compact convex set in a locally convex set V such that $0 \in \mathrm{ext}(K)$. An element $k \in K$ is called a *lead point* of K $(k \in \mathrm{lead}(K))$ if $k = \alpha k_1$ for some $k_1 \in K$ with $\alpha \in [0,1]$, then $\alpha = 1$.

We observe that $\mathrm{ext}(K) \setminus \{0\} \subseteq \mathrm{lead}(K)$.

PROPOSITION 3.2. *For each* $k \in K \setminus \{0\}$. *There is a unique* $\alpha \in (0,1]$ *and* $k_1 \in \mathrm{lead}(K)$ *such that* $k = \alpha k_1$.

Proof. Without any loss of generality, we may assume that $k \in K \setminus \mathrm{lead}(K)$. Then by the definition of lead, there is an $\alpha \in (0,1]$ and $k \in K$ such that $k = \alpha k_1$. Thus the set $R_K = \{\beta \geqslant 1 : \beta k \in K\}$ is non-empty. As K is a compact, R_K is bounded and we have $\beta_0 = \sup R_K \in R_K$. Let $k_0 = \beta_0 k \in K$ so that $k = \beta_0^{-1} k_0$. We show that $k_0 \in \mathrm{lead}(K)$. If possible, assume that $k_0 \notin \mathrm{lead}(K)$. Then by the definition of lead, there is a $\beta \in (0,1)$ and $k' \in K$ such that $k_0 = \beta k'$. But, then $\beta^{-1}\beta_0 k \in K$, where $\beta^{-1}\beta_0 > \beta$, which contradict $\beta_0 = \sup R_K$. Thus $k_0 \in \mathrm{lead}(K)$. Next we prove the uniqueness of k_0. Let $k = \alpha_1 k_1$ for some $k_1 \in \mathrm{lead}(K)$ and $\alpha_1 \in (0,1]$. We see that $k_1 = \alpha_1^{-1}\beta_0 k_0$. Thus $\alpha_1^{-1}\beta_0 = 1$ and hence $\alpha_1 = \beta_0, k_1 = k_0$.

By a $*$-*locally convex space*, we mean a locally convex space together with $*$-operation which is a homeomorphism. In this case, $M_n(V)$ is also a $*$-locally convex space with respect to product topology. Now introduce a new definition of matrix convex set. Note that the following definition is appeared in our recent paper [5] to characterize CM-ideals in terms of L^1-matricial split face.

DEFINITION 3.3. (L^1-matrix convex set) Let V be a $*$-locally convex space and let $\{K_n\}$ be a collection of compact convex sets and $K_n \subseteq M_n(V)_{sa}$ such that $0 \in \mathrm{ext}(K_n)$ for all n. Then $\{K_n\}$ is called an L^1*-matrix convex set* if the following conditions hold:

L_1 If $u \in K_n$ and $\gamma_i \in \mathbb{M}_{n,n_i}$ such that $\sum_{i=1}^k \gamma_i\gamma_i^* \leqslant I_n$, then $\oplus_{i=1}^k \gamma_i^* u \gamma_i \in K_{\sum_{i=1}^k n_i}$.

L_2 If $u \in K_{2n}$ so that $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ for some $u_{11}, u_{22} \in K_n$ and $u_{12} \in M_n(V)$, then $u_{12} + u_{12}^* \in \mathrm{co}(K_n \cup -K_n)$.

L_3 Let $u \in K_{m+n}$ with $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ so that $u_{11} \in K_m, u_{22} \in K_n$ and $u_{12} \in M_{m,n}(V)$ and if $u_{11} = \alpha_1 \widehat{u_{11}}, u_{22} = \alpha_{22}\widehat{u_{22}}$ with $\widehat{u_{11}} \in \mathrm{lead}(K_m), \widehat{u_{22}} \in \mathrm{lead}(K_n)$. Then $\alpha_1 + \alpha_2 \leqslant 1$.

REMARK 3.4. Let V be a C^*-ordered space. Then by Proposition 2.7, $\{Q_n(V)\}$ is an L^1-matrix convex set with $\mathrm{lead}(Q_n(V)) = S_n(V)$. In particular, $M_n(\mathscr{T}(H))_1^+$ is an L^1-matrix convex set. Here $\mathscr{T}(H)$ denote the set of all trace class operators on the complex Hilbert space H. Subsequently, $M_n(\mathscr{T}(H))$ is identified with $(\mathscr{T}(H^n))$ for each $n \in \mathbb{N}$.

3.2. Quantized $A_0(K)$ spaces

Let X be a $*$-locally convex space and let $\{K_n\}$ be an L^1- matrix convex set in X_{sa}. We assume that $M_n(X)^+ := \cup_{r=1}^\infty rK_n$ is a cone in $M_n(X)_{sa}$ for all n. Then using $\mathbf{L_1}$, we get that $(X, \{M_n(X)^+\})$ is a matrix ordered space such that X^+ is proper and generating. For each n, we define

$$A_0(K_n, M_n(X)) := \{a : K_n \mapsto \mathbb{C} | a \text{ is continuous and affine}; a(0) = 0; \text{and}$$
$$a \text{ extends to a continuous linear functional } \tilde{a} : M_n(X) \mapsto \mathbb{C}\}.$$

Let $a \in A_0(K_n, M_n(X))$. Since $\{K_n\}$ is an L_1-matrix convex set and since K_n spans $M_n(X)$, for $v \in M_n(X)$, we have $v = \sum_{j=1}^r \lambda_j v_j + i\sum_{k=1}^r \lambda'_k v'_k$ where $v_j, v'_j \in K_n$ and $\lambda_i, \lambda'_j \in \mathbb{R}$. Thus $\tilde{a}(v) = \sum_{j=1}^r \lambda_j a(v_j) + i\sum_{k=1}^r \lambda'_k a(v'_k)$. Therefore, such an extension is always unique.

We consider the following algebraic operations:

1. For $\alpha \in \mathbb{M}_{m,n}$, $\beta \in \mathbb{M}_{n,m}$ and $a \in A_0(K_n, M_n(X))$, we define

$$\alpha a \beta(v) := \tilde{a}(\alpha^T v \beta^T) \text{ for all } v \in K_m.$$

   Then $\alpha a \beta \in A_0(K_m, M_m(X))$. In fact, the map $v \mapsto \alpha^T v \beta^T$ from $M_m(X)$ to $M_n(X)$ is continuous so that the map $v \mapsto \tilde{a}(\alpha^T v \beta^T)$ from $M_m(X)$ into $\mathbb{C}$ is also continuous. Thus $\widetilde{\alpha v \beta} : M_n(X) \mapsto \mathbb{C}$ is continuous and hence $\alpha a \beta \in A_0(K_n, M_n(X))$.

2. For $a \in A_0(K_n, M_n(X))$ and $b \in A_0(K_m, M_m(X))$, we define

$$(a \oplus b)(v) := a(v_{11}) + b(v_{22})$$

   for all $v \in K_{n+m}$ where $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}$ with $v_{11} \in K_n, v_{22} \in K_m, v_{12} \in M_{n,m}(V)$. Then $a \oplus b \in A_0(K_{n+m}, M_{n+m}(X))$. In fact, the maps $v \mapsto v_{11}$ from K_{m+n} into K_n and $v \mapsto v_{22}$ from K_{m+n} into K_m are continuous so that $v \mapsto a(v_{11}) + b(v_{22})$ is also continuous. As $\widetilde{a \oplus b} = \tilde{a} \oplus \tilde{b}$, we see that $\widetilde{a \oplus b}$ is also continuous from $M_{m+n}(X) \mapsto \mathbb{C}$. Therefore, $a \oplus b \in A_0(K_{m+n}, M_{m+n}(X))$.

For $a \in A_0(K_n, M_n(X))$, we define $a^*(u) = \overline{a(u)}$ for all $u \in K_n$ so that $\widetilde{a^*}(u) = \overline{\tilde{a}(u^*)}$ for all $u \in M_n(X)$. Then $a \mapsto a^*$ is an involution. We set

$$A_0(K_n, M_n(X))_{sa} = \{a \in A_0(K_n, M_n(X)) : a^* = a\}.$$

We put

$$A_0(K_n, M_n(X))^+ := \{a \in A_0(K_n, M_n(X))_{sa} : a(f) \geqslant 0 \;\; \forall f \in K_n\}.$$

Next, for $a \in A_0(K_n, M_n(X))$, we define

$$\|a\|_{\infty,n} := \sup\left\{\left| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix}(u) \right| : u \in K_{2n}\right\} \text{ for } a \in A_0(K_n, M_n(X)).$$

Finally, for each $n \in \mathbb{N}$. we define $\Phi_n : M_n(A_0(K_1, X)) \to A_0(K_n, M_n(X))$ as follows: Let $a_{ij} \in A_0(K_1, X)$ for $1 \leqslant i, j \leqslant n$. We define

$$\Phi_n([a_{ij}]) : K_n \to \mathbb{C} \text{ given by } \Phi_n([a_{ij}])([v_{ij}]) = \sum_{i,j=1}^n \widetilde{a_{ij}}(v_{ij}) \text{ for all } [v_{ij}] \in K_n.$$

Now, it is routine to show that $\Phi_n([a_{ij}]) \in A_0(K_n, M_n(X))$. (Note that Φ_n is an amplification of Φ_1. That is, $\Phi_n([a_{ij}]) = [\Phi_1(a_{ij})]$, if $[a_{ij}] \in M_n(A_0(K_1, X)))$. In this identification, we note that $[a_{i,j}]^* = [a^*_{j,i}]$ is an involution in $M_n(A_0(K_1,X))$ so that Φ_1 is a $*$-isomorphism.

For each $n \in \mathbb{N}$, we set

$$M_n(A_0(K_1,X))^+ := \left\{ [a_{ij}] \in M_n(A_0(K_1,X))_{sa} : \sum_{i,j=1}^{n} \widetilde{a_{i,j}}(v_{i,j}) \geqslant 0 \text{ for all } [v_{i,j}] \in K_n \right\}$$

and transport the norm

$$\|[a_{i,j}]\|_n := \|\Phi_n([a_{i,j}])\|_{\infty,n}$$

for all $[a_{i,j}] \in M_n(A_0(K_1,X))$. Under these notions, we show affine representation of C*-ordered operator space which we call as quantization of $A_0(K)$-spaces.

THEOREM 3.5. *$(A_0(K_1,X), \{M_n(A_0(K_1,X))^+\}, \{\|\cdot\|_n\})$ is a* C**-ordered operator space.*

Proof. We prove the theorem in several steps:

It is easy to deduce from the definition that $(\alpha a \beta)^* = \beta^* a^* \alpha^*$ and that $(a \oplus b)^* = a^* \oplus b^*$.

1. Let $\alpha \in \mathbb{M}_{m,n}$, $a \in A_0(K_n, M_n(X))^+$ and let $v \in K_n$. Without any loss of generality, we may assume that $\|\alpha\| \leqslant 1$. Then, by the definition of an L^1-matrix convex set, we have $\alpha^{T^*} v \alpha^T \in K_n$. Thus $\alpha^* a \alpha(v) = a(\alpha^{T^*} v \alpha^T) \geqslant 0$ so that $\alpha^* a \alpha \in A_0(K_n, M_n(X))^+$.

2. Let $a \in A_0(K_m, M_m(X))^+, b \in A_0(K_n, M_n(X))^+$ and let $u \in K_{m+n}$ with $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$, for some $u_{11} \in K_m, u_{22} \in K_n$ and $u_{12} \in M_{m,n}(V)$. Then

$$(a \oplus b)(u) = a(u_{11}) + b(u_{22}) \geqslant 0$$

so that $a \oplus b \in A_0(K_{m+n}, M_{m+n}(X))^+$.

Now, it follows from (1) and (2) and the construction of $M_n(A_0(K_1,X))$ that the sequence of cones $\{M_n(A_0(K_1,X))\}$ is a matrix order on $A_0(K_1,X)$. Also, it is easy to verify that $A_0(K_1,X)^+$ is proper.

3. It is routine to verify that $\|\cdot\|_{\infty,n}$ is a semi-norm on $A_0(K_n, M_n(X))$. We show that it is a norm. Let $a \in A_0(K_n, M_n(X))$ such that $\|a\|_n = 0$. Let $u \in K_n$ and $\alpha = [\frac{1}{\sqrt{2}} I_n, \frac{1}{\sqrt{2}} I_n]$. Then $\alpha^* \alpha \leqslant I_{2n}$ so that $\alpha^* u \alpha = \begin{bmatrix} \frac{u}{2} & \frac{u}{2} \\ \frac{u}{2} & \frac{u}{2} \end{bmatrix} \in K_{2n}$. Also, then $\begin{bmatrix} \frac{u}{2} & i\frac{u}{2} \\ -i\frac{u}{2} & \frac{u}{2} \end{bmatrix} \in K_{2n}$. Thus, as $\|a\|_{\infty,n} = 0$, we get

$$0 = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{u}{2} & i\frac{u}{2} \\ -i\frac{u}{2} & \frac{u}{2} \end{bmatrix} \right) = \tilde{a}\left(\frac{iu}{2}\right) + \tilde{a^*}\left(\frac{-iu}{2}\right) = \frac{i}{2} a(u) + \frac{-i}{2} \overline{a(u)}.$$

Similarly,

$$0 = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{u}{2} & \frac{u}{2} \\ \frac{u}{2} & \frac{u}{2} \end{bmatrix} \right) = \frac{a(u)}{2} + \frac{\overline{a(u)}}{2}.$$

Therefore $a(u) \pm \overline{a(u)} = 0$ for all $u \in K_n$ and consequently $a(u) = 0$ for all $u \in K_n$. Hence $a = 0$.

4. Further, note that $\begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix} \in K_{2n}$ if and only if $\begin{bmatrix} v_{11} & v_{12}^* \\ v_{12} & v_{22} \end{bmatrix} \in K_{2n}$ and that

$$\begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \left(\begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix} \right) = \begin{bmatrix} 0 & a^* \\ a & 0 \end{bmatrix} \left(\begin{bmatrix} v_{11} & v_{12}^* \\ v_{12} & v_{22} \end{bmatrix} \right)$$

for $a \in A_0(K_n, M_n(X))$. Thus $\|a^*\|_{\infty,n} = \|a\|_{\infty,n}$ for all $a \in A_0(K_n, M_n(X))$.

5. Next, we show that If $a \in A_0(K_n, M_n(X))_{sa}$, then

$$\|a\|_{\infty,n} = \sup\{|a(v)| : v \in K_n\}.$$

In particular, we have

$$\|a\|_{\infty,n} = \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \right\|_{\infty,2n}$$

for every $a \in A_0(K_n, M_n(X))$.

To see this, we put $r_n(a) = \sup\{|a(v)| : v \in K_n\}$. Since K_{2n} is a compact set, we have $\|a\|_n = \left| \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} (v) \right|$ for some $v \in K_{2n}$. Let $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}$. Since $\{K_n\}$ is an L^1-matrix convex set, we have $v_{12} + v_{12}^* \in \mathrm{co}(K_n \cup (-K_n))$. As K_n is convex, there are $v, w \in K_n$ and $\lambda \in [0,1]$ such that $v_{12} + v_{12}^* = \lambda u - (1-\lambda) w$. Thus

$$\begin{aligned} \|a\|_{\infty,n} &= |\tilde{a}(v_{12}) + \tilde{a}(v_{12}^*)| = |\tilde{a}(v_{12} + v_{12}^*)| \\ &= |\tilde{a}(\lambda u - (1-\lambda) w)| = |\lambda a(u) - (1-\lambda) a(w)| \\ &\leqslant \lambda r_n(a) + (1-\lambda) r_n(a) = r_n(a). \end{aligned}$$

Again as K_n is a compact convex set, we have $r_n(a) = |a(v)|$ for some $v \in K_n$. Since $\{K_n\}$ is an L^1-matrix convex set, we have $\begin{bmatrix} \frac{v}{2} & \frac{v}{2} \\ \frac{v}{2} & \frac{v}{2} \end{bmatrix} \in K_{2n}$. Therefore,

$$r_n(a) = \left| \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{v}{2} & \frac{v}{2} \\ \frac{v}{2} & \frac{v}{2} \end{bmatrix} \right) \right| \leqslant \|a\|_{\infty,n}.$$

6. In particular, for $a \leqslant b \leqslant c$ in $A_0(K_n, M_n(X))_{sa}$, we have

$$\|b\|_{\infty,n} \leqslant \max\{\|a\|_{\infty,n}, \|c\|_{\infty,n}\}.$$

To prove this, let $a \leqslant b \leqslant c$ in $A_0(K_n, M_n(X))_{sa}$. Then $a(u) \leqslant b(u) \leqslant c(u)$ for all $u \in K_n$ so that $|b(u)| \leqslant \max\{|a(u)|, |c(u)|\}$. Thus by (5), we get $|b(u)| \leqslant \max\{\|a\|_{\infty,n}, \|b\|_{\infty,n}\}$ for all $u \in K_n$ so that $\|b\|_{\infty,n} \leqslant \max\{\|a\|_{\infty,n}, \|c\|_{\infty,n}\}$.

7. Now, we prove that $\|a\oplus b\|_{\infty,m+n} = \max\{\|a\|_{\infty,m}, \|b\|_{\infty,n}\}$ for all $a \in A_0(K_m, M_m(X))_{sa}$ and $b \in A_0(K_n, M_n(X))_{sa}$.

   Let $a \in A_0(K_m, M_m(X))_{sa}$ and $b \in A_0(K_n, M_n(X))_{sa}$. Now for every $v \in K_m$, we have
   $$|a(v)| = |(a\oplus b)(v\oplus 0)|.$$
   Since $\{K_n\}$ is an L^1-matrix convex set, we have $v\oplus 0 \in K_{m+n}$ whenever $v \in K_m$. Therefore by (5), we may conclude that $\|a\|_{\infty,m} \leqslant \|a\oplus b\|_{\infty,m+n}$. Similarly, we can show that $\|b\|_{\infty,n} \leqslant \|a\oplus b\|_{\infty,m+n}$.

   Conversely, let $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix} \in K_{m+n}$. Then there exist $\widehat{v_{11}} \in \mathrm{lead}(K_m), \widehat{v_{22}} \in \mathrm{lead}(K_n)$ and $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 + \alpha_2 \leqslant 1$ such that $v_{11} = \alpha_1 \widehat{v_{11}}, v_{22} = \alpha_2 \widehat{v_{22}}$. Thus
   $$\begin{aligned} |(a\oplus b)(v)| &= |a(v_{11}) + b(v_{22})| \\ &= |\alpha_1 a(\widehat{v_{11}}) + \alpha_2 b(\widehat{v_{22}})| \\ &\leqslant \alpha_1 \|a\|_{\infty,m} + \alpha_2 \|b\|_{\infty,n} \\ &\leqslant \max\{\|a\|_{\infty,m}, \|b\|_{\infty,n}\}. \end{aligned}$$
   Therefore $\|a\oplus b\|_{\infty,m+n} = \max\{\|a\|_{\infty,m}\|b\|_{\infty,n}\}$.

8. Next, we prove that for $a \in A_0(K_n, M_n(X))_{sa}$ and $\alpha \in \mathbb{M}_{m,n}$, we have $\|\alpha^* a \alpha\|_{\infty,n} \leqslant \|\alpha\|^2 \|a\|_{\infty,n}$.

   Let $a \in A_0(K_m, M_m(X))_{sa}$ and $\alpha \in \mathbb{M}_{m,n}$ such that $\|\alpha\| \leqslant 1$ and let $v \in K_n$. Since $\{K_n\}$ is an L^1-matrix convex set and $\alpha^{*T}\alpha^T \leqslant I_m$, we have $\alpha^{T^*} v \alpha^T \in K_m$. Also we know that
   $$|(\alpha^* a \alpha)(v)| = |a(\alpha^{T^*} v \alpha^T)|.$$
   Since a is self-adjoint, (5), we have $\|\alpha^* a \alpha\|_{\infty,n} \leqslant \|a\|_{\infty,n}$ for $a = a^*$. In particular, if $m = n$ and if $\alpha \in \mathbb{M}_m$ is unitary, then $\|\alpha^* a \alpha\|_{\infty,m} = \|a\|_{\infty,m}$. Also, in general, for $a \in A_0(K_n, M_n(X))_{sa}$ and $\alpha \in \mathbb{M}_{m,n}$, we have
   $$\|\alpha^* a \alpha\|_{\infty,n} \leqslant \|\alpha\|^2 \|a\|_{\infty,n}.$$

9. Let $a \in A_0(K_m, M_m(X))$ and $b \in A_0(K_n, M_n(X))$. Put $\gamma = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}$. Then $\gamma \in M_{2m+2n}$ is a unitary and
   $$\gamma^* \begin{bmatrix} 0 & a\oplus b \\ a^* \oplus b^* & 0 \end{bmatrix} \gamma = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix}$$

so that

$$\left\| \begin{bmatrix} 0 & a\oplus b \\ a^*\oplus b^* & 0 \end{bmatrix} \right\|_{\infty,2m+2n} = \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \right\|_{\infty,2m+2n}$$

by (8). Thus by (5), we have

$$\begin{aligned} \|a\oplus b\|_{m+n} &= \left\| \begin{bmatrix} 0 & a\oplus b \\ a^*\oplus b^* & 0 \end{bmatrix} \right\|_{\infty,2m+2n} \\ &= \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \right\|_{\infty,2m+2n} \\ &= \max\left\{ \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \right\|_{\infty,2m}, \left\| \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \right\|_{\infty,2n} \right\} \\ &= \max\{\|a\|_{\infty,m}, \|b\|_{\infty,n}\}. \end{aligned}$$

10. Let $\alpha \in \mathbb{M}_{m,n}, a \in A_0(K_n, M_n(X))$ and $\beta \in \mathbb{M}_{n,m}$. Then by (5), we have

$$\|\alpha a\beta\|_{\infty,m} = \left\| \begin{bmatrix} 0 & \alpha a\beta \\ \beta^* a^* \alpha & 0 \end{bmatrix} \right\|_{\infty,2m}$$

For $t \in \mathbb{R}^+ \setminus \{0\}$, we have

$$\begin{bmatrix} t\alpha & 0 \\ 0 & \frac{1}{t}\beta^* \end{bmatrix} \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \begin{bmatrix} t\alpha^* & 0 \\ 0 & \frac{1}{t}\beta \end{bmatrix} = \begin{bmatrix} 0 & \alpha a\beta \\ \beta^* a^* \alpha & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} \|\alpha a\beta\|_{\infty,m} &\leqslant \left\| \begin{bmatrix} t\alpha & 0 \\ 0 & \frac{1}{t}\beta^* \end{bmatrix} \right\| \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \right\|_{\infty,2n} \left\| \begin{bmatrix} t\alpha^* & 0 \\ 0 & \frac{1}{t}\beta \end{bmatrix} \right\| \\ &\leqslant \max\left\{ \|t\alpha\|, \|\frac{1}{t}\beta^*\| \right\}^2 \|a\|_{\infty,n} \\ &= \max\left\{ t^2\|\alpha\|^2, \frac{1}{t^2}\|\beta\|^2 \right\} \|a\|_{\infty,n}. \end{aligned}$$

Taking infimum over $t \in \mathbb{R}^+ \setminus \{0\}$, we may conclude that $\|\alpha a\beta\|_{\infty,m} \leqslant \|\alpha\| \|a\|_{\infty,n}$ $\|\beta\|$.

This completes the proof.

REMARK 3.6. Let $\{K_n\}$ be an L^1-matrix convex set of X. Then by Theorem 2.4, there is a complete order isometry $\varphi : A_0(K_1, X) \mapsto \mathscr{A}$ for some C*-algebra $\mathscr{A}$.

4. L^1-matricial cap and abstract operator systems

Let E be a real locally convex space, and M be a compact convex set in E. Then M is said to be *regularly embedded* in E if the following conditions hold:

1. M spans E;
2. there exists a hyperplane H containing M such that $0 \notin H$;
3. canonical embedding $x \mapsto \chi(x)$, mapping E to $A(M)^*_{w^*}$ is a topological isomorphism.

For more details, one can see [1, Chapter II.2]. We observe that a convex set M has a *universal cap* if and only if lead$M$ is convex (for more details see [2]). We use this idea to extend this definition to matricial version as follows:

DEFINITION 4.1. Let $\{K_n\}$ be an L^1-matrix convex set in a $*$-locally convex space X and L_n be the lead of K_n for each n. We call $\{L_n\}$ the *matricial lead* of $\{K_n\}$. We also assume that $M_n(X)^+ = \cup_{r=1}^{\infty} rK_n$ is a cone in $M_n(X)_{sa}$ for all n (so that $(X, \{M_n(X)^+\})$ is a matrix ordered space) such that X^+ is proper and generating. We call $\{K_n\}$ an L^1*-matricial cap* of V if

1. L_1 is convex; and
2. if $v \in L_{m+n}$ with $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}$ for some $v_{11} \in K_m, v_{22} \in K_n$ and $v_{12} \in M_{m,n}(V)$ and if $v_{11} = \alpha_1 \widehat{v_1}, v_{22} = \alpha_{22} \widehat{v_2}$ for some $\widehat{v_1} \in L_m, \widehat{v_2} \in L_n$ and $\alpha_1, \alpha_2 \in [0,1]$, then $\alpha_1 + \alpha_2 = 1$.

PROPOSITION 4.2. *Let $\{K_n\}$ be an L^1-matricial cap of V. Then L_n is convex for every n.*

Proof. We prove this result in several steps.

Step I. L_2 is convex.

Let $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}, w = \begin{bmatrix} w_{11} & w_{12} \\ w_{12}^* & w_{22} \end{bmatrix} \in L_2$ and let $\lambda \in [0,1]$. Then by (2), we have $v_{11} = \alpha_1 \widehat{v_1}, v_{22} = \alpha_2 \widehat{v_2}$ with $\alpha_1 + \alpha_2 = 1$, for some $\widehat{v_1}, \widehat{v_2} \in L_1$, and $w_{11} = \beta_1 \widehat{w_1}, w_{22} = \beta_2 \widehat{w_2}$ with $\beta_1 + \beta_2 = 1$, for some $\widehat{w_1}, \widehat{w_2} \in L_1$. Now

$$u := \lambda v + (1-\lambda) w = \begin{bmatrix} \lambda v_{11} + (1-\lambda) w_{11} & \lambda v_{12} + (1-\lambda) w_{12} \\ \lambda v_{12}^* + (1-\lambda) w_{12}^* & \lambda v_{22} + (1-\lambda) w_{22} \end{bmatrix} \in K_2.$$

Let $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ so that $u_{11} = \lambda v_{11} + (1-\lambda) w_{11} = \lambda \alpha_1 \widehat{v_1} + (1-\lambda) \beta_1 \widehat{w_1}$ and $u_{22} = \lambda v_{22} + (1-\lambda) w_{22} = \lambda \alpha_2 \widehat{v_2} + (1-\lambda) \beta_2 \widehat{w_2}$. Since L_1 is convex, we get $\widehat{u_1} := (\lambda \alpha_1 + (1-\lambda)\beta_1)^{-1} u_{11} \in L_1$ and $\widehat{u_2} := (\lambda \alpha_2 + (1-\lambda)\beta_2)^{-1} u_{22} \in L_1$. Put $(\lambda \alpha_1 + (1-\lambda)\beta_1) = \gamma_1$ and $(\lambda \alpha_2 + (1-\lambda)\beta_2) = \gamma_2$, then $u = \begin{bmatrix} \gamma_1 \widehat{u_1} & u_{12} \\ u_{12}^* & \gamma_2 \widehat{u_2} \end{bmatrix}$ and $\gamma_1 + \gamma_2 = \lambda(\alpha_1 +$

$\alpha_2)+(1-\lambda)(\beta_1+\beta_2)=\lambda+(1-\lambda)=1$. Let $u=\gamma\widehat{u}$, where $\widehat{u}\in L_2$ and $\gamma\in[0,1]$. We show that $\gamma=1$. Let $\widehat{u}=\begin{bmatrix} x_{11} & x_{12}\\ x_{12}^* & x_{22}\end{bmatrix}$. Then $x_{11},x_{22}\in K_1$ with $\gamma x_{11}=u_{11},\gamma x_{22}=u_{22}$. Thus $x_{11}=\gamma^{-1}\gamma_1\widehat{u_1}$ and $x_{22}=\gamma^{-1}\gamma_2\widehat{u_2}$. Now by (2), we get $1=\gamma^{-1}\gamma_1+\gamma^{-1}\gamma_2=\gamma^{-1}$ so that $\gamma=1$. Thus $u\in L_2$. Hence L_2 is convex.

Now, by induction, we may deduce that L_{2^n} is convex for every n.

Step II. For $m,n\in\mathbb{N}$, we have L_m is convex if L_{m+n} is convex.

First, we show that $v\mapsto v\oplus 0$ maps L_m into L_{m+n}. Let $v\in L_m$. Then $v\oplus 0\in K_{m+n}$ so that $v\oplus 0=\alpha\widehat{w}$ for some $\widehat{w}\in L_{m+n}$ and $\alpha\in[0,1]$. Thus

$$v=\begin{bmatrix} I_n & 0_{n,m}\end{bmatrix}(v\oplus 0)\begin{bmatrix} I_n\\ 0_{m,n}\end{bmatrix}=\alpha\begin{bmatrix} I_n & 0_{n,m}\end{bmatrix}\widehat{w}\begin{bmatrix} I_n\\ 0_{m,n}\end{bmatrix}=\alpha w_1.$$

where $w_1=\begin{bmatrix} I_n & 0_{n,m}\end{bmatrix}\widehat{w}\begin{bmatrix} I_n\\ 0_{m,n}\end{bmatrix}\in K_m$. Now, as L_m is the lead of K_m, we have $\alpha=1$ and $w_1=v$. Thus $v\oplus 0=\widehat{w}\in L_{m+n}$.

Fix $m\in\mathbb{N}$. Let $v,w\in L_m$ and $\alpha\in(0,1)$. As L_{2^m} is convex, we get

$$(\alpha v+(1-\alpha)w)\oplus 0=\alpha(v\oplus 0)+(1-\alpha)(w\oplus 0)\in L_{2^m}.$$

Put $u=\alpha v\oplus(1-\alpha)w$. Then $u\in K_m$ so that $u=\lambda\widehat{u}$ for some $\widehat{u}\in L_m$ and $\lambda\in[0,1]$. As $\widehat{u}\in L_m$, we get that $\widehat{u}\oplus 0\in L_{2^m}$. Now $\lambda(\widehat{u}\oplus 0)=u\oplus 0\in L_{2^m}$ so that $\lambda=1$ and $u=\widehat{u}\in L_m$. Thus L_m is also convex.

When L_1 is compact and convex, by $A(L_1)$ we denote the set of all complex valued continuous affine functions on L_1. Then $A(L_1)_{sa}$ is an order unit space so that $A(L_1)_{sa}^*$, the ordered Banach dual of $A(L_1)_{sa}$ is a base normed space [1, 6].

DEFINITION 4.3. Let $\{K_n\}$ be an L^1-matrix convex set in a $*$-locally convex space X. Then $\{K_n\}$ is called *regularly embedded* in X if L_1 is regularly embedded in X_{sa}. In other words,

1. L_1 is compact and convex; and
2. $\chi: X_{sa}\mapsto (A(L_1)_{sa}^*)_{w*}$ is an linear homeomorphism.

Here $\chi(w)(a)=\lambda a(u)-\mu a(v)$ for all for all $a\in A(L_1)_{sa}$ with $w=\lambda u-\mu v$ for some $u,v\in L_1$ and $\lambda,\mu\in\mathbb{R}^+$.

Notice that $\chi(w)$ is well defined. To see this, let $w=\lambda_1u_1-\mu_1v_1=\lambda_2u_2-\mu_2v_2$ for some $u_i,v_i\in L_1$ and $\lambda_i,\mu_i\in\mathbb{R}^+$ for $i=1,2$. As L_1 is convex and

$$\frac{\lambda_1+\mu_2}{\lambda_2+\mu_1}\left(\frac{\lambda_1u_1+\mu_2v_1}{\lambda_1+\mu_2}\right)=\frac{\lambda_2u_2+\mu_1v_1}{\lambda_2+\mu_1},$$

by Proposition 3.2, we have $\lambda_1+\mu_2=\lambda_2+\mu_1$. So if a is an affine function on L_1, then

$$\frac{\lambda_1a(u_1)+\mu_2a(v_2)}{\lambda_1+\mu_2}=a\left(\frac{\lambda_1u_1+\mu_2v_2}{\lambda_1+\mu_2}\right)=a\left(\frac{\lambda_2u_2+\mu_1v_1}{\lambda_2+\mu_1}\right)=\frac{\lambda_2a(u_2)+\mu_1a(v_1)}{\lambda_2+\mu_1}.$$

Thus $\lambda_1 a(u_1) - \mu_1 a(v_1) = \lambda_2 a(u_2) - \mu_2 a(v_2)$ so that $\chi(w)$ is well defined linear functional on $A(L_1)_{sa}$ for all $u,v \in L_n$ and $\lambda,\mu \in \mathbb{R}^+$.

THEOREM 4.4. *Let $\{K_n\}$ be a regularly embedded, L^1-matricial cap in X. Then $A_0(K_1,X)$ has an order unit, say e, and $(A_0(K_1,X),e)\}$ is an abstract operator system.*

Proof. As L_1 is the lead of K_1, there exists a mapping $e : K_1 \setminus \{0\} \to (0,1]$ given by $e(k) = \alpha$ if $k = \alpha\widehat{k}$ for some $\widehat{k} \in L_1$ and $\alpha \in (0,1]$. Since α and $\widehat{k}$ are uniquely determined by $k \in K_1 \setminus \{0\}$, e is well defined. We extend e to K by putting $e(0) = 0$. Since L_1 is convex, we may conclude that $e : K_1 \to [0,1]$ is affine. Again since K_1 spans X, we can extend e to a self-adjoint linear functional $\tilde{e} : X \to \mathbb{C}$. Following this way, for each $n \in \mathbb{N}$, we can construct a self-adjoint linear functional $\widetilde{e_n} : M_n(X) \to \mathbb{C}$ such that $\widetilde{e_n}(v) = 1$ for all $v \in L_n$. (We write e_n for $\widetilde{e_n}|_{L_n}$.)

We show that $\tilde{e}$ is continuous. It is suffices to show that $\tilde{e}|_{V_{sa}}$ is continuous at 0. Let $\{\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha\}$ be a net in X_{sa} for some $u_\alpha, v_\alpha \in L_1$ and $\lambda_\alpha, \mu_\alpha \in \mathbb{R}^+$ such that $\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha \to 0$. Since $\{K_n\}$ is L^1-regularly embedded in X, we get $\chi(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) \to 0$ in $(A(L_1)^*_{sa})_{w*}$. Let I_{L_1} be the constant map on L_1 such that $I_{L_1}(v) = 1$ for all $v \in L_1$. Then $I_{L_1} \in A(L_1)_{sa}$. Thus $\chi(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha)(I_{L_1}) \to 0$ so that $\tilde{e}(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) \to 0$. Now it follows that $e \in A_0(K_1,X)$.

Next, fix $n \in \mathbb{N}$ and consider $e^n \in M_n(A_0(K_1,X))$ so that by Theorem 3.5, $e_0^n := \Phi_n(e^n) \in A_0(K_n, M_n(X))$. We show that $e_0^n = e_n$. Let $[v_{i,j}] \in L_n$ so that $v_{i,i} \in K_1$ for $i = 1,\dots,n$. Let $v_{ii} = \alpha_i \widehat{v_i}$ for some $\alpha_i \in [0,1]$ and $\widehat{v_i} \in L_n$. Since $\{K_n\}$ is an L^1-matricial cap, we have $\sum_{i=1}^n \alpha_i = 1$. Thus

$$e_0(v) = \sum_{i=1}^n e_i(v_{i,i}) = \sum_{i=1}^n \alpha_i e_i(\widehat{v_i}) = \sum_{i=1}^n \alpha_i = 1$$

so that $e_0(v) = e_n(v)$ for all $v \in L_n$. Since L_n is the lead of K_n and since K_n spans $M_n(X)$, it follows that $\widetilde{e_n} = e_0$ and that $e_n \in A_0(K_n, M_n(X))$.

Note that $\|e\|_{\infty,1} = 1$. We show that e is an order unit for $A_0(K_1,X)_{sa}$. To see this, let $a \in A_0(K_1,X)_{sa}$. Then $|a(k)| \leqslant \|a\|_{\infty,1}$ for all $k \in K_1$. Let $k \in K_1$. If $k = 0$, then $a(0) = 0$ so that

$$-\|a\|_{\infty,1} e(0) = 0 = \|a\|_{\infty,1} e(0).$$

Let $k \neq 0$. Then there exist a unique $\widehat{k} \in L_1$ and $\alpha \in (0,1]$ such that $k = \alpha\widehat{k}$. Now

$$-\|a\|_{\infty,1} e(\hat{k}) = -\|a\|_{\infty,1} \leqslant a(\hat{k}) \leqslant \|a\|_{\infty,1} = \|a\|_{\infty,1} e(\hat{k}).$$

so that

$$-\|a\|_{\infty,1} e(k) \leqslant a(k) \leqslant \|a\|_{\infty,1} e(k)$$

for all $k \in K$. Thus we have $-\|a\|_{\infty,1} e \leqslant a \leqslant \|a\|_{\infty,1} e$ for all $a \in A_0(K_1,V)_{sa}$. In other words, e is an order unit for $A_0(K_1,X)_{sa}$ which determines $\|\cdot\|_{\infty,1}$ as an order unit norm on it. Similarly, we can show that for each $n \in \mathbb{N}$, e_n is an order unit for $A_0(K_n, M_n(X))_{sa}$ which determines $\|\cdot\|_{\infty,n}$ as an order unit norm on it. Again, being function space, $A_0(K_n, M_n(X))$ is Archimedean for every n. Hence $(A_0(K_1,X),e)$ is an abstract operator systems.

Next, we prove the completeness of $(A_0(K_1,X),e)$.

PROPOSITION 4.5. *Let $\{K_n\}$ be an L^1-matrix convex set in a $*$-locally convex space X. Then $\overline{A_0(K_n, M_n(X))_{sa}} = A_0(K_n)_{sa}$ for every $n \in \mathbb{N}$.*

Proof. By the definition, $A_0(K_n, M_n(X))_{sa} \subset A_0(K_n)_{sa}$. Also, since $A_0(K_n)_{sa}$ is norm complete, we get $\overline{A_0(K_n, M_n(X))_{sa}} \subset A_0(K_n)_{sa}$. Conversely, let $a \in A_0(K_n)_{sa}$ and $\varepsilon > 0$. Then $G_{K_n}(a)$ and $G_{K_n}(a+\varepsilon)$ are compact convex set in $M_n(X)_{sa} \times \mathbb{R}$ where

$$G_{K_n}(b+\lambda) := \{(k, b(k)+\lambda) : k \in K_n\}$$

with $b \in A_0(K_n)_{sa}$ and $\lambda \in [0,\infty)$. Thus $G_{K_n}(a) \cap G_{K_n}(a+\varepsilon) = \emptyset$. Therefore, by the Hahn Banach separation theorem, there are $f \in (M_n(X)_{sa})^* (= (M_n(X)^*)_{sa})$ and $\lambda \in \mathbb{R}$ such that

$$(f,\lambda)(u, a(u)) < (f,\lambda)(v, a(v)+\varepsilon) \ \forall u, v \in K_n.$$

Simplifying this, we get

$$f(u) + \lambda a(u) < f(v) + \lambda(a(v)+\varepsilon) \ \forall u, v \in K_n.$$

In particular, when $u = v = 0$, we get $\lambda > 0$. Similarly, for $u = 0$ and $v = 0$ separately, we have

$$\lambda^{-1} f(u) + a(u) < \varepsilon \text{ and } \lambda^{-1} f(v) + a(v) > -\varepsilon \ \forall u, v \in K_n.$$

Let us put $a_1 = -\lambda^{-1} f$, then $a_1 \in A_0(K_n, M_n(X))_{sa}$ and $|a_1(u) - a(u)| < \varepsilon$ for all $u \in K_n$. Now, by (5) of the proof of Theorem 3.5, we have $\|a_1 - a\|_{\infty,n} \leqslant \varepsilon$. This completes the proof.

PROPOSITION 4.6. *Under the assumptions of Theorem 4.4, $A_0(K_n, M_n(X)) = A_0(K_n)$ for each $n \in \mathbb{N}$.*

Proof. We know that $A_0(K_1, X) \subseteq A_0(K_1)$. Let $a \in A_0(K_1)$ so that $a = a_1 + ia_2$ for some $a_1, a_2 \in A_0(K_1)_{sa}$ and let $\{\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha\}$ be a net in X_{sa} for some $u_\alpha, v_\alpha \in L_1$ and $\lambda_\alpha, \mu_\alpha \geqslant 0$ such that $\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha \to 0$. Since K_1 generates V, a_i has a unique linear extension $\widetilde{a_i}$ for $i = 1, 2$. Since $\{K_n\}$ is L^1-regularly embedded in X, $\chi(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) \to 0$ in $(A(L_1)^*_{sa})_{w*}$. Thus

$$\begin{aligned} \widetilde{a_i}(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) &= \lambda_\alpha a_i(u_\alpha) - \mu_\alpha a_i(v_\alpha) \\ &= \lambda_\alpha a_i|_{L_1}(u_\alpha) - \mu_\alpha a_i|_{L_1}(v_\alpha) \\ &= \chi(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha)(a_i|_{L_1}) \to 0 \end{aligned}$$

Put $\widetilde{a} = \widetilde{a_1} + i\widetilde{a_2}$. Then $\widetilde{a}|_{K_1} = a$ and $\widetilde{a}(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) \to 0$. Thus $\widetilde{a}$ is continuous on X and consequently, $a \in A_0(K_1, X)$. Therefore we have $A_0(K_1) = A_0(K_1, X)$. It follows that $A_0(K_1, X)$ is $\|\cdot\|_1$-complete so that $(A_0(K_n, M_n(X))$ is $\|\cdot\|_{\infty,n}$-complete. Since $\overline{A_0(K_n, M_n(X))_{sa}} = A_0(K_n)_{sa}$ by Proposition 4.5, we may conclude that

$$A_0(K_n) = \overline{A_0(K_n, M_n(X))} = A_0(K_n, M_n(X))$$

for $A_0(K_n, M_n(X))$ is $\|\cdot\|_{\infty,n}$-complete.

REMARK 4.7. Under the assumptions of Theorem 4.4, L_n is compact for each $n \in \mathbb{N}$. To see this, let $\{u_\alpha\}$ be a net in L_n. Since $L_n \subseteq K_n$ and K_n is compact, u_α has subnet $\{u_\beta\}$ that convergent $u_0 \in K_n$. Since $e_n \in A_0(K_n)$. Therefore $1 = e_n(u_\beta) \to e_n(u_0)$ so that $e_n(u_0) = 1$. Hence $u_0 \in L_n$.

PROPOSITION 4.8. *$A_0(K_n)$ is order isomorphic to $A(L_n)$.*

Proof. It suffices to prove that the map $a \mapsto a|_{L_n}$ from $A_0(K_n)$ into $A(L_n)$ is surjective. Let $a \in A(L_n)$. Since L_n is convex, there is an affine map b on K_n such that $b|_{L_n} = a$ and $b(0) = 0$. Let u_α be a net in K_n such that $u_\alpha \to u_0$ in K_n. Since $e_n \in A_0(K_n)$, $e_n(u_\alpha) \to e_n(u_0)$, by Proposition 3.2, we have $u_\alpha = \lambda_\alpha \widehat{u_\alpha}$ for some $\widehat{u_\alpha} \in L_n$ and $\lambda_\alpha \in [0,1]$. If $u_0 = 0$, then $\lambda_\alpha = \lambda_\alpha e_n(\widehat{u_\alpha}) = e_n(u_\alpha) \to e(0) = 0$. Therefore, $b(u_\alpha) = \lambda_\alpha a(\widehat{u_\alpha}) \to 0 = b(0)$. Again if $u_0 \neq 0$, then by Proposition 3.2, we have $u_0 = \lambda_0 \widehat{u_0}$ for some $\lambda_0 \in (0,1]$ and $\widehat{u_0} \in L_n$. Thus $\lambda_\alpha = \lambda_\alpha e_n(\widehat{u_\alpha}) = e_n(u_\alpha) \to e_n(u_0) = \lambda_0$ so that $\widehat{u_\alpha} \to \widehat{u_0}$. Since $b(u_\alpha) = \lambda_\alpha a(\widehat{u_\alpha})$, we have $b(u_\alpha) \to \lambda_0 a(u_0) = b(u_0)$.

REMARK 4.9. In Proposition 4.8, we note that $a \mapsto a|_L$ is an isometry from $A_0(K_n)_{sa}$ onto $A(L_n)$ as well. Hence $(A_0(K_1), e)$ is unitally, complete isometrically, completely order isomorphic to $(A(L_1), e)$ as abstract operator systems.

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