Operators and Matrices
Volume 14, Number 2 (2020), 401–416 doi:10.7153/oam-2020-14-29

ON THE SPECTRUM OF THE SYLVESTER–ROSENBLUM OPERATOR ACTING ON TRIANGULAR ALGEBRAS

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(*Communicated by S. McCullough*)

Abstract. Let $\mathscr{A}$ and $\mathscr{B}$ be algebras and $\mathscr{M}$ be an $\mathscr{A}$-$\mathscr{B}$-bimodule. For $A \in \mathscr{A}$, $B \in \mathscr{B}$, we define the Sylvester-Rosenblum operator $\tau_{A,B} : \mathscr{M} \to \mathscr{M}$ via $\tau_{A,B}(M) = AM + MB$ for all $M \in \mathscr{M}$. We investigate the spectrum of $\tau_{A,B}$ in three settings, namely: (a) when $\mathscr{A} = \mathscr{B} = \mathscr{T}_n(\mathbb{F})$, the set of upper-triangular matrices over an algebraically closed field $\mathbb{F}$ and $\mathscr{M} \subseteq \mathbb{M}_n(\mathbb{F})$; (b) when $\mathscr{A} = \mathscr{B} = \mathscr{M}$ is a unital triangular Banach algebra; and (c), when $\mathscr{M} = \mathscr{T}(\mathscr{N})$ is the nest algebra associated to a nest $\mathscr{N}$ on a complex, separable Hilbert space and $\mathscr{A} = \mathscr{B} = \mathbb{C}I + \mathscr{K}(\mathscr{N})$ consists of the unitization of the algebra of compact operators in $\mathscr{T}(\mathscr{N})$.

1. Introduction

Let $\mathscr{A}$ and $\mathscr{B}$ be algebras over a field $\mathbb{F}$ and $\mathscr{M}$ be a $\mathscr{A}$-$\mathscr{B}$-bimodule. Fixing $A \in \mathscr{A}$ and $B \in \mathscr{B}$, it is interesting to study properties of the **Sylvester-Rosenblum operator**

$$\begin{aligned} \tau_{A,B} : \mathscr{M} &\to \mathscr{M} \\ T &\mapsto AT + TB. \end{aligned}$$

In the context of operators and matrices (for example, taking $\mathscr{M} = \mathbb{M}_n(\mathbb{F})$, the algebra of $n \times n$ matrices over a field $\mathbb{F}$), the equation $AX + XB = Y$ has been studied extensively for many years. Solvability of the equation has several striking consequences in operator theory, linear algebra and the theory of differential equations. An excellent resource is the expository article [2]. First results are due to Sylvester [11] who studied solutions of the equation in matrices. The equation were later investigated for bounded operators on infinite-dimensional spaces by several mathematicians including M.G. Krein, Dalecki [3, 4], Rosenblum [10] and Kleineke (cited in [10]).

We state two of the results we need in a form suitable for our discussion. For vector spaces V and W, denote the space of linear maps from W to V by $\mathscr{L}(W,V)$. As usual $\mathscr{L}(V,V)$ is written as $\mathscr{L}(V)$. When $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces over $\mathbb{C}$, we denote by $\mathscr{B}(\mathfrak{Y},\mathfrak{X})$ the space of bounded linear maps from $\mathfrak{Y}$ into $\mathfrak{X}$, and we write $\mathscr{B}(\mathfrak{X})$ for $\mathscr{B}(\mathfrak{X},\mathfrak{X})$. The spectrum of a linear transformation A is denoted by $\sigma(A)$. If $\mathscr{A}$ and $\mathscr{B}$ are two unital algebras and $A \in \mathscr{A} \subseteq \mathscr{B}$, then it is possible that the spectrum

Mathematics subject classification (2010): 15A06, 15A24, 46H25, 47L35.
Keywords and phrases: Sylvester equation, Sylvester-Rosenblum operator, triangular algebra, nest algebra.
Research supported in part by NSERC (Canada).

Paper OaM-14-29

of A relative to these algebras will differ. For this reason, we shall also write $\sigma_{\mathscr{A}}(A)$ (resp. $\sigma_{\mathscr{B}}(A)$) to highlight the fact that we are considering the spectrum of A relative to $\mathscr{A}$ (resp. relative to $\mathscr{B}$).

THEOREM 1. *(a)* **(Sylvester [11])** *Let m and n be positive integers, $\mathbb{F}$ an algebraically closed field, $A \in \mathbb{M}_m(\mathbb{F})$, $B \in \mathbb{M}_n(\mathbb{F})$, and let $\tau_{A,B}$ be the linear transformation on the space $\mathbb{M}_{m,n}(\mathbb{F})$ of $m \times n$ matrices over $\mathbb{F}$ defined by $\tau_{A,B}(T) = AT + TB$. Then $\sigma(\tau) = \sigma(A) + \sigma(B)$.*

(b) **(Dalecki, Rosenblum, Kleineke)** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces over the complex field. Let $A \in \mathscr{B}(\mathfrak{X})$, $B \in \mathscr{B}(\mathfrak{Y})$, and consider the linear transformation $\tau_{A,B}$ on $\mathscr{B}(\mathfrak{Y},\mathfrak{X})$ defined by $\tau_{A,B}(T) = AT + TB$. Then $\sigma(\tau_{A,B}) = \sigma(A) + \sigma(B)$.*

In the infinite-dimensional setting above (part (b)), the proof of the inclusion

$$\sigma(\tau_{A,B}) \subseteq \sigma(A) + \sigma(B)$$

may be found in [8] (see also [2]) and follows immediately from general spectral theory of commuting elements in a Banach algebra applied to left-multiplication by A and right-multiplication by B. (See Section 3 below as well.) The less obvious reverse inclusion comes from an unpublished work by Kleineke as stated by Rosenblum [10]. A proof may be found in [1] .

The purpose of the present article is to initiate the study of the spectrum of the Sylvester-Rosenblum operator in the setting where the algebras $\mathscr{A}$ and $\mathscr{B}$ are in some sense "triangular". More explicitly, in Section 2 below, we consider the case where $\mathscr{A}$ and $\mathscr{B}$ consist of all block upper-triangular $n \times n$ matrices over the field $\mathbb{F}$, and $\mathscr{M} \subseteq \mathbb{M}_n(\mathbb{F})$ is a $\mathscr{A}$-$\mathscr{B}$-bimodule. In Section 4, we turn our attention to the case where $\mathscr{A}, \mathscr{B}$ and $\mathscr{M}$ are all *nest algebras* acting on complex Hilbert spaces. Nest algebras are the natural generalizations to the infinite-dimensional setting of block upper-triangular matrices over $\mathbb{C}$. The analysis of the spectrum of the operator $\tau_{A,B}$ is made more difficult by the fact that if $\mathscr{H}$ is a Hilbert space and $\mathscr{A}$ is a closed subalgebra of $\mathscr{B}(\mathscr{H})$, then the spectrum of an element $A \in \mathscr{A}$ need not agree with its spectrum considered as an element of $\mathscr{B}(\mathscr{H})$. For this reason, we first set up the machinery we shall need in Section 3, where we consider the spectrum of the Sylvester-Rosenblum operator $\tau_{A,B}$, where $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $\mathscr{A} = \mathscr{B} = \mathscr{M} = \mathscr{T}_p$, a *triangular Banach algebra* as defined in that section.

2. Purely algebraic results

Let $\mathbb{F}$ a field and n a positive integer. We denote by $\mathbb{M}_n(\mathbb{F})$ be the algebra of $n \times n$ matrices with entries in $\mathbb{F}$ and let $\mathscr{T}_n(\mathbb{F})$ be the algebra of $n \times n$ upper triangular matrices over F. One of the special cases of the results we shall prove is that if $A, B \in \mathscr{T}_n(\mathbb{F})$ with $\mathbb{F}$ algebraically closed, and if $\tau_{A,B}$ is the corresponding Sylvester-Rosenblum operator on $\mathscr{T}_n(\mathbb{F})$, then

$$\sigma(\tau_{A,B}) = \{a_{ii} + b_{jj} : i \leqslant j\}.$$

We present an illustrative example. For simplicity we consider diagonal A, B. As usual E_{ij} denotes the basic matrix units on $\mathscr{T}_n(\mathbb{F})$.

EXAMPLE 1. Let $A = [a_{ij}]$, $B = [b_{ij}]$ be $n \times n$ diagonal matrices over a field $\mathbb{F}$ and let τ be the linear transformation on the $\frac{n(n+1)}{2}$-dimensional space $\mathscr{T}_n(\mathbb{F})$ defined by

$$\tau(X) = AX + XB.$$

It is then routine to calculate that

$$\tau(E_{ij}) = (a_{ii} + b_{jj})E_{ij} \text{ for all } 1 \leqslant i \leqslant j \leqslant n.$$

Thus $\tau_{A,B}$ admits a spanning set of eigenvectors in $\mathscr{T}_n(\mathbb{F})$. From this it easily follows that $\sigma(\tau_{A,B}) = \{a_{ii} + b_{jj} : 1 \leqslant i \leqslant j \leqslant n\}$.

Next we describe "block upper-triangular matrix algebras". To every finite sequence of positive integers $n_1, n_2, \dots n_k$, satisfying $n_1 + n_2 + \dots + n_k = n$, we associate an algebra $\mathscr{T}(n_1, n_2, \dots n_k)$ consisting of all $n \times n$ matrices over $\mathbb{F}$ of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ 0 & A_{22} & \dots & A_{2k} \\ \vdots & & & \\ 0 & 0 & \dots & A_{kk} \end{bmatrix} \tag{1}$$

where A_{ij} is an $n_i \times n_j$ matrix. We call such an algebra a **block upper-triangular algebra.** The algebra may be identified as the set of all linear transformation on $V = \mathbb{F}^n$ leaving a nest of subspaces invariant. The nest consists of the subspaces $V_1, \cdots V_k$ defined by

$$V_j = \operatorname{span}\{e_t : 1 \leqslant t \leqslant n_1 + n_2 + \cdots + n_j\},$$

where $\{e_i\}$ denotes the standard basis of $\mathbb{F}^n$.

THEOREM 2. *Let $\mathfrak{A} = \mathscr{T}(n_1, n_2, \dots, n_k)$ be a block triangular algebra over an algebraically closed field $\mathbb{F}$. Fix $A, B \in \mathfrak{A}$ and define $\tau = \tau_{A,B} \in \mathscr{L}(\mathfrak{A})$ by $\tau(T) = AT + TB$, $T \in \mathfrak{A}$. Then*

$$\sigma(\tau) = \{\alpha + \beta : \alpha \in \sigma(A_{ii}), \beta \in \sigma(B_{jj}), 1 \leqslant i \leqslant j \leqslant n\}.$$

Proof. We prove this by induction on k, the number of blocks. If $k = 1$, the result follows by Sylvester's Theorem. If $k > 1$, we partition the matrices in $\mathfrak{A}$ as $Z = \begin{bmatrix} Z_{11} & Z_0 \\ 0 & \hat{Z} \end{bmatrix}$ and we write $\mathfrak{A} = \begin{bmatrix} M_{n_1} & M_{n_1 m} \\ 0 & \hat{\mathfrak{A}} \end{bmatrix}$. We decompose $\mathfrak{A}$ as a direct sum of the two subspaces

$$\mathfrak{S}_1 = \left\{ \begin{bmatrix} Z_{11} & Z_0 \\ 0 & 0 \end{bmatrix} \right\} \text{ and } \mathfrak{S}_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \hat{Z} \end{bmatrix} \right\}.$$

The space $\mathfrak{S}_1$ is invariant under τ. The matrix of τ with respect to the decomposition $\mathfrak{A} = \mathfrak{S}_1 \oplus \mathfrak{S}_2$ is of the form $\begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix}$. The matrix R_1 is the matrix of the Sylvester-Rosenblum operator $X \mapsto A_{11}X + XB$ as an operator on $\mathscr{L}(V, V_1)$. By Sylvester's Theorem [11], the set of eigenvalues of R_1 is the set $\{\alpha_1 + \beta : \alpha_1 \in \sigma(A_{11}), \beta \in \sigma(B)\}$. The matrix R_3 is the matrix of the compression of τ to $\mathfrak{S}_2$, i.e., $Q\tau|_{\mathfrak{S}_2}$ where Q is the projection on $\mathfrak{S}_2$ along $\mathfrak{S}_1$. This is exactly the spectrum of the Sylvester operator $\tau_{\hat{A},\hat{B}}$ as an operator on the block triangular algebra $\hat{\mathfrak{A}}$. By the induction hypothesis

$$\sigma(R_3) = \{\alpha + \beta : \alpha \in \sigma(A_{ii}), \beta \in \sigma(B_{jj}),\ 2 \leqslant i \leqslant j\}.$$

As $\sigma(\tau) = \sigma(R_1) \cup \sigma(R_3)$, the result follows.

REMARK 1. The requirement that $\mathbb{F}$ be algebraically closed may be relaxed to the requirement that $\mathbb{F}$ include all of the eigenvalues of A and B. This explains the fact that we may take $\mathbb{F}$ to be an arbitrary field in the following Corollary.

COROLLARY 1. *Let $A, B \in \mathscr{T}_n(\mathbb{F})$ over an arbitrary field $\mathbb{F}$ and consider $\tau = \tau_{A,B}$ as an operator on $\mathscr{T}_n(\mathbb{F})$. Then*

$$\sigma(\tau) = \{a_{ii} + b_{jj} : 1 \leqslant i \leqslant j \leqslant n\}.$$

As we did earlier in the case where A and B are diagonal matrices, we may identify eigenvectors for each eigenvalue of $\tau_{A,B}$ acting on block upper-triangular algebras.

EXAMPLE 2. With the notation of Theorem 2, let $\alpha \in \sigma(A_{ii}), \beta \in \sigma(B_{jj}), i \leqslant j$. As above we let $V_j = \operatorname{span}\{e_t : 1 \leqslant t \leqslant n_1 + \cdots n_j\}$ and we define $V_j^0 = \operatorname{span}\{e_t : n_1 + \cdots n_j < t \leqslant n\}$, the natural complement to V_j. Then α is an eigenvalue of the matrix

$$A_{[i]} := \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1i} \\ 0 & A_{22} & \dots & A_{2i} \\ \vdots & & & \\ 0 & 0 & \dots & A_{ii} \end{bmatrix}$$

and β is an eigenvalue of the matrix

$$B_{[j]} := \begin{bmatrix} B_{jj} & \dots & B_{jn} \\ \vdots & & \\ 0 & \dots & B_{nn} \end{bmatrix}.$$

Thus there exists a nonzero vector $u \in V_i$ and a nonzero vector $v \in V_{j-1}^0$ such that $A_{[j]}u = \alpha u$ and $B_{[j]}^t v = \beta v$. It follows that $Au = \alpha u$ and $B^t v = \beta v$. Let $T = uv^t$. Then clearly $T \in \mathfrak{A}$ and

$$\tau_{A,B}(T) = Auv^t + uv^t B = (Au)v^t + u(B^t v)^t = \alpha uv^t + \beta uv^t = (\alpha + \beta)T.$$

Therefore T is an eigenvector of τ.

Let $\mathscr{M} \subseteq \mathbb{M}_{m,n}(\mathbb{F})$ be a $(\mathscr{T}_m(\mathbb{F}), \mathscr{T}_n(\mathbb{F}))$-bimodule, and $A \in \mathscr{T}_m(\mathbb{F})$, $B \in \mathscr{T}_n(\mathbb{F})$. We will next identify the spectrum of $\tau_{A,B} \in \mathscr{L}(\mathscr{M})$. We start by characterizing $(\mathscr{T}_m(\mathbb{F}), \mathscr{T}_n(\mathbb{F}))$-bimodules of $\mathbb{M}_{mn}(\mathbb{F})$. The following characterization is similar to the characterization of weakly closed ideals in nest algebras of operators on Hilbert space [6].

LEMMA 1. *A subspace $\mathscr{M}$ of $\mathbb{M}_{mn}(\mathbb{F})$ is a $(\mathscr{T}_m(\mathbb{F}), \mathscr{T}_n(\mathbb{F}))$-bimodule if and only if there exists a monotone increasing function (not necessarily strictly increasing) $f : \{1, \cdots n\} \longrightarrow \{0, 1, \cdots n\}$ such that $\mathscr{M} = \operatorname{span}\{E_{ij} : 1 \leqslant i \leqslant f(j)\}$.*

Proof. If f is a function that satisfies the assertion, $A \in \mathscr{T}_m(\mathbb{F}), B \in \mathscr{T}_n(\mathbb{F})$ and $i \leqslant f(j)$, then clearly $AE_{ij} \in \operatorname{span}\{E_{i'j} : 1 \leqslant i' \leqslant j\}$ and $E_{ij}B \in \operatorname{span}\{E_{ij'} : j \leqslant j' \leqslant n\}$. Therefore $AE_{ij}, E_{ij}B \in \mathscr{M}$, the former since $i' \leqslant i \leqslant j$ and the latter due to the monotonicity of f.

For the converse assume that $\mathscr{M}$ is a bimodule and define $f(s)$ to be the maximum index r such that $\mathscr{M}$ includes a matrix $C = [c_{ij}]$ with $c_{rs} \neq 0$. We observe that if $C \in \mathscr{M}$ with $c_{rs} \neq 0$ and $r' \leqslant r$, $s' \geqslant s$ then each of the following matrices belong to the bimodule:

$$E_{rs} = E_{rr}CE_{ss}, \quad E_{r's'} = E_{r'r}CE_{ss'}.$$

The fact that $E_{rs'} \in \mathscr{M}$ proves the monotonicity of f and the fact that $E_{r's} \in \mathscr{M}$ proves that $\mathscr{M} = \operatorname{span}\{E_{ij} : i \leqslant f(j)\}$.

THEOREM 3. *Let $\mathbb{F}$ be an algebraically closed field, $A \in \mathscr{T}_m(\mathbb{F}), B \in \mathscr{T}_n(\mathbb{F})$ and $\mathscr{M} \subseteq \mathbb{M}_{mn}(\mathbb{F})$ be a nonzero $(\mathscr{T}_m(\mathbb{F}), \mathscr{T}_n(\mathbb{F}))$-bimodule. If $\tau = \tau_{A,B}$ is considered as an operator on $\mathscr{M}$, then $\sigma(\tau) = \{a_{ii} + b_{jj} : 1 \leqslant i \leqslant f(j)\}$, where f is the function affiliated with $\mathscr{M}$ described in Lemma 1.*

Proof. We prove this by induction on m. If $m = 1$, then $\mathscr{M} = \{[0, \cdots 0, *, \cdots *]\}$ where the number of initial zeros is $t = \max\{s : f(s) = 0\}$ (which may be zero). The map τ is then recognized as the Sylvester-Rosenblum operator $\tau_{a,B'}$ where $a = A$ and B' is the compression of B to the last $n - t$ rows and columns. By Sylvester's Theorem $\sigma(\tau) = \{a\} + \sigma(B') = \{a + b_{jj} : t + 1 \leqslant j \leqslant n\} = \{a + b_{jj} : 1 \leqslant f(j)\}$.

For $m > 1$ we decompose $\mathfrak{A}$ as a direct sum of the two subspaces $\mathscr{S}_1$ consisting of the first $m - 1$ rows of $\mathscr{M}$ and $\mathscr{S}_2$ consisting of the last row of $\mathscr{M}$. The subspace $\mathscr{S}_1$ is invariant under τ and so as in the proof of Theorem 2, the spectrum of τ is the union of the spectrum of $\tau|_{\mathscr{S}_1}$ and the spectrum of the compression of τ to $\mathscr{S}_2$. By the induction hypothesis $\sigma(\tau|_{\mathscr{S}_1}) = \{a_{ii} + b_{jj} : 1 \leqslant i \leqslant m - 1, i \leqslant f(j)\}$. As in the proof of the case $m = 1$, the spectrum of the compression of τ to $\mathscr{S}_2$ equals $\{a_{mm} + b_{jj} : f(j) = m\}$.

3. Triangular Banach algebras

Let $\mathscr{A}$ and $\mathscr{B}$ be unital Banach algebras, and suppose that $\mathscr{M}$ is a Banach $\mathscr{A}$-$\mathscr{B}$-bimodule. That is, $\mathscr{M}$ is a Banach space, a left-$\mathscr{A}$ module and a right $\mathscr{B}$-module, and the (continuous) module actions satisfy:

$$\|AMB\|_{\mathscr{M}} \leqslant \|A\|_{\mathscr{A}} \, \|M\|_{\mathscr{M}} \|B\|_{\mathscr{B}},$$

for all $A \in \mathscr{A}$, $B \in \mathscr{B}$, and $M \in \mathscr{M}$.

Let us now fix $A \in \mathscr{A}$ and $B \in \mathscr{B}$, and denote by L_A (resp. R_B) the left multiplication operator $L_A(M) = AM$ (resp. the *right* multiplication operator $R_B(M) = MB$) for all $M \in \mathscr{M}$. It is easily seen that $L_A, R_B \in \mathscr{B}(\mathscr{M})$ and that $\|L_A\| \leqslant \|A\|_{\mathscr{A}}$ and that $\|R_B\| \leqslant \|B\|_{\mathscr{B}}$. The corresponding **Sylvester-Rosenblum operator**

$$\begin{aligned} \tau_{A,B} : \mathscr{M} &\to \quad \mathscr{M} \\ T \; &\mapsto AT + TB \end{aligned}$$

is bounded (indeed $\tau_{A,B} = L_A + R_B$), and as noted in the introduction – observing that $L_A R_B = R_B L_A$ – it follows from the general theory of abelian Banach algebras that

$$\sigma_{\mathscr{B}(\mathscr{M})}(\tau_{A,B}) \subseteq \sigma_{\mathscr{B}(\mathscr{M})}(L_A) + \sigma_{\mathscr{B}(\mathscr{M})}(R_B) \subseteq \sigma_{\mathscr{A}}(A) + \sigma_{\mathscr{B}}(B).$$

Our main goal is to consider a particular example of this phenomenon where $\mathscr{N} \subseteq \mathscr{H}$ is a nest, $\mathscr{M} = \mathscr{T}(\mathscr{N})$ is the corresponding **nest algebra**, and $\mathscr{A} = \mathscr{B} = \mathbb{C}I + \mathscr{K}(\mathscr{N})$ is the unitization of the algebra of compact operators in $\mathscr{T}(\mathscr{N})$.

It will be useful, however, to frame our first results in terms of so-called **triangular Banach algebras**, which we now define.

Let $p \geqslant 1$ be an integer. Suppose that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_p$ are Banach algebras and that $\mathscr{M}_{ij}$, $1 \leqslant i < j \leqslant p$, are $\mathscr{A}_i - \mathscr{A}_j$ bimodules with the additional multiplication that satisifies $\mathscr{M}_{ij}\mathscr{M}_{jk} \subseteq \mathscr{M}_{ik}$, $1 \leqslant i < j < k \leqslant p$. Then we construct **triangular Banach algebras** of the form

$$\mathscr{T}_p = \begin{bmatrix} \mathscr{A}_1 & \mathscr{M}_{12} & \mathscr{M}_{13} & \ldots & \mathscr{M}_{1p} \\ & \mathscr{A}_2 & \mathscr{M}_{23} & \ldots & \mathscr{M}_{2p} \\ & & \ddots & & \\ & & & & \mathscr{A}_p \end{bmatrix},$$

where for

$$T = \begin{bmatrix} a_1 & m_{12} & m_{13} & \ldots & m_{1p} \\ & a_2 & m_{23} & \ldots & m_{2p} \\ & & \ddots & & \\ & & & & a_p \end{bmatrix} \in \mathscr{T}_p,$$

we define $\|T\| = \sum_{k=1}^{p} \|a_k\|_{\mathscr{A}_k} + \sum_{1 \leqslant i < j \leqslant p} \|m_{ij}\|_{\mathscr{M}_{ij}}$. For our purposes, the norm should be interpreted as a conveyor of topological information, rather than isometric information. That is to say, any topology on $\mathscr{T}_p$ equivalent to the norm topology we have just defined would work just as well. Linear combinations and products of elements of $\mathscr{T}_p$ are provided by the natural matrix linear combinations and products - where the entrywise-products are defined through module actions where appropriate. For example, if $p = 2$, then the multiplication in $\mathscr{T}$ is the natural one defined by

$$\begin{bmatrix} a_1 & m \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} b_1 & n \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 n + m b_2 \\ 0 & b_1 b_2 \end{bmatrix}.$$

Since $\mathscr{T}_p$ is a Banach algebra, it is clearly a Banach bimodule over itself. Given $X = [x_{ij}]$ and $Y = [y_{ij}] \in \mathscr{T}_p$ (so that it is understood that $x_{ij} = 0 = y_{ij}$ if $i > j$), we can once again consider the corresponding Sylvester-Rosenblum operator $\tau_{X,Y} = L_X + R_Y$, and observe that $\|\tau_{X,Y}\| \leqslant \|X\| + \|Y\| < \infty$. (We emphasize that for our purposes, all that matters is that $\tau_{X,Y}$ is continuous.)

We saw above that $\sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{X,Y}) \subseteq \sigma_{\mathscr{B}(\mathscr{T}_p)}(X) + \sigma_{\mathscr{B}(\mathscr{T}_p)}(Y)$. By setting $\mathbb{F} = \mathbb{C}$ in Example 1, we see that this containment is, in general, strict. Our goal is to precisely determine $\sigma(\tau_{X,Y})$ in the setting of nest algebras acting on a Hilbert space.

In analogy to both the matrix and nest algebra settings, we shall define the **diagonal** $\mathscr{D}_p$ of $\mathscr{T}_p$ to be the algebra $\mathscr{D}_p = \oplus_{k=1}^p \mathscr{A}_k$. The map $\Delta : \mathscr{T}_p \to \mathscr{D}_p \subseteq \mathscr{T}_p$, $\Delta(Z = [z_{ij}]) = \oplus_{k=1}^p z_{kk}$ is a contractive (hence continuous), linear, idempotent homomorphism.

We refer to Δ as an "expectation" of $\mathscr{T}_p$ onto the diagonal $\mathscr{D}_p$.

Observe that for all $T = [t_{ij}], X = [x_{ij}]$, and $Y = [y_{ij}] \in \mathscr{T}_p$, we have that $\tau_{x_{ii},y_{jj}} : \mathscr{M}_{ij} \to \mathscr{M}_{ij}$ is a Sylvester-Rosenblum operator, and that

$$\tau_{\Delta(X),\Delta(Y)}(T) = [\tau_{x_{ii},y_{jj}}(t_{ij})].$$

It is easy to see from this that $\tau_{\Delta(X),\Delta(Y)}$ is invertible (or equivalently bijective) if and only if each $\tau_{x_{ii},y_{jj}}$ is invertible , $1 \leqslant i \leqslant j \leqslant p$.

Our first goal is to show that $\sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{X,Y}) \subseteq \sigma_{\mathscr{B}(\mathscr{T}_p))}(\tau_{\Delta(X),\Delta(Y)})$, which will simplify many of our computations later on.

We do this by first introducing a family of natural ideals of $\mathscr{T}_p$, consisting of those upper triangular matrices whose first k diagonals (starting at the main diagonal and proceeding "upwards") are all zero.

For $0 \leqslant k \leqslant p$, set

$$\mathscr{J}_k := \{T \in \mathscr{T}_p : t_{ij} = 0 \text{ if } j - i < k\}.$$

Then $\mathscr{J}_0 = \mathscr{T}_p$, $\mathscr{J}_p = \{0\}$, and $\mathscr{J}_k \lhd \mathscr{T}_p$, $0 \leqslant k \leqslant p$. By virtue of the fact that each such $\mathscr{J}_k$ is an ideal, it is clear that $\tau_{X,Y}(\mathscr{J}_k) \subseteq \mathscr{J}_k$ and $\tau_{\Delta(X),\Delta(Y)}(\mathscr{J}_k) \subseteq \mathscr{J}_k$ for all k.

Note also that if $T = [t_{ij}] \in \mathscr{J}_k$, then a routine calculation shows that

$$[\tau_{X,Y}(T)]_{i\,i+k} = \tau_{x_{ii},y_{i+k\,i+k}}(t_{i\,i+k}) = \tau_{\Delta(X),\Delta(Y)}(t_{i\,i+k}).$$

From this it in turn follows that $\tau_{X,Y}(T) = \tau_{\Delta(X),\Delta(Y)}(T)$ for all $T \in \mathscr{J}_{p-1}$.

PROPOSITION 1. *Let $\mathscr{T}_p$ be a triangular Banach algebra and $X, Y \in \mathscr{T}_p$. If $\tau_{\Delta(X),\Delta(Y)}$ is invertible, then so is $\tau_{X,Y}$.*

Proof. Suppose that $\tau_{\Delta(X),\Delta(Y)}$ is invertible. Let $T = [t_{ij}] \in \ker \tau_{X,Y}$. For $0 \leqslant i \leqslant p$,

$$0 = [\tau_{X,Y}(T)]_{ii} = \tau_{x_{ii},y_{ii}}(t_{ii}).$$

But as observed above, the fact that $\tau_{\Delta(X),\Delta(Y)}$ is injective implies that each $\tau_{x_{ii},y_{ii}}$ is injective, and thus $t_{ii} = 0$, $1 \leqslant i \leqslant p$. In other words, $T \in \mathscr{J}_1$.

Let $k = \max\{1 \leqslant \ell \leqslant p : T \in \mathscr{J}_\ell\}$. If $k < p$, then $t_{ij} = 0$ for all $i \leqslant j < i+k$, but there exists $1 \leqslant i_0 \leqslant p-k$ such that $t_{i_0\, i_0+k} \neq 0$. From our observation above,

$$0 = [\tau_{X,Y}(T)]_{i_0\, i_0+k} = \tau_{x_{i_0\, i_0}, y_{i_0+k\, i_0+k}}(t_{i_0\, i_0+k}).$$

Since each $\tau_{x_{ii}, y_{jj}}$ is injective, $t_{i_0\, i_0+k} = 0$, a contradiction. Thus $k = p$; i.e. $T = 0$.

Thus shows that $\tau_{X,Y}$ is injective. (Note that this argument only required the injectivity of $\tau_{\Delta(X),\Delta(Y)}$.)

Next, let us show that $\tau_{X,Y}$ is surjective. We will show that $\mathscr{J}_k \subseteq \operatorname{ran} \tau_{X,Y}$, $0 \leqslant k \leqslant p-1$. Since $\mathscr{J}_0 = \mathscr{T}_p$, this clearly suffices.

Note that if $0 \neq W = [w_{ij}] \in \mathscr{J}_{p-1}$, then by hypothesis there exists $T = [t_{ij}] \in \mathscr{T}_p$ such that $\tau_{\Delta(X),\Delta(Y)}(T) = W$. Thus

$$\tau_{x_{ii}, y_{jj}}(t_{ij}) = \begin{cases} 0 & \text{if } (i,j) \neq (1,p) \\ w_{1,p} & \text{if } (i,j) = (1,p). \end{cases}$$

Since $\tau_{\Delta(X),\Delta(Y)}$ is injective, each $\tau_{x_{ii}, y_{jj}}$ is injective and thus $t_{ij} = 0$ for all $(i,j) \neq (1,p)$. In particular, $T \in \mathscr{J}_{p-1}$. But then

$$\tau_{X,Y}(T) = \tau_{\Delta(X),\Delta(Y)}(T) = W,$$

and thus $\mathscr{J}_{p-1} \subseteq \operatorname{ran} \tau_{X,Y}$. In fact, we have just argued that $\mathscr{J}_{p-1} = \tau_{X,Y}(\mathscr{J}_{p-1})$.

Let $k = \min\{0 \leqslant \ell \leqslant p : \mathscr{J}_\ell = \tau_{X,Y}(\mathscr{J}_\ell)\}$. Suppose that $k > 0$, and let $Z = [z_{ij}] \in \mathscr{J}_{k-1}$. Since $\tau_{\Delta(X),\Delta(Y)}$ is surjective, there exists $S = [s_{ij}] \in \mathscr{T}_p$ such that $\tau_{\Delta(X),\Delta(Y)}(S) = Z$. By injectivity of each $\tau_{x_{ii}, y_{jj}}$, $1 \leqslant i \leqslant j \leqslant p$, we see that $S \in \mathscr{J}_{k-1}$. But then

$$[\tau_{\Delta(X),\Delta(Y)}(S)]_{i\, i+(k-1)} = [\tau_{X,Y}(S)]_{i\, i+(k-1)}, \quad 1 \leqslant i \leqslant p-(k-1),$$

and thus $\tau_{X,Y}(S) \in \mathscr{J}_{k-1}$. Furthermore, $Z - \tau_{X,Y}(S) \in \mathscr{J}_k$. Choose $S_0 \in \mathscr{J}_k$ such that $\tau_{X,Y}(S_0) = Z - \tau_{X,Y}(S) \in \mathscr{J}_k$.

Then $V := S - S_0 \in \mathscr{J}_{k-1}$ and $\tau_{X,Y}(V) = Z$, so that $Z \in \operatorname{ran} \tau_{X,Y}$. Hence $\mathscr{J}_{k-1} = \tau_{X,Y}(\mathscr{J}_{k-1})$, contradicting our choice of k. Thus $k = 0$, and so $\tau_{X,Y}$ is surjective.

This completes the proof.

COROLLARY 2. *Let $\mathscr{T}_p$ be a triangular Banach algebra and $X, Y \in \mathscr{T}_p$. Then*

$$\sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{X,Y}) \subseteq \sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{\Delta(X),\Delta(Y)}).$$

Proof. Let $\lambda \in \mathbb{C}$ and denote by $I = \oplus_{i=1}^p I_i$ the identity of $\mathscr{T}_p$, where I_i in turn denotes the identity of $\mathscr{A}_i$, $1 \leqslant i \leqslant p$.

It is clear that $\Delta(I) = I$, and that $\lambda I - \tau_{X,Y} = \tau_{\lambda I - X, Y}$ and $\lambda I - \tau_{\Delta(X),\Delta(Y)} = \tau_{\lambda I - \Delta(X), \Delta(Y)}$.

Thus $\lambda \in \sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{X,Y})$ (resp. $\lambda \in \sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{\Delta(X),\Delta(Y)})$) if and only if $0 \in \sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{\lambda I - X, Y})$ (resp. $0 \in \sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{\Delta(\lambda I - X),\Delta(Y)})$).

The result now follows immediately from Proposition 1.

PROPOSITION 2. *Let $\mathscr{T}_p$ be a triangular Banach algebra and $X,Y \in \mathscr{D}_p$. Then*

$$\sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{X,Y}) \subseteq \Lambda := \{\alpha_i + \beta_j : \alpha_i \in \sigma_{\mathscr{A}_i}(x_{ii}), \beta_j \in \sigma_{\mathscr{B}_j}(y_{jj}), 1 \leqslant i \leqslant j \leqslant n\}.$$

Proof. Suppose that $\lambda \notin \Lambda$. For $T = [t_{ij}] \in \mathscr{T}_p$,

$$(\lambda I - \tau_{X,Y})(T) = [(\lambda - \tau_{x_{ii},y_{jj}})(t_{ij})].$$

But as observed above, for each $1 \leqslant i \leqslant j \leqslant p$, $\tau_{x_{ii},y_{jj}} = L_{x_{ii}} + R_{y_{jj}}$ and by the theory of commutative Banach algebras,

$$\sigma_{\mathscr{B}(\mathscr{M}_{ij})}(\tau_{x_{ii},y_{jj}}) \subseteq \sigma_{\mathscr{A}_i}(x_{ii}) + \sigma_{\mathscr{B}_j}(y_{jj}).$$

Thus each $\lambda - \tau_{x_{ii},y_{jj}}$ is invertible, $1 \leqslant i \leqslant j \leqslant p$, whence $\lambda \notin \sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{X,Y})$.

By combining Corollary 2 and Proposition 2 we arrive at the following result.

COROLLARY 3. *Let $\mathscr{T}_p$ be a triangular Banach algebra and $X,Y \in \mathscr{T}_p$. Then*

$$\sigma_{\mathscr{B}(\mathscr{T}_p)}(\tau_{X,Y}) \subseteq \Lambda := \{\alpha + \beta : \alpha \in \sigma_{\mathscr{A}_i}(x_{ii}), \beta \in \sigma_{\mathscr{B}_j}(y_{jj}), 1 \leqslant i \leqslant j \leqslant n\}.$$

In the case where $\mathbb{F} = \mathbb{C}$, the block upper-triangular algebras $\mathscr{T}(n_1, n_2, \ldots, n_k)$ of $\mathbb{M}_n(\mathbb{C})$ (where $n = n_1 + n_2 + \cdots + n_k$) defined in Section 2 are the prototype of a **nest algebra** acting on a finite-dimensional Hilbert space. We saw in Theorem 2 that in this setting, the containment described in Corollary 3 is actually an equality. Indeed, Example 2 allows us to produce an eigenvector for each $\alpha + \beta$ in the set Λ above.

We now turn our attention to the case of nest algebras acting on infinite-dimensional, complex, separable Hilbert spaces.

4. Nest algebras

Let $\mathscr{H}$ denote a complex, infinite-dimensional, separable Hilbert space. Given a closed subspace $M \subseteq \mathscr{H}$, we denote by $P(M)$ the orthogonal projection of $\mathscr{H}$ onto M. A **nest** on $\mathscr{H}$ is a chain $\mathscr{N}$ of closed subspaces of $\mathscr{H}$, which is closed under the operations of taking closed linear spans, arbitrary intersections, and which contains $\{0\}$ and $\mathscr{H}$. Given $N_1 < N_2 \in \mathscr{N}$, the subspace $N_2 \ominus N_1$ is referred to as an **interval** in $\mathscr{N}$. A **partition** of $\mathscr{N}$ is a finite set $\mathscr{E} := \{E_1, E_2, \ldots, E_r\}$ of pairwise orthogonal intervals for which $\mathscr{H} = \oplus_{m=1}^{r} E_m$. Thus $\mathscr{E}$ is a partition precisely when there exist $0 = N_0 < N_1 < N_2 \cdots < N_r = \mathscr{H}$ such that (after reindexing the E_j's if necessary) $E_m = N_m \ominus N_{m-1}$, $1 \leqslant m \leqslant r$.

For each $N \in \mathscr{N}$, we may define $N_- := \vee\{M \in \mathscr{N} : M < N\}$ (here $\vee$ denotes the "closed linear span"). If $N_- \neq N$, we refer to N_- as the **immediate predecessor** of N, and we refer to $N \ominus N_-$ as an **atom** of $\mathscr{N}$. (These are the minimal non-zero intervals of $\mathscr{N}$.) As $\mathscr{H}$ is separable, it is clear that $\mathscr{N}$ admits at most countably many atoms. We write $\mathbb{A}_{\mathscr{N}} = \{A_\alpha : \alpha \in \Lambda\}$ to denote the set of atoms of $\mathscr{N}$. Given atoms $A_\alpha = N_\alpha \ominus (N_\alpha)_-$ and $A_\beta = N_\beta \ominus (N_\beta)_-$ in $\mathbb{A}_{\mathscr{N}}$, we set

$$A_\alpha \leqslant A_\beta$$

if $N_\alpha \leqslant N_\beta$. It follows that $(\mathbb{A}_{\mathscr{N}}, \leqslant)$ is a totally ordered set. More generally, given two intervals $E_1 = N_1 \ominus M_1$ and $E_2 = N_2 \ominus M_2$ for some $M_k \leqslant N_k \in \mathscr{N}$, $k = 1,2$, we shall define

$$E_1 \leqslant E_2$$

if $N_1 \leqslant M_2$.

Corresponding to $\mathscr{N}$ is a WOT-closed algebra

$$\mathscr{T}(\mathscr{N}) = \{T \in \mathscr{B}(\mathscr{H}) : TN \subseteq N \text{ for all } N \in \mathscr{N}\}.$$

We refer to $\mathscr{T}(\mathscr{N})$ as a **nest algebra**, and denote by $\mathscr{K}(\mathscr{N}) = \mathscr{T}(\mathscr{N}) \cap \mathscr{K}(\mathscr{H})$ the closed, two-sided ideal of compact operators in $\mathscr{T}(\mathscr{N})$.

4.1. Fine picture of the spectrum

Let $K, L \in \mathscr{K}(\mathscr{N})$. Our goal is to calculate the spectrum of the Sylvester-Rosenblum operator

$$\begin{aligned} \tau_{K,L} : \mathscr{T}(\mathscr{N}) &\to \mathscr{T}(\mathscr{N}) \\ T &\mapsto KT + TL. \end{aligned}$$

To that end, we shall define three sets associated with $\tau_{K,L}$.

- $\Omega_{\text{atom}} := \{\kappa + \lambda : \kappa \in \sigma_{\mathscr{B}(A_\alpha)}(P(A_\alpha)K|_{A_\alpha}), \lambda \in \sigma_{\mathscr{B}(A_\beta)}(P(A_\beta)L|_{A_\beta}), A_\alpha \leqslant A_\beta \in \mathbb{A}_{\mathscr{N}}\}$;
- $\Omega_{\text{left}} := \{\kappa : \kappa \in \sigma_{\mathscr{B}(A_\alpha)}(P(A_\alpha)K|_{A_\alpha}), A_\alpha = N_\alpha \ominus (N_\alpha)_- \in \mathbb{A}_{\mathscr{N}}, \dim (N_\alpha)^\perp = \infty\}$; and
- $\Omega_{\text{right}} := \{\lambda : \lambda \in \sigma_{\mathscr{B}(A_\beta)}(P(A_\beta)L|_{A_\beta}), A_\beta = N_\beta \ominus (N_\beta)_- \in \mathbb{A}_{\mathscr{N}}, \dim (N_\beta)_- = \infty\}$.

Obviously each of these sets depends upon K and L. We set $\Omega (= \Omega_{K,L}) := \Omega_{\text{atom}} \cup \Omega_{\text{left}} \cup \Omega_{\text{right}} \cup \{0\}$. We shall demonstrate that

$$\sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L}) = \overline{\Omega}.$$

EXAMPLE 3. Let us illustrate what Ω looks like in an example.

Consider the complex Hilbert space $\mathscr{H} = L^2([0,1], dx) \oplus \mathbb{C}^2 \oplus L^2([0,1], dx) \oplus \mathbb{C}$, where dx represents Lebesgue measure on the interval $[0,1]$. For each $0 \leqslant t \leqslant 1$, let $N_t := \{f \in \mathscr{H} : f = \chi_{[0,t]} f\}$, where $\chi_{[0,t]}$ is the characteristic function of the interval $[0,t]$, and let $\mathscr{N}$ be the nest

$$\mathscr{N} = \{N_t \oplus 0 \oplus 0 \oplus 0, N_1 \oplus \mathbb{C}^2 \oplus 0 \oplus 0, N_1 \oplus \mathbb{C}^2 \oplus N_t \oplus 0, \mathscr{H} : t \in [0,1]\}.$$

Thus the atoms of $\mathscr{N}$ are $A_1 = 0 \oplus \mathbb{C}^2 \oplus 0 \oplus 0$ and $A_2 = 0 \oplus 0 \oplus 0 \oplus \mathbb{C}$, and $A_1 \leqslant A_2$.

Let $K, L \in \mathscr{K}(\mathscr{N})$, and suppose that $\sigma_{\mathscr{B}(\mathbb{C}^2)}(K_1) = \{1,2\}$, $\sigma_{\mathscr{B}(\mathbb{C})}(K_2) = \{4\}$, $\sigma_{\mathscr{B}(\mathbb{C}^2)}(L_1) = \{8,16\}$, $\sigma_{\mathscr{B}(\mathbb{C})}(L_2) = \{32\}$.

Then:

- $\Omega_{\text{atom}} := \{1+8, 1+16, 2+8, 2+16, 1+32, 2+32, 4+32\} = \{9, 10, 17, 18, 33, 34, 36\}$;
- $\Omega_{\text{left}} := \{1, 2\}$; and
- $\Omega_{\text{right}} := \{8, 16, 32\}$.

Thus $\Omega = \{0, 1, 2, 8, 9, 10, 16, 17, 18, 32, 33, 34, 36\}$.

4.2. Finite partitions

It is now time to relate the theory of Sylvester-Rosenblum operators on nest algebras to our work in the previous section. Suppose that $\mathscr{E} := \{E_1, E_2, \dots, E_p\}$ is a partition of a given nest $\mathscr{N}$. Relative to the decomposition $\mathscr{H} = E_1 \oplus E_2 \oplus \cdots \oplus E_p$, any $T \in \mathscr{T}(\mathscr{N})$ admits a block upper-triangular form

$$T = \begin{bmatrix} T_{1,1} & T_{1,2} & \cdots & \cdots & T_{1,p} \\ & T_{2,2} & T_{2,3} & \cdots & T_{2,p} \\ & & \ddots & \ddots & \vdots \\ & & & T_{p-1,p-1} & T_{p-1,p} \\ & & & & T_{p,p} \end{bmatrix}.$$

Moreover, for each $1 \leqslant i \leqslant p$, the set $\mathscr{A}_i := \{T_{i,i} : T = [T_{i,j}] \in \mathscr{T}(\mathscr{N})\}$ forms a nest algebra on the space E_i, while each of the spaces $\mathscr{M}_{i,j} = \mathscr{B}(E_j, E_i)$, $1 \leqslant i \leqslant j \leqslant p$, forms a $\mathscr{A}_i$-$\mathscr{A}_j$ bimodule. Using the construction of Section 3, we obtain a triangular Banach algebra $\mathscr{T}_{\mathscr{E}}$ which coincides (as a set of operators on $\mathscr{H}$) with $\mathscr{T}(\mathscr{N})$. It is not too difficult to verify that the norm we associated to $\mathscr{T}_{\mathscr{E}}$ in the previous section is equivalent to the original operator norm on $\mathscr{T}(\mathscr{N})$.

The **diagonal** of the nest algebra $\mathscr{T}(\mathscr{N})$ is the set $\mathscr{D}(\mathscr{N}) := \mathscr{T}(\mathscr{N}) \cap \mathscr{T}(\mathscr{N})^*$. The set $\mathscr{D}_{\text{atom}} := \oplus_{\alpha\in\Lambda} \mathscr{B}(A_\alpha) \subseteq \mathscr{D}(\mathscr{N})$ is called the **atomic part** of the diagonal. It is known that the map $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{D}_{\text{atom}}$ defined by $\Phi(T) = \sum_{\alpha\in\Lambda} P(A_\alpha) T P(A_\alpha)$ is a contractive projection of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{D}_{\text{atom}}$, and that $\Phi|_{\mathscr{T}(\mathscr{N})}$ is multiplicative.

We shall also require the following two results. The second appears in [5], Lemma 3.5.

THEOREM 4. **(Ringrose [9])** *Let $\mathscr{H}$ be an infinite-dimensional, separable, complex Hilbert space and $K \in \mathscr{K}(\mathscr{H})$ be a compact operator. Let $\mathscr{N}$ denote a maximal nest of invariant subspaces of $\mathscr{K}$, and $\mathbb{A}_{\mathscr{N}} = \{A_\alpha : \alpha \in \Lambda\}$ denote the atoms of $\mathscr{N}$. Then each atom is one-dimensional, and*

$$\sigma_{\mathscr{B}(\mathscr{H})}(K) = \sigma_{\mathscr{T}(\mathscr{N})}(K) = \{0\} \cup \{K_\alpha : A_\alpha \in \mathbb{A}_{\mathscr{N}}\},$$

where $K_\alpha = P(A_\alpha) K P(A_\alpha) \in \mathbb{C}$. Moreover, the non-zero eigenvalues are repeated according to their algebraic multiplicity.

PROPOSITION 3. *Let $\mathscr{N}$ be a nest and $K \in \mathscr{K}(\mathscr{N})$. Given $\varepsilon > 0$, there exists a partition $\mathscr{E} = \{E_1, E_2, \dots, E_r\}$ of $\mathscr{N}$ such that for each $1 \leqslant m \leqslant r$, either E_m is an atom, or $\|P(E_m) K P(E_m)\| < \varepsilon$.*

For a subset $Z \subseteq \mathbb{C}$ and $\varepsilon > 0$, we write $Z_\varepsilon := \{w \in \mathbb{C} : |w - z| < \varepsilon \text{ for some } z \in Z\}$.

THEOREM 5. *Let $\mathscr{N}$ be a nest and $K, L \in \mathscr{K}(\mathscr{N})$. Let $\Omega = \Omega_{K,L}$ be the set defined in section 4.1. Then $\sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L}) \subseteq \overline{\Omega}$.*

Proof. Let $\varepsilon > 0$. By applying Proposition 3 to each of K and L, and then choosing a "common refinement" of each partition thereby obtained, we can find a partition $\mathscr{E} = \{E_m\}_{m=1}^r$ of $\mathscr{N}$ such that for each $1 \leqslant m \leqslant r$, either E_m is an atom, or $\|P(E_m)KP(E_m)\| < \varepsilon$ and $\|P(E_m)LP(E_m)\| < \varepsilon$. By refining our partition once more if necessary, we may also assume without loss of generality that any finite-dimensional subspace E_m is an atom. As seen in section 4.2, we may then decompose $\mathscr{H} = \oplus_{m=1}^r E_m$ and write $K = [K_{ij}]$, $L = [L_{ij}]$ as block-upper triangular operator matrices relative to this decomposition.

By Corollary 3,

$$\sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L}) \subseteq \Phi := \{\alpha_i + \beta_j : \alpha_i \in \sigma_{\mathscr{A}_i}(K_{ii}), \beta_j \in \sigma_{\mathscr{A}_j}(L_{jj}), 1 \leqslant i \leqslant j \leqslant r\}.$$

We now show that $\Phi \subseteq \Omega_{2\varepsilon}$. To do this, we proceed by a case-by-case analysis. Fix $1 \leqslant i \leqslant j \leqslant r$, and let $\alpha_i \in \sigma_{\mathscr{A}_i}(K_{ii}), \beta_j \in \sigma_{\mathscr{A}_j}(L_{jj})$.

CASE 1. Suppose that both E_i and E_j are atoms. Then $\mathscr{A}_i = \mathscr{B}(A_{\alpha_i})$ and $\mathscr{A}_j = \mathscr{B}(A_{\alpha_j})$ for some $\alpha_i, \alpha_j \in \Lambda$. Moreover, $i \leqslant j$ implies that $A_{\alpha_i} \leqslant A_{\alpha_j}$, and so $\alpha_i + \beta_j \in \Omega_{\text{atom}} \subseteq \Omega_{2\varepsilon}$.

CASE 2. Suppose that E_i is an atom, but that E_j is not. Here, $\mathscr{A}_i = \mathscr{B}(A_{\alpha_i})$ for some $\alpha_i \in \Lambda$, and this time $\|L_j\| < \varepsilon$, by our choice of $\mathscr{E}$. As such, $\beta_j \in \sigma_{\mathscr{A}_j}(L_{jj})$ implies that $|\beta_j| < \varepsilon$.

By our choice of the partition $\mathscr{E}$, the fact E_j is not an atom implies that $\dim E_j = \infty$. But then $\alpha_i \in \Omega_{\text{left}}$, and so $\alpha_i + \beta_j \in (\Omega_{\text{left}})_\varepsilon \subseteq \Omega_{2\varepsilon}$.

CASE 3. Suppose that E_i is not an atom, but that E_j is. This case is analogous to the previous case. This time, $\mathscr{A}_j = \mathscr{B}(A_{\alpha_j})$ for some $\alpha_j \in \Lambda$, and $\|K_i\| < \varepsilon$. As such, $\alpha_i \in \sigma_{\mathscr{A}_i}(K_{ii})$ implies that $|\alpha_i| < \varepsilon$.

By our choice of the partition $\mathscr{E}$, the fact E_i is not an atom implies that $\dim E_i = \infty$. Arguing as above, $\beta_j \in \Omega_{\text{right}}$, and so $\alpha_i + \beta_j \in (\Omega_{\text{right}})_\varepsilon \subseteq \Omega_{2\varepsilon}$.

CASE 4. Suppose that neither E_i nor E_j is an atom. By our choice of $\mathscr{E}$, this implies that $\|K_{ii}\| < \varepsilon$ and $\|L_{jj}\| < \varepsilon$. Thus $\alpha_i \in \sigma_{\mathscr{A}_i}(K_{ii}), \beta_j \in \sigma_{\mathscr{A}_j}(L_{jj})$ implies that $|\alpha_i| < \varepsilon$ and $|\beta_j| < \varepsilon$. But then $|\alpha_i + \beta_j| < 2\varepsilon$, so $\alpha_i + \beta_j \in \{0\}_{2\varepsilon} \subseteq \Omega_{2\varepsilon}$.

Hence $\sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L}) \subseteq \Phi \subseteq \Omega_{2\varepsilon}$. But $\varepsilon > 0$ was arbitrary, and so

$$\sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L}) \subseteq \cap_{\varepsilon>0}\Omega_{2\varepsilon} = \overline{\Omega}.$$

There remains to show that the reverse inclusion holds, namely that $\overline{\Omega} \subseteq \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$. Note that since the latter set is closed, it suffices to prove that $\Omega \subseteq \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$.

Before proceeding to the main result of this paper, we pause to remind the reader that if $K \in \mathscr{K}(\mathscr{H})$ is a compact operator, then $\kappa \in \sigma_{\mathscr{B}(\mathscr{H})}(K)$ implies that κ is an approximate eigenvalue of K; that is, there exists a sequence $(x_n)_{n=1}^\infty$ of unit vectors in $\mathscr{H}$ such that $\lim_{n\to\infty}(K-\kappa I)x_n = 0$. Indeed, any eigenvalue of K is clearly an approximate eigenvalue of K, while the only other possibility is that $\kappa = 0$ is not an eigenvalue, in which case $\mathscr{H}$ must be infinite-dimensional. But compact operators acting on infinite-dimensional Hilbert spaces are not bounded below, which is the statement that 0 is an approximate eigenvalue of K as well.

THEOREM 6. *Let $\mathscr{N}$ be a nest and $K, L \in \mathscr{K}(\mathscr{N})$. Let $\Omega = \Omega_{K,L}$ be the set defined in section 4.1. Then*

$$\sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L}) = \overline{\Omega}.$$

Proof. As noted above, we have reduced the problem to showing that $\Omega \subseteq \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$.

CASE 1. Suppose that $\gamma \in \Omega_{\text{atom}}$, and write $\gamma = \kappa + \lambda$ where $\kappa \in \sigma_{\mathscr{B}(A_\alpha)}(K_\alpha), \lambda \in \sigma_{\mathscr{B}(A_\beta)}(L_\beta)$, and $A_\alpha \leqslant A_\beta \in \mathbb{A}_{\mathscr{N}}$.

Let $\mathscr{E} = \{E_1, E_2, \dots, E_r\}$ be a partition of $\mathscr{N}$ such that A_α, $A_\beta \in \mathscr{E}$, say $A_\alpha = E_i$ and $A_\beta = E_j$. Since $A_\alpha \leqslant A_\beta$, we have that $i \leqslant j$. Write $K = [K_{st}]$ and $L = [L_{st}]$ with respect to the decomposition $\mathscr{H} = \oplus_{m=1}^r E_m$. Let $V = \oplus_{s=1}^i E_s$ and $W = \oplus_{t=j}^r E_t$.

Since K and L are compact, $\kappa \in \sigma_{\mathscr{B}(E_i)}(K_{ii})$ implies that $\kappa \in \sigma_{\mathscr{B}(V)}(P(V)K|_V)$, and as such it is an approximate eigenvalue of $P(V)K|_V$. Similarly, $\lambda \in \sigma_{\mathscr{B}(E_j)}(L_{jj})$ implies that $\overline{\lambda} \in \sigma_{\mathscr{B}(W)}((P(W)L|_W)^*)$, and as such it is an approximate eigenvalue of $(P(W)L|_W)^*$. Thus we can find unit vectors $(v_n)_{n=1}^\infty$ in V and $(w_n)_{n=1}^\infty$ in W such that

$$\lim_n (K-\kappa I)v_n = \lim_n (P(V)K|_V - \kappa P(V))v_n = 0,$$

and similarly

$$\lim_n (L-\lambda I)^* w_n = \lim_n (P(W)L|_W)^* - \overline{\lambda} P(W))w_n = 0.$$

By Lemma 2.8 of [5], it follows that $v_n \otimes w_n^* \in \mathscr{T}(\mathscr{N})$. Observe that

$$\begin{aligned}\lim_n (\tau_{K,L} - \gamma I)(v_n \otimes w_n^*) &= \lim_n \tau_{K-\kappa I, L-\lambda I}(v_n \otimes w_n^*) \\ &= \lim_n (K-\kappa I)(v_n \otimes w_n^*) + (v_n \otimes w_n^*)(L-\lambda I) \\ &= \lim_n ((K-\kappa I)v_n) \otimes w_n^* + v_n \otimes ((L-\lambda I)^* w_n)^* \\ &= 0.\end{aligned}$$

Thus $\tau_{K,L} - \gamma I$ is not bounded below, and so $\gamma \in \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$.

CASE 2. Suppose that $\gamma \in \Omega_{\text{left}}$, and write $\gamma = \kappa$, with $\kappa \in \sigma_{\mathscr{B}(A_\alpha)}(K_\alpha)$ for some $A_\alpha = N_\alpha \ominus (N_\alpha)_-, \dim N_\alpha^\perp = \infty$. Consider $\mathscr{E} = \{E_1, E_2, E_3\}$, where $E_1 = (N_\alpha)_-$, $E_2 = A_\alpha$, and $E_3 = N_\alpha^\perp$.

Write $K = [K_{st}]_{1 \leqslant s,t \leqslant 3}$ and $L = [L_{st}]_{1 \leqslant s,t \leqslant 3}$ relative to the decomposition $\mathscr{H} = E_1 \oplus E_2 \oplus E_3$. Then $\kappa \in \sigma_{\mathscr{B}(E_2)}(K_{22})$ implies that $\kappa \in \sigma_{\mathscr{B}(E_1 \oplus E_2)}(\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix})$. As noted above, this implies that there exists a sequence of unit vectors $(v_n)_{n=1}^\infty$ in $E_1 \oplus E_2$ such that $\lim_n (K - \kappa I) v_n = 0$.

Also, $\dim E_3 = \infty$ implies that there exists a sequence $(w_n)_n$ of unit vectors in E_3 such that $\lim_n L^* w_n = 0$. Using Lemma 2.8 of [5] once again, we find that $v_n \otimes w_n^* \in \mathscr{T}(\mathscr{N})$ for all $n \geqslant 1$, and

$$\begin{aligned}
\lim_n (\tau_{K,L} - \kappa I)(v_n \otimes w_n^*) &= \lim_n \tau_{K-\kappa I, L}(v_n \otimes w_n^*) \\
&= \lim_n (K - \kappa I)(v_n \otimes w_n^*) + (v_n \otimes w_n^*)(L) \\
&= \lim_n ((K - \kappa I) v_n) \otimes w_n^* + v_n \otimes (L^* w_n)^* \\
&= 0.
\end{aligned}$$

As before, $\kappa \in \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$.

CASE 3. The proof that $\Omega_{\text{right}} \subseteq \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$ is analogous to that of CASE 2, and is left to the reader.

CASE 4. That $0 \in \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$ follows from the fact that any nest algebra on an infinite dimensional space admits a sequence $(F_n)_n$ of norm-one, rank-one operators which converge to 0 in the weak-operator topology. But then $\lim_n K F_n = 0 = \lim_n F_n L$ in the norm topology. Hence

$$\lim_n (\tau_{K,L} - \kappa I)(F_n) = \lim_n K F_n + F_n L = 0.$$

As before, $\tau_{K,L}$ is not bounded below, and so $0 \in \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$.

This proves that $\Omega \subseteq \sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{K,L})$ which we showed was sufficient to complete the proof of the Theorem.

COROLLARY 4. *Let $\mathscr{H}$ be an infinite-dimensional, separable Hilbert space and $\mathscr{N}$ be a nest on $\mathscr{H}$. Let $K, L \in \mathscr{K}(\mathscr{N})$ and let Ω denote the set defined in paragraph 4.1. Given $\alpha, \beta \in \mathbb{C}$, set $R = \alpha I + K$ and $S = \beta I + L$. Then*

$$\sigma_{\mathscr{B}(\mathscr{T}(\mathscr{N}))}(\tau_{R,S}) = \{\alpha + \beta + \omega : \omega \in \overline{\Omega}\}.$$

Proof. This follows immediately from Theorem 6, combined with the routine observation that

$$\tau_{R,S} = (\alpha + \beta) I + \tau_{K,L}.$$

EXAMPLE 4. Consider the standard ONB $\{e_n\}_{n=1}^{\infty}$ for ℓ_2, and let $\mathcal{N}$ be the nest on ℓ_2 given by

$$N_k = \operatorname{span}\{e_1, e_2, \dots, e_k\}, \quad k \geqslant 0,$$

and $N_\infty = \ell_2$. Let $K = [k_{ij}]$ and $L = [l_{ij}] \in \mathcal{K}(\mathcal{N})$. Let $\Omega_{\text{atom}}, \Omega_{\text{left}}$ and Ω_{right} be the sets defined in Section 4.1. Then

(a) For each $n \geqslant 1$, $A_n = \mathbb{C}e_n$ is an atom of $\mathcal{N}$. Thus

$$\Omega_{\text{atom}} = \{k_{ii} + l_{jj} : 1 \leqslant i \leqslant j < \infty\}.$$

(b) For each $n \geqslant 1$, $A_n = \mathbb{C}e_n$ is an atom of $\mathcal{N}$, and $N_n^{\perp}$ is infinite-dimensional. Thus

$$\Omega_{\text{left}} = \{k_{ii} : 1 \leqslant i < \infty\}.$$

Note, however, that since $L \in \mathcal{K}(\mathcal{N})$, we know that $\lim_n l_{nn} = 0$, and so $k_{ii} = \lim_j k_{ii} + l_{jj} \in \overline{\Omega_{\text{atom}}}$ for all $1 \leqslant i$. In particular,

$$\Omega_{\text{left}} \subseteq \overline{\Omega_{\text{atom}}}.$$

But $K \in \mathcal{K}(\mathcal{N})$ also implies that $\lim_i k_{ii} = 0$, whence

$$\{0\} \subseteq \overline{\Omega_{\text{atom}}}.$$

(c) Given any atom $A_n = \mathbb{C}e_n$ of $\mathcal{N}$, $\dim N_{n-1} = n-1 < \infty$, and so

$$\Omega_{\text{right}} = \varnothing.$$

It follows from Theorem 6 that

$$\sigma_{\mathcal{B}(\mathcal{T}(\mathcal{N}))}(\tau_{K,L}) = \overline{\{k_{ii} + l_{jj} : 1 \leqslant i \leqslant j < \infty\}}.$$

The situation is not as simple as that above when the operators $X, Y \in \mathcal{T}(\mathcal{N})$ implementing the Sylvester-Rosenblum operator $\tau_{X,Y}$ are not compact. For example, let $S \in \mathcal{T}(\mathcal{N})$ denote the unilateral backward shift which satisfies $Se_k = e_{k-1}$, $k \geqslant 1$, $Se_1 = 0$, and let $Y = 2I \in \mathcal{T}(\mathcal{N})$.

In this case, $s_{ii} = 0$ for all $i \geqslant 1$, and $y_{jj} = 2$ for all $j \geqslant 1$. Thus

$$\overline{\{s_{ii} + y_{jj} : 1 \leqslant i \leqslant j < \infty\}} = \{2\}.$$

On the other hand, $\tau_{S,Y} = \tau_{S-2I,0} = L_{S-2I}$. But $S - 2I \in \mathcal{T}(\mathcal{N})$ is invertible in $\mathcal{B}(\mathcal{H})$, and since $\mathcal{T}(\mathcal{N})$ is inverse-closed (see, for example [7], Remark 1), we have that

$$\sigma_{\mathcal{B}(\mathcal{T}(\mathcal{N}))}(\tau_{S,Y}) = \sigma_{\mathcal{B}(\mathcal{T}(\mathcal{N}))}(L_{S-2I}) = \{z \in \mathbb{C} : |z-2| \leqslant 1\}.$$

Acknowledgement. We would like to thank the referee for a very careful reading of the manuscript, and for a number of helpful suggestions.

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(Received March 4, 2019)

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