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INEQUALITIES FOR WEIGHTED GEOMETRIC MEAN IN HERMITIAN UNITAL BANACH $*$–ALGEBRAS VIA A RESULT OF CARTWRIGHT AND FIELD

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Abstract. Consider the *quadratic weighted geometric mean*

$$x\circledS_{\nu}y := \left|\left|yx^{-1}\right|^{\nu}x\right|^{2}$$

for invertible elements x, y in a *Hermitian unital Banach $*$-algebra* and real number ν. In this paper, by utilizing a result of Cartwright and Field, we obtain various upper and lower bounds for the positive difference

$$(1-\nu)|x|^{2}+\nu|y|^{2}-x\circledS_{\nu}y,$$

where $\nu\in[0,1]$, under various assumptions for the elements involved. Applications for the classical *weighted geometric mean*

$$a\sharp_{\nu}b := a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^{\upsilon}a^{1/2}$$

of positive elements a, b that satisfy the condition $0<ka\leqslant b\leqslant Ka$ for certain numbers $0<k<K$, are also given.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a\in A$ is called *selfadjoint* if $a^{*}=a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a)\subset\mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a\geqslant 0$ if $a^{*}=a$ and $\sigma(a)\subset[0,\infty)$. We say that a is *positive* and write $a>0$ if $a\geqslant 0$ and $0\notin\sigma(a)$. Thus $a>0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a,b\in\mathrm{Inv}(A)$, then $ab\in\mathrm{Inv}(A)$ and $(ab)^{-1}=b^{-1}a^{-1}$. Also, saying that $a\geqslant b$ means that $a-b\geqslant 0$ and, similarly $a>b$ means that $a-b>0$.

The *Shirali-Ford theorem* asserts that [14] (see also [2, Theorem 41.5])

$$a^{*}a\geqslant 0 \text{ for every } a\in A. \tag{SF}$$

Based on this fact, Okayasu [13], Tanahashi and Uchiyama [15] proved the following fundamental properties (see also [7]):

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(i) If $a,\ b\in A$, then $a\geqslant 0,\ b\geqslant 0$ imply $a+b\geqslant 0$ and $\alpha\geqslant 0$ implies $\alpha a\geqslant 0$;

(ii) If $a,\ b\in A$, then $a>0,\ b\geqslant 0$ imply $a+b>0$;

(iii) If $a,\ b\in A$, then either $a\geqslant b>0$ or $a>b\geqslant 0$ imply $a>0$;

(iv) If $a>0$, then $a^{-1}>0$;

(v) If $c>0$, then $0<b<a$ if and only if $cbc<cac$, also $0<b\leqslant a$ if and only if $cbc\leqslant cac$;

(vi) If $0<a<1$, then $1<a^{-1}$;

(vii) If $0<b<a$, then $0<a^{-1}<b^{-1}$, also if $0<b\leqslant a$, then $0<a^{-1}\leqslant b^{-1}$.

Okayasu [13] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a,\ b\in A$ and $p\in[0,1]$ then $a>b\ \ (a\geqslant b)$ implies that $a^p>b^p\ \ (a^p\geqslant b^p)$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let $a\in A$ and $a>0$, then $0\notin\sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z:z\in\sigma(a)\}>0$ and $\sup\{z:z\in\sigma(a)\}<\infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z>0\}$, the right half open plane of the complex plane, such that $\sigma(a)\subset\operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a):=\frac{1}{2\pi i}\int_{\gamma}f(z)(z-a)^{-1}dz.$$

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a))=f(\sigma(a))$$

holds.

For any $\alpha\in\mathbb{R}$ we define for $a\in A$ and $a>0$, the real power

$$a^{\alpha}:=\frac{1}{2\pi i}\int_{\gamma}z^{\alpha}(z-a)^{-1}dz,$$

where z^{α} is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^{\alpha}\in A$. Moreover, since z^{α} is analytic in $\{\operatorname{Re} z>0\}$, then by (SMT) we have

$$\sigma(a^{\alpha})=(\sigma(a))^{\alpha}=\{z^{\alpha}:z\in\sigma(a)\}\subset(0,\infty).$$

Following [7], we list below some important properties of real powers:

(viii) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $a^{\alpha}\in A$ with $a^{\alpha}>0$ and $\left(a^2\right)^{1/2}=a$, [15, Lemma 6];

(ix) If $0 < a \in A$ and $\alpha,\ \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;

(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = \left(a^{-1}\right)^\alpha = a^{-\alpha}$;

(xi) If $0 < a,\ b \in A$, $\alpha,\ \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

We define the following means for $\nu \in [0,1]$, see also [7] for different notations:

$$a\nabla_\nu b := (1-\nu)a + \nu b,\ a,\ b \in A \tag{A}$$

the *weighted arithmetic mean* of (a,b),

$$a!_\nu b := \left((1-\nu)a^{-1} + \nu b^{-1}\right)^{-1},\ a,\ b > 0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a,b) and

$$a\sharp_\nu b := a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^\upsilon a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a,b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu = \frac{1}{2}$, we use the simpler notations $a\nabla b$, $a!b$ and $a\sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [7], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a,\ b \in A$, then $a!b = b!a$ and $a\sharp b = b\sharp a$;

(xiii) If $0 < a,\ b \in A$ and $c \in \mathrm{Inv}(A)$, then

$$c^*(a!b)c = (c^*ac)!(c^*bc) \text{ and } c^*(a\sharp b)c = (c^*ac)\sharp(c^*bc);$$

(xiv) If $0 < a,\ b \in A$ and $\nu \in [0,1]$, then

$$(a!_\nu b)^{-1} = \left(a^{-1}\right)\nabla_\nu\left(b^{-1}\right) \text{ and } \left(a^{-1}\right)\sharp_\nu\left(b^{-1}\right) = (a\sharp_\nu b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [7] the following inequality between the weighted means introduced above:

$$a\nabla_\nu b \geqslant a\sharp_\nu b \geqslant a!_\nu b \tag{HGA}$$

for any $0 < a,\ b \in A$ and $\nu \in [0,1]$.

In [15], Tanahashi and Uchiyama obtained the following identity of interest:

LEMMA 1. *If $0 < c,\ d$ and λ is a real number, then*

$$(dcd)^\lambda = dc^{1/2}\left(c^{1/2}d^2c^{1/2}\right)^{\lambda-1}c^{1/2}d. \tag{1.1}$$

Using this equality we can prove the following fact [6]:

PROPOSITION 1. *For any* $0 < a,\ b \in A$ *we have*

$$b\sharp_{1-\nu}a = a\sharp_\nu b \tag{1.2}$$

for any real number ν.

In [6] we introduced the *quadratic weighted mean* of (x,y) with $x,\ y \in \operatorname{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$, as the positive element denoted by $x \circledS_\nu y$ and defined by

$$x \circledS_\nu y := x^*\left((x^*)^{-1}y^*yx^{-1}\right)^\nu x = x^*\left|yx^{-1}\right|^{2\nu}x = \left|\left|yx^{-1}\right|^\nu x\right|^2. \tag{S}$$

When $\nu = 1/2$, we denote $x \circledS_{1/2} y$ by $x \circledS y$ and we have

$$x \circledS y = x^*\left((x^*)^{-1}y^*yx^{-1}\right)^{1/2} x = x^*\left|yx^{-1}\right|x = \left|\left|yx^{-1}\right|^{1/2} x\right|^2.$$

We can also introduce the $1/2$-*quadratic weighted mean* of (x,y) with $x,\ y \in \operatorname{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$ by

$$x \circledS_\nu^{1/2} y := (x \circledS_\nu y)^{1/2} = \left|\left|yx^{-1}\right|^\nu x\right|. \tag{1/2-S}$$

Correspondingly, when $\nu = 1/2$ we denote $x \circledS^{1/2} y$ and we have

$$x \circledS^{1/2} y = \left|\left|yx^{-1}\right|^{1/2} x\right|.$$

The following equalities hold [6]:

PROPOSITION 2. *For any* $x,\ y \in \operatorname{Inv}(A)$ *and* $\nu \in \mathbb{R}$ *we have*

$$(x \circledS_\nu y)^{-1} = (x^*)^{-1} \circledS_\nu (y^*)^{-1}$$

and

$$\left(x^{-1}\right) \circledS_\nu \left(y^{-1}\right) = (x^* \circledS_\nu y^*)^{-1}.$$

If we take in (S) $x = a^{1/2}$ and $y = b^{1/2}$ with $a,\ b > 0$ then we get

$$a^{1/2} \circledS_\nu b^{1/2} = a\sharp_\nu b$$

for any $\nu \in \mathbb{R}$ that shows that the quadratic weighted mean can be seen as an extension of the weighted geometric mean for positive elements considered in the introduction.

Let $x,\ y \in \operatorname{Inv}(A)$. If we take in the definition of "$\sharp_\nu$" the elements $a = |x|^2 > 0$ and $b = |y|^2 > 0$ we also have for real ν

$$|x|^2 \sharp_\nu |y|^2 = |x|\left(|x|^{-1}|y|^2|x|^{-1}\right)^\upsilon |x| = |x|\left||y|\,|x|^{-1}\right|^{2\upsilon}|x| = \left|\left||y|\,|x|^{-1}\right|^\upsilon |x|\right|^2.$$

It is then natural to ask how the positive elements $x \circledS_\nu y$ and $|x|^2 \sharp_\nu |y|^2$ do compare, when $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in \mathbb{R}$?

In [6] we proved the following lemma that provides a slight generalization of Lemma 1.

LEMMA 2. *If* $0 < c,\ d \in \mathrm{Inv}(A)$ *and* λ *is a real number, then*

$$(dcd^*)^\lambda = dc^{1/2}\left(c^{1/2}|d|^2 c^{1/2}\right)^{\lambda-1} c^{1/2}d^*. \tag{1.3}$$

REMARK 1. The identity (1.3) was proved by. T. Furuta in [8] for positive operator c and invertible operator d in the Banach algebra of all bonded linear operators on a Hilbert space by using the polar decomposition of the invertible operator $dc^{1/2}$.

The following fundamental fact that connects the quadratic weighted geometric mean $\circledS_\nu$ to the weighted geometric mean $\sharp_\nu$ holds [6]:

THEOREM 1. *If* $x,\ y \in \mathrm{Inv}(A)$ *and* λ *is a real number, then*

$$x \circledS_\nu y = |x|^2 \sharp_\nu |y|^2 \tag{1.4}$$

Now, assume that $f(z)$ is analytic in the right half open plane $\{\mathrm{Re}\, z > 0\}$ and for the interval $I \subset (0,\infty)$ assume that $f(z) \geqslant 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0,\infty)$$

meaning that $f(u) \geqslant 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A.

LEMMA 3. *Let* $f(z)$ *and* $g(z)$ *be analytic in the right half open plane* $\{\mathrm{Re}\, z > 0\}$ *and for the interval* $I \subset (0,\infty)$ *assume that* $f(z) \geqslant g(z)$ *for any* $z \in I$. *Then for any* $u \in A$ *with* $\sigma(u) \subset I$ *we have* $f(u) \geqslant g(u)$ *in the order of* A.

We have the following inequalities between means [6]:

THEOREM 2. *For any* $x,\ y \in \mathrm{Inv}(A)$ *and* $\nu \in [0,1]$ *we have*

$$|x|^2 \nabla_\nu |y|^2 \geqslant x \circledS_\nu y \geqslant |x|^2 !_\nu |y|^2. \tag{1.5}$$

In particular,

$$|x|^2 \nabla |y|^2 \geqslant x \circledS y \geqslant |x|^2 ! |y|^2. \tag{1.6}$$

We can define the weighted means for $\nu \in [0,1]$ and the elements $x,\ y \in \mathrm{Inv}(A)$ and $\nu \in [0,1]$ by

$$x \nabla_\nu^{1/2} y := \left(|x|^2 \nabla_\nu |y|^2\right)^{1/2} = \left((1-\nu)|x|^2 + \nu |y|^2\right)^{1/2}$$

and

$$x !_\nu^{1/2} y := \left(|x|^2 !_\nu |y|^2\right)^{1/2} = \left((1-\nu)|x|^{-2} + \nu |y|^{-2}\right)^{-1/2}.$$

For $\nu = 1/2$ we consider

$$x\nabla^{1/2}y := \left(|x|^2 \nabla |y|^2\right)^{1/2} = \frac{\sqrt{2}}{2}\left(|x|^2 + |y|^2\right)^{1/2}$$

and

$$x!^{1/2}y := \left(|x|^2 ! |y|^2\right)^{1/2} = \sqrt{2}\left(|x|^{-2} + |y|^{-2}\right)^{-1/2}.$$

COROLLARY 1. *Let A be a Hermitian unital Banach $*$-algebra with continuous involution. Then for any $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$x\nabla_\nu^{1/2}y \geqslant x\circledS_\nu^{1/2}y \geqslant x!_\nu^{1/2}y. \tag{1.7}$$

In particular, we have

$$x\nabla^{1/2}y \geqslant x\circledS^{1/2}y \geqslant x!^{1/2}y. \tag{1.8}$$

Recall that a C^**-algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^*a\| = \|a\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [12, 2.2.5 Theorem])

$$b \geqslant a \geqslant 0 \text{ implies that } \|b\| \geqslant \|a\|.$$

COROLLARY 2. *Let A be a unital C^*-algebra. Then for any $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$(1-\nu)\|x\|^2 + \nu\|y\|^2 \geqslant \left\|(1-\nu)|x|^2 + \nu|y|^2\right\| \geqslant \left\|\left|yx^{-1}\right|^\nu x\right\|^2. \tag{1.9}$$

In particular,

$$\frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) \geqslant \frac{1}{2}\left\||x|^2 + |y|^2\right\| \geqslant \left\|\left|yx^{-1}\right|^{1/2} x\right\|^2. \tag{1.10}$$

Motivated by the above facts, in this paper we obtain various upper and lower bounds for the positive difference

$$(1-\nu)|x|^2 + \nu|y|^2 - x\circledS_\nu y,$$

where $\nu \in [0,1]$, under various assumptions for the elements involved. Applications for the classical geometric mean $a\sharp_\nu b := a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^\upsilon a^{1/2}$ of positive elements $a,\ b$ that satisfy the condition $0 < ka \leqslant b \leqslant Ka$ for certain numbers $0 < k < K$, are also given.

2. Refinements and reverses

We have the following inequality that provides a refinement and a reverse for the celebrated scalar Young's inequality

$$\frac{1}{2}\nu(1-\nu)\frac{(\beta-\alpha)^2}{\max\{\alpha,\beta\}} \leqslant (1-\nu)\alpha+\nu\beta-\alpha^{1-\nu}\beta^{\nu} \leqslant \frac{1}{2}\nu(1-\nu)\frac{(\beta-\alpha)^2}{\min\{\alpha,\beta\}} \quad (2.1)$$

for any $\alpha,\ \beta > 0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [3] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval. For other similar inequalities, see [1], [5] and [9, 10, 11].

Assume that $x,\ y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that

$$M \geqslant \left|yx^{-1}\right| \geqslant m. \quad (2.2)$$

The inequality (2.2) is equivalent to

$$M^2 \geqslant \left|yx^{-1}\right|^2 = (x^*)^{-1}|y|^2 x^{-1} \geqslant m^2. \quad (2.3)$$

If we multiply at left with x^* and at right with x we get the equivalent relation

$$M^2|x|^2 \geqslant |y|^2 \geqslant m^2|x|^2. \quad (2.4)$$

For $[k,K] \subset (0,\infty)$ we consider the coefficients

$$c(k,K) := \begin{cases} (K-1)^2 \text{ if } K<1, \\ 0 \text{ if } k \leqslant 1 \leqslant K, \\ \frac{(k-1)^2}{K} \text{ if } 1<k \end{cases} \quad (2.5)$$

and

$$C(k,K) := \begin{cases} \frac{(k-1)^2}{k} \text{ if } K<1, \\ \frac{1}{k}\max\left\{(k-1)^2,(K-1)^2\right\} \text{ if } k \leqslant 1 \leqslant K, \\ (K-1)^2 \text{ if } 1<k. \end{cases} \quad (2.6)$$

We have:

THEOREM 3. *Assume that $x,\ y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that (2.2) is true. Then we have the inequalities*

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)c\left(m^2,M^2\right)|x|^2 &\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\max\{M^2,1\}}\left|\left(\left|yx^{-1}\right|^2-1\right)x\right|^2 \\ &\leqslant |x|^2 \nabla_\nu |y|^2 - x Ⓢ_\nu y \\ &\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}}\left|\left(\left|yx^{-1}\right|^2-1\right)x\right|^2 \end{aligned} \quad (2.7)$$

$$\leqslant \frac{1}{2}\nu(1-\nu)C\left(m^2,M^2\right)|x|^2$$

for any $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}c\left(m^2,M^2\right)|x|^2 \leqslant \frac{1}{8}\frac{1}{\max\{M^2,1\}}\left|\left(\left|yx^{-1}\right|^2-1\right)x\right|^2 \tag{2.8}$$
$$\leqslant |x|^2 \nabla |y|^2 - x \circledS y$$
$$\leqslant \frac{1}{8}\frac{1}{\min\{m^2,1\}}\left|\left(\left|yx^{-1}\right|^2-1\right)x\right|^2$$
$$\leqslant \frac{1}{8}C\left(m^2,M^2\right)|x|^2.$$

Proof. If we write the inequality (2.1) for $\alpha = 1$ and $\beta = \tau$ we get

$$\frac{1}{2}\nu(1-\nu)\frac{(\tau-1)^2}{\max\{\tau,1\}} \leqslant 1-\nu+\nu\tau-\tau^\nu \leqslant \frac{1}{2}\nu(1-\nu)\frac{(\tau-1)^2}{\min\{\tau,1\}} \tag{2.9}$$

for any $\tau > 0$ and for any $\nu \in [0,1]$.

If $\tau \in [k,K] \subset (0,\infty)$, then $\max\{\tau,1\} \leqslant \max\{K,1\}$ and $\min\{k,1\} \leqslant \min\{\tau,1\}$ and by (2.9) we get

$$\frac{1}{2}\nu(1-\nu)\frac{\min_{\tau\in[k,K]}(\tau-1)^2}{\max\{K,1\}} \leqslant \frac{1}{2}\nu(1-\nu)\frac{(\tau-1)^2}{\max\{K,1\}} \tag{2.10}$$
$$\leqslant 1-\nu+\nu\tau-\tau^\nu$$
$$\leqslant \frac{1}{2}\nu(1-\nu)\frac{(\tau-1)^2}{\min\{k,1\}}$$
$$\leqslant \frac{1}{2}\nu(1-\nu)\frac{\max_{\tau\in[k,K]}(\tau-1)^2}{\min\{k,1\}}$$

for any $\tau \in [k,K]$ and for any $\nu \in [0,1]$.

Observe that

$$\min_{\tau\in[k,K]}(\tau-1)^2 = \begin{cases} (K-1)^2 \text{ if } K<1, \\ 0 \text{ if } k \leqslant 1 \leqslant K, \\ (k-1)^2 \text{ if } 1<k \end{cases}$$

and

$$\max_{\tau\in[k,K]}(\tau-1)^2 = \begin{cases} (k-1)^2 \text{ if } K<1, \\ \max\left\{(k-1)^2,(K-1)^2\right\} \text{ if } k \leqslant 1 \leqslant K, \\ (K-1)^2 \text{ if } 1<k. \end{cases}$$

Then

$$\frac{\min_{\tau\in[k,K]}(\tau-1)^2}{\max\{K,1\}} = c(k,K)$$

and

$$\frac{\max_{\tau\in[k,K]}(\tau-1)^2}{\min\{k,1\}} = C(k,K)$$

as defined by (2.5) and (2.6).

Using the inequality (2.10) we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)c(k,M) &\leqslant \frac{1}{2}\nu(1-\nu)\frac{(z-1)^2}{\max\{M,1\}} \\ &\leqslant 1-\nu+\nu z - z^{\nu} \\ &\leqslant \frac{1}{2}\nu(1-\nu)\frac{(z-1)^2}{\min\{k,1\}} \\ &\leqslant \frac{1}{2}\nu(1-\nu)C(k,M)\end{aligned} \tag{2.11}$$

for any real $z\in[k,K]\subset(0,\infty)$ and for any $\nu\in[0,1]$.

Let $u\in A$ with spectrum $\sigma(u)\subset[k,K]\subset(0,\infty)$. Then by applying Lemma 3 for the corresponding analytic functions in the right half open plane $\{\mathrm{Re}\,z>0\}$ involved in the inequality (2.11) we conclude that we have in the order of A that

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)c(k,K) &\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\max\{K,1\}}(u-1)^2 \\ &\leqslant 1-\nu+\nu u - u^{\nu} \\ &\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\min\{k,1\}}(u-1)^2 \\ &\leqslant \frac{1}{2}\nu(1-\nu)C(k,K)\end{aligned} \tag{2.12}$$

for any $\nu\in[0,1]$.

If $x,\ y\in\mathrm{Inv}(A)$ satisfy the condition (2.2) then, by (2.3), the element $u=\left|yx^{-1}\right|^2\in\mathrm{Inv}(A)$ and $\sigma(u)\subset\left[m^2,M^2\right]\subset(0,\infty)$.

By (2.12) we then have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)c\left(m^2,M^2\right) &\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\max\{M^2,1\}}\left(\left|yx^{-1}\right|^2-1\right)^2 \\ &\leqslant 1-\nu+\nu\left|yx^{-1}\right|^2-\left(\left|yx^{-1}\right|^2\right)^{\nu} \\ &\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}}\left(\left|yx^{-1}\right|^2-1\right)^2 \\ &\leqslant \frac{1}{2}\nu(1-\nu)C\left(m^2,M^2\right)\end{aligned} \tag{2.13}$$

for any $\nu\in[0,1]$.

If we multiply this inequality at left with x^* and at right with x we get

$$\frac{1}{2}\nu(1-\nu)c\left(m^2,M^2\right)|x|^2 \leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\max\{M^2,1\}}x^*\left(\left|yx^{-1}\right|^2-1\right)^2 x \tag{2.14}$$

$$\leqslant (1-\nu)|x|^2 + \nu x^* \left|yx^{-1}\right|^2 x - x^* \left(\left|yx^{-1}\right|^2\right)^\nu x$$
$$\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}} x^* \left(\left|yx^{-1}\right|^2 - 1\right)^2 x$$
$$\leqslant \frac{1}{2}\nu(1-\nu) C\left(m^2,M^2\right)|x|^2$$

for any $\nu \in [0,1]$.

Since

$$x^* \left|yx^{-1}\right|^2 x = x^* \left((x^*)^{-1} y^* y x^{-1}\right) x = y^* y = |y|^2,$$
$$x^* \left(\left|yx^{-1}\right|^2\right)^\nu x = x \circledS_\nu y$$

and

$$x^* \left(\left|yx^{-1}\right|^2 - 1\right)^2 x = \left|\left(\left|yx^{-1}\right|^2 - 1\right) x\right|^2$$

for $x,\ y \in \mathrm{Inv}(A)$, then by (2.14) we get the desired result (2.7).

COROLLARY 3. *Let A be a unital C^*-algebra. Assume that $x,\ y \in \mathrm{Inv}(A)$ and the constants $M > m > 0$ are such that (2.2) holds, then we have*

$$\frac{1}{2}\nu(1-\nu)c\left(m^2,M^2\right)\|x\|^2 \leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\max\{M^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right)x\right\|^2 \tag{2.15}$$
$$\leqslant \left\||x|^2 \nabla_\nu |y|^2 - x \circledS_\nu y\right\|$$
$$\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right)x\right\|^2$$
$$\leqslant \frac{1}{2}\nu(1-\nu)C\left(m^2,M^2\right)\|x\|^2$$

for any $\nu \in [0,1]$.

In particular,

$$\frac{1}{8}c\left(m^2,M^2\right)\|x\|^2 \leqslant \frac{1}{8}\frac{1}{\max\{M^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right)x\right\|^2 \tag{2.16}$$
$$\leqslant \left\||x|^2 \nabla |y|^2 - x \circledS y\right\|$$
$$\leqslant \frac{1}{8}\frac{1}{\min\{m^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right)x\right\|^2$$
$$\leqslant \frac{1}{8}C\left(m^2,M^2\right)\|x\|^2.$$

REMARK 2. Using the triangle inequality we have

$$0 \leqslant \left\||x|^2 \nabla_\nu |y|^2\right\| - \|x \circledS_\nu y\| \leqslant \left\||x|^2 \nabla_\nu |y|^2 - x \circledS_\nu y\right\|$$

and by (2.15) we get the following reverse of the second inequality in (1.9)

$$\begin{aligned} &\left\| (1-\nu)|x|^2 + \nu|y|^2 \right\| \\ &\leqslant \left\| \left|yx^{-1}\right|^{\nu} x \right\|^2 + \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}} \left\| \left(\left|yx^{-1}\right|^2 - 1 \right) x \right\|^2 \\ &\leqslant \left\| \left|yx^{-1}\right|^{\nu} x \right\|^2 + \frac{1}{2}\nu(1-\nu) C\left(m^2, M^2\right) \|x\|^2 \end{aligned} \tag{2.17}$$

provided that $x,\ y$ and ν are as in Corollary 3.

In particular,

$$\begin{aligned} \frac{1}{2}\left\| |x|^2 + |y|^2 \right\| &\leqslant \left\| \left|yx^{-1}\right|^{1/2} x \right\|^2 + \frac{1}{8}\frac{1}{\min\{m^2,1\}} \left\| \left(\left|yx^{-1}\right|^2 - 1 \right) x \right\|^2 \\ &\leqslant \left\| \left|yx^{-1}\right|^{1/2} x \right\|^2 + \frac{1}{8} C\left(m^2, M^2\right) \|x\|^2 . \end{aligned} \tag{2.18}$$

COROLLARY 4. *If* $0 < a,\ b \in A$ *and* $0 < k < K$ *are such that*

$$ka \leqslant b \leqslant Ka, \tag{2.19}$$

then

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)c(k,K)a &\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\max\{K,1\}} \left| \left(\left|b^{1/2}a^{-1/2}\right|^2 - 1 \right) a^{1/2} \right|^2 \\ &\leqslant a\nabla_{\nu} b - a\sharp_{\nu} b \\ &\leqslant \frac{1}{2}\frac{\nu(1-\nu)}{\min\{k,1\}} \left| \left(\left|b^{1/2}a^{-1/2}\right|^2 - 1 \right) a^{1/2} \right|^2 \\ &\leqslant \frac{1}{2}\nu(1-\nu) C(k,K) a \end{aligned} \tag{2.20}$$

for any $\nu \in [0,1]$, *where* $c(k,K)$ *and* $C(k,K)$ *are given by (2.5) and (2.6).*

In particular, we have

$$\begin{aligned} \frac{1}{8}c(k,K)a &\leqslant \frac{1}{8}\frac{1}{\max\{K,1\}} \left| \left(\left|b^{1/2}a^{-1/2}\right|^2 - 1 \right) a^{1/2} \right|^2 \\ &\leqslant a\nabla b - a\sharp b \\ &\leqslant \frac{1}{8}\frac{1}{\min\{k,1\}} \left| \left(\left|b^{1/2}a^{-1/2}\right|^2 - 1 \right) a^{1/2} \right|^2 \\ &\leqslant \frac{1}{8} C(k,K) a. \end{aligned} \tag{2.21}$$

The proof follows by Theorem 3 applied for $x = a^{1/2},\ y = b^{1/2},\ M = \sqrt{K}$ and $m = \sqrt{k}$.

3. Some related results

We observe that since

$$\max\{\alpha,\beta\}\min\{\alpha,\beta\}=\alpha\beta \text{ for } \alpha,\beta>0,$$

then the inequality (2.1) can be written in an equivalent form as

$$\frac{1}{2}\nu(1-\nu)\min\{\alpha,\beta\}\frac{(\beta-\alpha)^2}{\alpha\beta}\leqslant(1-\nu)\alpha+\nu\beta-\alpha^{1-\nu}\beta^{\nu} \leqslant\frac{1}{2}\nu(1-\nu)\max\{\alpha,\beta\}\frac{(\beta-\alpha)^2}{\alpha\beta} \tag{3.1}$$

for any $\alpha,\ \beta>0$ and $\nu\in[0,1]$.

We define the following coefficients associated with the interval $[k,K]\subset(0,\infty)$:

$$d(k,K):=\begin{cases}\frac{k(K-1)^2}{K} \text{ if } K<1,\\ 0 \text{ if } k\leqslant 1\leqslant K,\\ \frac{(k-1)^2}{k} \text{ if } 1<k\end{cases} \tag{3.2}$$

and

$$D(k,K):=\begin{cases}\frac{(k-1)^2}{k} \text{ if } K<1,\\ \max\left\{\frac{K(k-1)^2}{k},(K-1)^2\right\} \text{ if } k\leqslant 1\leqslant K,\\ (K-1)^2 \text{ if } 1<k.\end{cases} \tag{3.3}$$

THEOREM 4. *Assume that $x,\ y\in \operatorname{Inv}(A)$ and the constants $M>m>0$ are such that (2.2) is true. Then we have the inequalities*

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)d\left(m^2,M^2\right)|x|^2&\leqslant\frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left||y|^{-1}\left(|y|^2-|x|^2\right)\right|^2\\&\leqslant|x|^2\nabla_\nu|y|^2-x \circledS_\nu y\\&\leqslant\frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left||y|^{-1}\left(|y|^2-|x|^2\right)\right|^2\\&\leqslant\frac{1}{2}\nu(1-\nu)D\left(m^2,M^2\right)|x|^2,\end{aligned} \tag{3.4}$$

for any $\nu\in[0,1]$, where the coefficients $d(\cdot,\cdot)$ and $D(\cdot,\cdot)$ are defined by (3.2) and (3.3).

In particular, we have

$$\begin{aligned}\frac{1}{8}d\left(m^2,M^2\right)|x|^2&\leqslant\frac{1}{8}\min\left\{m^2,1\right\}\left||y|^{-1}\left(|y|^2-|x|^2\right)\right|^2\\&\leqslant|x|^2\nabla|y|^2-x\circledS y\\&\leqslant\frac{1}{8}\max\left\{M^2,1\right\}\left||y|^{-1}\left(|y|^2-|x|^2\right)\right|^2\\&\leqslant\frac{1}{8}D\left(m^2,M^2\right)|x|^2.\end{aligned} \tag{3.5}$$

Proof. If we write the inequality (3.1) for $\alpha = 1$ and $\beta = \tau$ we get

$$\frac{1}{2}\nu(1-\nu)\min\{\tau,1\}\frac{(\tau-1)^2}{\tau} \leqslant 1-\nu+\nu\tau-\tau^\nu \leqslant \frac{1}{2}\nu(1-\nu)\max\{\tau,1\}\frac{(\tau-1)^2}{\tau} \tag{3.6}$$

for any $\tau > 0$ and for any $\nu \in [0,1]$.

If $\tau \in [k,K] \subset (0,\infty)$, then $\max\{\tau,1\} \leqslant \max\{K,1\}$ and $\min\{k,1\} \leqslant \min\{\tau,1\}$ and by (3.6) we get

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)\min\{k,1\}\min_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau}\\ \leqslant&\frac{1}{2}\nu(1-\nu)\min\{k,1\}\frac{(\tau-1)^2}{\tau}\\ \leqslant&1-\nu+\nu\tau-\tau^\nu\\ \leqslant&\frac{1}{2}\nu(1-\nu)\max\{K,1\}\frac{(\tau-1)^2}{\tau}\\ \leqslant&\frac{1}{2}\nu(1-\nu)\max\{K,1\}\max_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau}.\end{aligned} \tag{3.7}$$

Consider the function $\delta : (0,\infty) \to (0,\infty)$, $\delta(\tau) = \frac{(\tau-1)^2}{\tau}$. Then

$$\delta'(\tau) = \frac{2(\tau-1)\tau-(\tau-1)^2}{\tau^2} = \frac{(\tau-1)(\tau+1)}{\tau^2}.$$

This shows that the function δ is strictly decreasing on $(0,1)$, strictly increasing on $(1,\infty)$, $\delta(1) = 0$ and

$$\lim_{\tau\to 0+}\delta(\tau) = \lim_{\tau\to\infty}\delta(\tau) = \infty.$$

By taking into account all possible locations of the interval $[k,K]$ and the number 1 we have

$$\min_{\tau\in[k,K]}\delta(\tau) = \begin{cases}\frac{(K-1)^2}{K} \text{ if } K<1,\\ 0 \text{ if } k\leqslant 1\leqslant K,\\ \frac{(k-1)^2}{k} \text{ if } 1<k\end{cases}$$

and

$$\max_{\tau\in[k,K]}\delta(\tau) = \begin{cases}\frac{(k-1)^2}{k} \text{ if } K<1,\\ \max\left\{\frac{(k-1)^2}{k},\frac{(K-1)^2}{K}\right\} \text{ if } k\leqslant 1\leqslant K,\\ \frac{(K-1)^2}{K} \text{ if } 1<k.\end{cases}$$

Since

$$\min\{k,1\}\min_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau} = \begin{cases}\frac{k(K-1)^2}{K} \text{ if } K<1,\\ 0 \text{ if } k\leqslant 1\leqslant K,\\ \frac{(k-1)^2}{k} \text{ if } 1<k\end{cases}$$

and

$$\max\{K,1\}\max_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau}=\begin{cases}\frac{(k-1)^2}{k}\text{ if }K<1,\\ \max\left\{\frac{K(k-1)^2}{k},(K-1)^2\right\}\text{ if }k\leqslant 1\leqslant K,\\ (K-1)^2\text{ if }1<k,\end{cases}$$

then by (3.7) we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)d(k,K)&\leqslant\frac{1}{2}\nu(1-\nu)\min\{k,1\}\left(z+z^{-1}-2\right)\\&\leqslant 1-\nu+\nu z-z^{\nu}\\&\leqslant\frac{1}{2}\nu(1-\nu)\max\{K,1\}\left(z+z^{-1}-2\right)\\&\leqslant\frac{1}{2}\nu(1-\nu)D(k,K)\end{aligned}\tag{3.8}$$

for any $z\in[k,K]$ and for any $\nu\in[0,1]$.

Let $u\in A$ with spectrum $\sigma(u)\subset[k,K]\subset(0,\infty)$. Then by applying Lemma 3 for the corresponding analytic functions in the right half open plane $\{\mathrm{Re}\,z>0\}$ involved in the inequality (3.8) we conclude that we have in the order of A that

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)d(k,K)&\leqslant\frac{1}{2}\nu(1-\nu)\min\{k,1\}\left(u+u^{-1}-2\right)\\&\leqslant 1-\nu+\nu u-u^{\nu}\\&\leqslant\frac{1}{2}\nu(1-\nu)\max\{K,1\}\left(u+u^{-1}-2\right)\\&\leqslant\frac{1}{2}\nu(1-\nu)D(k,K)\end{aligned}\tag{3.9}$$

for any $\nu\in[0,1]$.

If $x,\ y\in\mathrm{Inv}(A)$ satisfy the condition (2.2) then, by (2.3), the element $u=\left|yx^{-1}\right|^2\in\mathrm{Inv}(A)$ and $\sigma(u)\subset\left[m^2,M^2\right]\subset(0,\infty)$.

By (3.9) we then have

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)d\left(m^2,M^2\right)\\&\leqslant\frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left(\left|yx^{-1}\right|^2+\left(\left|yx^{-1}\right|^2\right)^{-1}-2\right)\\&\leqslant 1-\nu+\nu\left|yx^{-1}\right|^2-\left(\left|yx^{-1}\right|^2\right)^{\nu}\\&\leqslant\frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left(\left|yx^{-1}\right|^2+\left(\left|yx^{-1}\right|^2\right)^{-1}-2\right)\\&\leqslant\frac{1}{2}\nu(1-\nu)D\left(m^2,M^2\right)\end{aligned}\tag{3.10}$$

for any $\nu\in[0,1]$.

If we multiply this inequality at left with x^* and at right with x we get

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)d\left(m^2,M^2\right)|x|^2 \qquad (3.11)\\
&\leqslant\frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left(x^*\left|yx^{-1}\right|^2x+x^*\left(\left|yx^{-1}\right|^2\right)^{-1}x-2|x|^2\right)\\
&\leqslant(1-\nu)|x|^2+\nu x^*\left|yx^{-1}\right|^2x-x^*\left(\left|yx^{-1}\right|^2\right)^{\nu}x\\
&\leqslant\frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left(x^*\left|yx^{-1}\right|^2x+x^*\left(\left|yx^{-1}\right|^2\right)^{-1}x-2|x|^2\right)\\
&\leqslant\frac{1}{2}\nu(1-\nu)D\left(m^2,M^2\right)|x|^2
\end{aligned}$$

for any $\nu\in[0,1]$.

Since

$$x^*\left|yx^{-1}\right|^2x=|y|^2,\ x^*\left(\left|yx^{-1}\right|^2\right)^{\nu}x=x\circledS_{\nu}y$$

and

$$\begin{aligned}
x^*\left(\left|yx^{-1}\right|^2\right)^{-1}x&=x^*\left(\left(x^*\right)^{-1}y^*yx^{-1}\right)^{-1}x=x^*\left(xy^{-1}\left(y^*\right)^{-1}x^*\right)x\\
&=x^*xy^{-1}\left(y^*\right)^{-1}x^*x=|x|^2|y|^{-2}|x|^2,
\end{aligned}$$

then by (3.11) we get

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)d\left(m^2,M^2\right)|x|^2 \qquad (3.12)\\
&\leqslant\frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left(|y|^2+|x|^2|y|^{-2}|x|^2-2|x|^2\right)\\
&\leqslant|x|^2\nabla_{\nu}|y|^2-x\circledS_{\nu}y\\
&\leqslant\frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left(|y|^2+|x|^2|y|^{-2}|x|^2-2|x|^2\right)\\
&\leqslant\frac{1}{2}\nu(1-\nu)D\left(m^2,M^2\right)|x|^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
|y|^2+|x|^2|y|^{-2}|x|^2-2|x|^2&=\left(|y|^2-|x|^2\right)\left(1-|y|^{-2}|x|^2\right)\\
&=\left(|y|^2-|x|^2\right)|y|^{-2}\left(|y|^2-|x|^2\right)\\
&=\left||y|^{-1}\left(|y|^2-|x|^2\right)\right|^2
\end{aligned}$$

and by (3.12) we get the desired result (3.4).

COROLLARY 5. *Let A be a unital C^*-algebra. Assume that $x,\ y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that (2.2) holds, then we have*

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)d\left(m^2,M^2\right)\|x\|^2 \\ \leqslant&\frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2 \\ \leqslant&\left\||x|^2\nabla_\nu|y|^2-x \circledS_\nu y\right\| \\ \leqslant&\frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2 \\ \leqslant&\frac{1}{2}\nu(1-\nu)D\left(m^2,M^2\right)\|x\|^2\end{aligned} \tag{3.13}$$

for any $\nu \in [0,1]$.

In particular, we have

$$\begin{aligned}\frac{1}{8}d\left(m^2,M^2\right)\|x\|^2 &\leqslant \frac{1}{8}\min\left\{m^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2 \\ &\leqslant \left\||x|^2\nabla|y|^2-x\circledS y\right\| \\ &\leqslant \frac{1}{8}\max\left\{M^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2 \\ &\leqslant \frac{1}{8}D\left(m^2,M^2\right)\|x\|^2.\end{aligned} \tag{3.14}$$

REMARK 3. We also have the following reverse of the second inequality in (1.9)

$$\begin{aligned}&\left\|(1-\nu)|x|^2+\nu|y|^2\right\| \\ \leqslant&\left\|\left|yx^{-1}\right|^\nu x\right\|^2+\frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2 \\ \leqslant&\left\|\left|yx^{-1}\right|^\nu x\right\|^2+\frac{1}{2}\nu(1-\nu)D\left(m^2,M^2\right)\|x\|^2\end{aligned} \tag{3.15}$$

provided that $x,\ y$ and ν are as in Corollary 3.

In particular,

$$\begin{aligned}\frac{1}{2}\left\||x|^2+|y|^2\right\| &\leqslant \left\|\left|yx^{-1}\right|^{1/2}x\right\|^2+\frac{1}{8}\max\left\{M^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2 \\ &\leqslant \left\|\left|yx^{-1}\right|^{1/2}x\right\|^2+\frac{1}{8}D\left(m^2,M^2\right)\|x\|^2.\end{aligned} \tag{3.16}$$

COROLLARY 6. *With the assumptions of Corollary 4 we have*

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)d(k,K)a &\leqslant \frac{1}{2}\nu(1-\nu)\min\{k,1\}\left|b^{-1/2}(b-a)\right|^2 \\ &\leqslant a\nabla_\nu b-a\sharp_\nu b\end{aligned} \tag{3.17}$$

$$\leqslant \frac{1}{2}\nu(1-\nu)\max\{K,1\}\left|b^{-1/2}(b-a)\right|^2$$
$$\leqslant \frac{1}{2}\nu(1-\nu)D(k,K)a$$

for any $\nu\in[0,1]$, *where* $d(k,K)$ *and* $D(k,K)$ *are given by (3.2) and (3.3).*

In particular,

$$\frac{1}{8}d(k,K)a \leqslant \frac{1}{8}\min\{k,1\}\left|b^{-1/2}(b-a)\right|^2 \leqslant a\nabla b - a\sharp b \leqslant \frac{1}{8}\max\{K,1\}\left|b^{-1/2}(b-a)\right|^2 \leqslant \frac{1}{8}D(k,K)a. \tag{3.18}$$

For an interval $[k,K]$, define the coefficients

$$f(k,K) := \begin{cases} (K-1)^2 \text{ if } K<1, \\ 0 \text{ if } k\leqslant 1\leqslant K, \\ \frac{(k-1)^2}{k} \text{ if } 1<k \end{cases} \tag{3.19}$$

and

$$F(k,K) := \begin{cases} \frac{(k-1)^2}{k} \text{ if } K<1, \\ \max\left\{\frac{(k-1)^2}{k},(K-1)^2\right\} \text{ if } k\leqslant 1\leqslant K, \\ (K-1)^2 \text{ if } 1<k. \end{cases} \tag{3.20}$$

THEOREM 5. *Assume that* $x,\ y\in \operatorname{Inv}(A)$ *and the constants* $M>m>0$ *are such that (2.2) is true. Then we have the inequalities*

$$\frac{1}{2}\nu(1-\nu)f\left(m^2,M^2\right)|x|^2 \leqslant |x|^2\nabla_\nu |y|^2 - x \circledS_\nu y \leqslant \frac{1}{2}\nu(1-\nu)F\left(m^2,M^2\right)|x|^2 \tag{3.21}$$

for any $\nu\in[0,1]$, *where* $f(\cdot,\cdot)$ *and* $F(\cdot,\cdot)$ *are defined in (3.19) and (3.20).*

In particular, we have

$$\frac{1}{8}f\left(m^2,M^2\right)|x|^2 \leqslant |x|^2\nabla|y|^2 - x\circledS y \leqslant \frac{1}{8}F\left(m^2,M^2\right)|x|^2. \tag{3.22}$$

Proof. From (2.9) we get

$$\frac{1}{2}\nu(1-\nu)\psi(\tau) \leqslant 1-\nu+\nu\tau-\tau^\nu \leqslant \frac{1}{2}\nu(1-\nu)\Psi(\tau) \tag{3.23}$$

for any $\tau>0$ and for any $\nu\in[0,1]$, where $\psi(\tau) := \frac{(\tau-1)^2}{\max\{\tau,1\}}$ and $\Psi(\tau) := \frac{(\tau-1)^2}{\min\{\tau,1\}}$.

Observe that

$$\psi(\tau)=\begin{cases}(\tau-1)^2 \text{ if } \tau\in(0,1),\\ \\ \frac{(\tau-1)^2}{\tau} \text{ if } \tau\in[1,\infty)\end{cases}$$

and

$$\Psi(\tau)=\begin{cases}\frac{(\tau-1)^2}{\tau} \text{ if } \tau\in(0,1),\\ \\ (\tau-1)^2 \text{ if } \tau\in[1,\infty).\end{cases}$$

We observe that the functions ψ and Ψ are strictly decreasing on $(0,1)$ and strictly increasing on $[1,\infty)$ with $\psi(1)=\Psi(1)=0$.

If we consider all possible locations of the interval $[k,K]$ and the number 1 then we get

$$\min_{\tau\in[k,K]}\psi(\tau)=\begin{cases}\psi(K) \text{ if } K<1,\\ 0 \text{ if } k\leqslant 1\leqslant K,\\ \psi(k) \text{ if } 1<k\end{cases}=f(k,K)$$

and

$$\max_{\tau\in[k,K]}\Psi(\tau)=\begin{cases}\Psi(k) \text{ if } K<1,\\ \max\{\Psi(k),\Psi(K)\} \text{ if } k\leqslant 1\leqslant K,\\ \Psi(K) \text{ if } 1<k\end{cases}=F(k,K),$$

then by (3.23) we get

$$\frac{1}{2}\nu(1-\nu)f(k,K)\leqslant 1-\nu+\nu\tau-\tau^{\nu}\leqslant\frac{1}{2}\nu(1-\nu)F(k,K) \tag{3.24}$$

for any $\tau\in[k,K]$ and for any $\nu\in[0,1]$.

By making use of a similar argument as in the proof of Theorem 4 we deduce the desired result (3.21).

REMARK 4. For $0<k\leqslant 1\leqslant K$ we have from (2.6), (3.3) and (3.20) that

$$C(k,K)=\frac{1}{k}\max\left\{(k-1)^2,(K-1)^2\right\},$$

$$D(k,K)=\max\left\{\frac{K(k-1)^2}{k},(K-1)^2\right\}$$

and

$$F(k,K)=\max\left\{\frac{(k-1)^2}{k},(K-1)^2\right\}.$$

We observe that

$$F(k,K)\leqslant C(k,K),\ D(k,K)$$

for $0<k\leqslant 1\leqslant K$, which means that the upper bound for the difference $|x|^2\nabla_\nu|y|^2-x\circledS_\nu y$ provided by (3.21) is better than the corresponding upper bounds from (2.7) and (3.4).

COROLLARY 7. *With the assumptions of Corollary 5 we have*

$$\frac{1}{2}\nu(1-\nu)f\left(m^2,M^2\right)\|x\|^2 \leqslant \left\||x|^2\nabla_\nu|y|^2 - x\circledS_\nu y\right\| \leqslant \frac{1}{2}\nu(1-\nu)F\left(m^2,M^2\right)\|x\|^2 \tag{3.25}$$

for any $\nu\in[0,1]$.

In particular, we have

$$\frac{1}{8}f\left(m^2,M^2\right)\|x\|^2 \leqslant \left\||x|^2\nabla|y|^2 - x\circledS y\right\| \leqslant \frac{1}{8}F\left(m^2,M^2\right)\|x\|^2. \tag{3.26}$$

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