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AN INDEFINITE RANGE INCLUSION THEOREM FOR TRIPLETS OF BOUNDED LINEAR OPERATORS ON A HILBERT SPACE

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Abstract. We study triplets of Hilbert space operators satisfying a certain inequality. A range inclusion theorem with norm estimate for those triplets is given with the language of Kreĭn space geometry and de Branges-Rovnyak space theory.

1. Introduction

In Wu-Seto-Yang [10], we encountered the family of triplets consisting of Toeplitz operators whose twisted sum

$$T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*$$

is an orthogonal projection. Those triplets are closely related to the Hilbert module structure of the Hardy space over the bidisk (see Example 2.2). In this paper, we would like to focus on those triplets with some generalization. More precisely, we consider bounded linear operators T_1, T_2 and T_3 on a Hilbert space $\mathscr{H}$, and assume that triplet (T_1,T_2,T_3) satisfies the following operator inequality:

$$0 \leqslant T_1T_1^* + T_2T_2^* - T_3T_3^* \leqslant I. \tag{1.1}$$

Let $\mathfrak{T}(\mathscr{H})$ denote the set of triplets satisfying (1.1) on a Hilbert space $\mathscr{H}$. For any triplet (T_1,T_2,T_3) in $\mathfrak{T}(\mathscr{H})$, we set

$$T = (T_1T_1^* + T_2T_2^* - T_3T_3^*)^{1/2}.$$

In order to study the structure of T, we introduce de Branges-Rovnyak spaces. Let $\mathscr{M}(A)$ denote the de Branges-Rovnyak space induced by a bounded linear operator A, that is, $\mathscr{M}(A)$ is the Hilbert space consisting of all vectors in $\operatorname{ran} A$ with the pull-back norm

$$\|Ax\|_{\mathscr{M}(A)} = \|P_{(\ker A)^\perp}x\|_{\mathscr{H}} = \min\{\|y\|_{\mathscr{H}} : Ay = Ax\},$$

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where $P_{(\ker A)^\perp}$ is the orthogonal projection from $\mathscr{H}$ onto $(\ker A)^\perp$. Then, applying de Branges-Rovnyak space theory to (1.1), it is immediate that

$$\mathscr{M}(T) \hookrightarrow \mathscr{M}(T) + \mathscr{M}(T_3) = \mathscr{M}(\sqrt{T_1 T_1^* + T_2 T_2^*}) = \mathscr{M}(T_1) + \mathscr{M}(T_2).$$

Thus, $\mathscr{M}(T)$ can be captured as a Hilbert space embedded contractively into $\mathscr{M}(T_1) + \mathscr{M}(T_2)$. However, in this context, it is rather vague how T_3 is involved with $\mathscr{M}(T)$. The purpose of this paper is to study the structure of $\mathscr{M}(T)$ from this point of view, and to explore hidden inner product spaces behind the rather formal identity

$$\mathscr{M}(T) = \mathscr{M}(T_1) + \mathscr{M}(T_2) - \mathscr{M}(T_3).$$

As the main theorem of this paper, we will show the following: for any vector u in $\mathscr{M}(T)$, there exists some vector $\mathbf{z}_\varepsilon = (z_1(\varepsilon), z_2(\varepsilon), z_3(\varepsilon))^t$ in $\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ such that:

(i) $T_1 z_1(\varepsilon) + T_2 z_2(\varepsilon) - T_3 z_3(\varepsilon) \to u \ (\varepsilon \to 0)$ in the strong topology of $\mathscr{H}$,

(ii) $0 \leqslant \|z_1(\varepsilon)\|_{\mathscr{H}}^2 + \|z_2(\varepsilon)\|_{\mathscr{H}}^2 - \|z_3(\varepsilon)\|_{\mathscr{H}}^2 \uparrow \|u\|_{\mathscr{M}(T)}^2 \ (\varepsilon \downarrow 0)$,

where $\|u\|_{\mathscr{M}(T)}$ denotes the norm of u in $\mathscr{M}(T)$. We would like to emphasize that the above norm estimate is nontrivial, and this result might be written as follows:

$$\mathscr{M}(T) \hookrightarrow \mathscr{M}(T_1) + \mathscr{M}(T_2) - \mathscr{M}(T_3).$$

This paper is organized as follows. In Section 2, by giving examples from operator theory on Hardy spaces, it is shown that $\mathfrak{T}(\mathscr{H})$ is nontrivial. In Section 3, we study inner product spaces induced by triplets in $\mathfrak{T}(\mathscr{H})$, and prove the main theorem (Theorem 3.1). In Section 4, we investigate the local structure of range spaces of operators appearing in Section 3. In Section 5, we consider an indefinite Toeplitz corona problem.

2. Examples

Trivial examples of triplets in $\mathfrak{T}(\mathscr{H})$ are easily obtained from Douglas' range inclusion theorem. We shall see that $\mathfrak{T}(\mathscr{H})$ is nontrivial. Let H^2 be the Hardy space over the open unit disk $\mathbb{D}$ in the complex plane, and let H^∞ be the Banach algebra consisting of all bounded analytic functions on $\mathbb{D}$. For any function φ in H^∞, T_φ denotes the Toeplitz operator with symbol φ.

EXAMPLE 2.1. We choose φ_1 and φ_2 from H^∞ satisfying $\|\begin{pmatrix} T_{\varphi_1} & T_{\varphi_2} \end{pmatrix}\| \leqslant 1$. Then this norm inequality is equivalent to that $0 \leqslant T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* \leqslant I$. Further, we choose ψ_1 and ψ_2 from H^∞ satisfying

$$\left\| \begin{pmatrix} T_{\psi_1} \\ T_{\psi_2} \end{pmatrix} \right\| \leqslant 1.$$

Then, setting

$$\varphi_3 = \varphi_1 \psi_1 + \varphi_2 \psi_2 = \begin{pmatrix} \varphi_1 & \varphi_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

by the generalized Toeplitz-corona theorem (see Theorem 8.57 in Agler-McCarthy [1]), we have that

$$0 \leqslant T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^* \leqslant T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* \leqslant I.$$

Our study has been motivated by the next example.

EXAMPLE 2.2. (Wu-Seto-Yang [10]) We consider the tensor product Hilbert space $H^2 \otimes H^2$, which is isomorphic to the Hardy space over the bidisk $\mathbb{D}^2$. Let z and w denote coordinate functions, and let T_z and T_w be Toeplitz operators with symbols z and w, respectively. We note that T_z and T_w are doubly commuting isometries on $H^2 \otimes H^2$. In fact, T_z and T_w are identified with $T_z \otimes I$ and $I \otimes T_w$, respectively. Now, since orthogonal projections $T_zT_z^*$ and $T_w(I-T_zT_z^*)T_w^*$ are commuting,

$$T_zT_z^* + T_wT_w^* - T_{zw}T_{zw}^* = T_zT_z^* + T_w(I-T_zT_z^*)T_w^*,$$

is the orthogonal projection onto $(H^2 \otimes H^2) \ominus \mathbb{C}$. Hence (T_z, T_w, T_{zw}) belongs to $\mathfrak{T}(H^2 \otimes H^2)$. Further nontrivial examples can be obtained from the module structure of $H^2 \otimes H^2$. Let $\mathscr{M}$ be a closed subspace of $H^2 \otimes H^2$. Then $\mathscr{M}$ is called a submodule if $\mathscr{M}$ is invariant for T_z and T_w. For many examples of submodules in $H^2 \otimes H^2$, there exist bounded analytic functions φ_1, φ_2 and φ_3 on $\mathbb{D}^2$ such that

$$T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^* = P_{\mathscr{M}},$$

where $P_{\mathscr{M}}$ denotes the orthogonal projection onto $\mathscr{M}$, and

$$T_{\varphi_1}^*T_{\varphi_1} + T_{\varphi_2}^*T_{\varphi_2} - T_{\varphi_3}^*T_{\varphi_3} = I.$$

3. Indefinite range inclusion

Setting

$$\mathscr{H}_+ = \mathscr{H} \oplus \mathscr{H}, \quad \mathscr{H}_- = \mathscr{H} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we consider the Kreĭn space $\mathscr{K} = (\mathscr{H}_+ \oplus \mathscr{H}_-, J)$, that is, for any vectors $\mathbf{x} = (x_1, x_2, x_3)^t$ and $\mathbf{y} = (y_1, y_2, y_3)^t$ in $\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$, the inner product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathscr{K}$ is defined to be

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathscr{K}} = \langle J\mathbf{x}, \mathbf{y} \rangle_{\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}} = \langle x_1, y_1 \rangle_{\mathscr{H}} + \langle x_2, y_2 \rangle_{\mathscr{H}} - \langle x_3, y_3 \rangle_{\mathscr{H}}.$$

For basic Kreĭn space geometry, see Dritschel-Rovnyak [7].

Let (T_1, T_2, T_3) be a triplet in $\mathfrak{T}(\mathscr{H})$. Then we define a linear operator $\mathbb{T}$ as follows:

$$\mathbb{T} : \mathscr{K} \to \mathscr{H}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto T_1x_1 + T_2x_2 - T_3x_3.$$

The adjoint operator $\mathbb{T}^\sharp$ of $\mathbb{T}$ with respect to inner products of $\mathscr{H}$ and $\mathscr{K}$ is obtained as follows:

$$\begin{aligned}\langle x, \mathbb{T}(x_1,x_2,x_3)^t\rangle_{\mathscr{H}} &= \langle x, T_1x_1+T_2x_2-T_3x_3\rangle_{\mathscr{H}} = \langle T_1^*x,x_1\rangle_{\mathscr{H}}+\langle T_2^*x,x_2\rangle_{\mathscr{H}}-\langle T_3^*x,x_3\rangle_{\mathscr{H}}\\ &= \langle (T_1^*x,T_2^*x,T_3^*x)^t,(x_1,x_2,x_3)^t\rangle_{\mathscr{K}},\end{aligned}$$

that is, we have that

$$\mathbb{T}^\sharp : \mathscr{H} \to \mathscr{K}, \quad x \mapsto \begin{pmatrix} T_1^*x \\ T_2^*x \\ T_3^*x \end{pmatrix}.$$

In particular, we have that

$$\mathbb{T}\mathbb{T}^\sharp x = T_1T_1^*x + T_2T_2^*x - T_3T_3^*x.$$

For any (T_1,T_2,T_3) in $\mathfrak{T}(\mathscr{H})$, we set

$$T = (T_1T_1^* + T_2T_2^* - T_3T_3^*)^{1/2}.$$

Note that T is positive and contractive. Consider the operator $V : \operatorname{ran} T \to \mathbb{T}^\sharp(\ker T)^\perp$ defined by

$$VTx = \begin{pmatrix} T_1^*x \\ T_2^*x \\ T_3^*x \end{pmatrix} \quad (x \in (\ker T)^\perp).$$

Then it follows from the identity

$$\|Tx\|_{\mathscr{H}}^2 = \|T_1^*x\|_{\mathscr{H}}^2 + \|T_2^*x\|_{\mathscr{H}}^2 - \|T_3^*x\|_{\mathscr{H}}^2 = \langle \mathbb{T}^\sharp x, \mathbb{T}^\sharp x\rangle_{\mathscr{K}} \tag{3.1}$$

that V is an isometry and $\mathbb{T}^\sharp(\ker T)^\perp$ is a pre-Hilbert space. Let $\mathscr{K}_0$ be the completion of $\mathbb{T}^\sharp(\ker T)^\perp$ with the norm induced by (3.1), and let $\widetilde{V} : \overline{\operatorname{ran}} T \to \mathscr{K}_0$ denote the isometric extension of V, in fact, $\widetilde{V}$ is unitary. Then $\mathbb{T}^\sharp = \widetilde{V}T$ on $(\ker T)^\perp$ gives the polar decomposition of $\mathbb{T}^\sharp$, that is, the following diagram is commutative. Further, it follows

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\mathbb{T}^\sharp} & \mathscr{K}_0 \\ {\scriptstyle T}\downarrow & \nearrow_{\widetilde{V}} & \\ \mathscr{L} & & \end{array} \qquad (\mathscr{L} := (\ker T)^\perp = \overline{\operatorname{ran}} T) \tag{3.2}$$

from (3.1) that $\mathbb{T}^\sharp$ is bounded in (3.2). Hence, we can take the Hilbert space adjoint $\widetilde{\mathbb{T}}$ of $\mathbb{T}^\sharp$ in (3.2). We summarize basic properties of $\widetilde{\mathbb{T}}$ in the following proposition:

PROPOSITION 3.1. *Let $\widetilde{\mathbb{T}} : \mathscr{K}_0 \to \mathscr{L}$ be the Hilbert space adjoint of $\mathbb{T}^\sharp$ in the sense of* (3.2)*. Then:*

(i) $\widetilde{\mathbb{T}}$ *is injective,*

(ii) $\widetilde{\mathbb{T}}$ *is the extension of* $\mathbb{T}|_{\mathbb{T}^\sharp \mathscr{L}}$,

(iii) $\operatorname{ran}\widetilde{\mathbb{T}}$ *is dense in* $\mathscr{L}$.

Proof. Suppose that $\widetilde{\mathbb{T}}\mathbf{x} = 0$ for some $\mathbf{x}$ in $\mathscr{K}_0$. Then there exists a sequence $\{w_n\}_n$ in $\mathscr{L} = (\ker T)^\perp$ such that $\|\mathbf{x} - \mathbb{T}^\sharp w_n\|_{\mathscr{K}_0} \to 0$ as n tends to infinity. Hence we have that

$$\|\mathbf{x}\|^2_{\mathscr{K}_0} = \langle \mathbf{x}, \mathbf{x}\rangle_{\mathscr{K}_0} = \lim_{n\to\infty} \langle \mathbf{x}, \mathbb{T}^\sharp w_n\rangle_{\mathscr{K}_0} = \lim_{n\to\infty} \langle \widetilde{\mathbb{T}}\mathbf{x}, w_n\rangle_{\mathscr{H}} = 0.$$

Thus we have (i). Further, since

$$\langle \mathbb{T}\mathbb{T}^\sharp x, y\rangle_{\mathscr{H}} = \langle \mathbb{T}^\sharp x, \mathbb{T}^\sharp y\rangle_{\mathscr{K}} = \langle \mathbb{T}^\sharp x, \mathbb{T}^\sharp y\rangle_{\mathscr{K}_0} = \langle \widetilde{\mathbb{T}}\mathbb{T}^\sharp x, y\rangle_{\mathscr{H}} \quad (x, y \in \mathscr{L}),$$

we have (ii). It follows from (ii) that

$$\mathscr{L} \supset \widetilde{\mathbb{T}}\mathscr{K}_0 \supset \mathbb{T}\mathbb{T}^\sharp \mathscr{L} = \operatorname{ran} T^2.$$

This concludes (iii). □

Let $\mathscr{M}(T)$ denote the de Branges-Rovnyak space induced by T, that is, $\mathscr{M}(T)$ is the Hilbert space consisting of all vectors in $\operatorname{ran} T$ with the pull-back norm

$$\|Tx\|_{\mathscr{M}(T)} = \|P_{(\ker T)^\perp} x\|_{\mathscr{H}} = \min\{\|y\|_{\mathscr{H}} : Ty = Tx\},$$

where $P_{(\ker T)^\perp}$ is the orthogonal projection from $\mathscr{H}$ onto $(\ker T)^\perp$.

THEOREM 3.1. *Let $\mathscr{H}$ be a Hilbert space. For any triplet (T_1, T_2, T_3) in $\mathfrak{T}(\mathscr{H})$, we set*

$$T = (T_1T_1^* + T_2T_2^* - T_3T_3^*)^{1/2}.$$

If u belongs to $\mathscr{M}(T)$, then, for any $\varepsilon > 0$, there exists some vector $\mathbf{z}_\varepsilon = (z_1(\varepsilon), z_2(\varepsilon), z_3(\varepsilon))^t$ in $\mathbb{T}^\sharp(\ker T)^\perp$ such that:

(i) $T_1z_1(\varepsilon) + T_2z_2(\varepsilon) - T_3z_3(\varepsilon) \to u$ $(\varepsilon \to 0)$ *in the strong topology of* $\mathscr{H}$,

(ii) $0 \leqslant \|z_1(\varepsilon)\|^2_{\mathscr{H}} + \|z_2(\varepsilon)\|^2_{\mathscr{H}} - \|z_3(\varepsilon)\|^2_{\mathscr{H}} \uparrow \|u\|^2_{\mathscr{M}(T)}$ $(\varepsilon \downarrow 0)$,

(iii) $\mathbf{z}_\varepsilon$ *converges to some vector* $\mathbf{z}$ *in the strong topology of* $\mathscr{K}_0$ *and* $u = \widetilde{\mathbb{T}}\mathbf{z}$.

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of T, and we set

$$E_\lambda^\perp = I_{\mathscr{H}} - E_\lambda = E((\lambda, \infty)).$$

Suppose that $u = Tx$ where x is in $\mathscr{L} = (\ker T)^\perp$. Then, for arbitrary $\varepsilon > 0$, put

$$x_\varepsilon = E_\varepsilon^\perp x, \quad y_\varepsilon = \left(\int_\varepsilon^\infty \frac{1}{\lambda}\, dE_\lambda\right) x_\varepsilon \quad \text{and} \quad \mathbf{z}_\varepsilon = \mathbb{T}^\sharp y_\varepsilon.$$

We note that y_ε and $\mathbf{z}_\varepsilon$ belong to $\mathscr{L}$ and $\mathscr{K}_0$, respectively. Then we have that

$$\mathbb{T}\mathbf{z}_\varepsilon = \mathbb{T}\mathbb{T}^\sharp y_\varepsilon = T^2 y_\varepsilon = T^2 \left(\int_\varepsilon^\infty \frac{1}{\lambda}\, dE_\lambda \right) x_\varepsilon = T x_\varepsilon.$$

Hence we have that

$$\|u - \mathbb{T}\mathbf{z}_\varepsilon\|_{\mathscr{H}} = \|Tx - Tx_\varepsilon\|_{\mathscr{H}} \leqslant \|x - x_\varepsilon\|_{\mathscr{H}} \to 0 \quad (\varepsilon \to 0).$$

Thus we have (i). Further, it follows from $Ty_\varepsilon = x_\varepsilon$ that

$$\begin{aligned} \|z_1(\varepsilon)\|_{\mathscr{H}}^2 + \|z_2(\varepsilon)\|_{\mathscr{H}}^2 - \|z_3(\varepsilon)\|_{\mathscr{H}}^2 &= \|T_1^* y_\varepsilon\|_{\mathscr{H}}^2 + \|T_2^* y_\varepsilon\|_{\mathscr{H}}^2 - \|T_3^* y_\varepsilon\|_{\mathscr{H}}^2 = \|T y_\varepsilon\|_{\mathscr{H}}^2 \\ &= \|x_\varepsilon\|_{\mathscr{H}}^2 \leqslant \|x\|_{\mathscr{H}}^2. \end{aligned}$$

This concludes (ii). Finally, since

$$\begin{aligned} \|\mathbf{z}_\varepsilon - \mathbf{z}_\delta\|_{\mathscr{K}_0}^2 &= \|\mathbb{T}^\sharp y_\varepsilon - \mathbb{T}^\sharp y_\delta\|_{\mathscr{K}_0} \\ &= \|T_1^*(y_\varepsilon - y_\delta)\|_{\mathscr{H}}^2 + \|T_2^*(y_\varepsilon - y_\delta)\|_{\mathscr{H}}^2 - \|T_3^*(y_\varepsilon - y_\delta)\|_{\mathscr{H}}^2 \\ &= \|T(y_\varepsilon - y_\delta)\|_{\mathscr{H}}^2 = \|x_\varepsilon - x_\delta\|_{\mathscr{H}}^2 \to 0 \quad (\varepsilon, \delta \to 0), \end{aligned}$$

$\mathbf{z}_\varepsilon$ converges to some vector $\mathbf{z}$ in $\mathscr{K}_0$, and

$$u = \lim_{\varepsilon \to 0} \mathbb{T}\mathbf{z}_\varepsilon = \lim_{\varepsilon \to 0} \widetilde{\mathbb{T}}\mathbf{z}_\varepsilon = \widetilde{\mathbb{T}}\mathbf{z}.$$

Thus we have (iii). □

COROLLARY 3.1. *Suppose that T is of finite rank. If u belongs to $\mathscr{M}(T)$, then there exists some $\mathbf{z} = (z_1, z_2, z_3)^t$ in $\mathbb{T}^\sharp(\ker T)^\perp$ such that:*

$$T_1 z_1 + T_2 z_2 - T_3 z_3 = u$$

and

$$\|z_1\|_{\mathscr{H}}^2 + \|z_2\|_{\mathscr{H}}^2 - \|z_3\|_{\mathscr{H}}^2 = \|u\|_{\mathscr{M}(T)}^2.$$

Proof. If $\varepsilon > 0$ is sufficiently small, then $E_\varepsilon^\perp = P_{(\ker T)^\perp}$. □

REMARK 3.1. It is easy to see that Theorem 3.1 can be generalized to any finite operator tuple $(T_1, \dots, T_m, T_{m+1}, \dots, T_n)$ satisfying

$$\sum_{j=1}^m T_j T_j^* - \sum_{k=m+1}^n T_k T_k^* \geqslant 0.$$

In the proof of Theorem 3.1, we essentially showed that $\mathscr{M}(T)$ is contractively embedded into $\mathscr{M}(\widetilde{\mathbb{T}})$. Moreover, applying the same method in the proof of Theorem 4.3 stated in the next section, we can conclude that the converse is also true, that is, $\mathscr{M}(T) = \mathscr{M}(\widetilde{\mathbb{T}})$ as Hilbert spaces. However, $\mathscr{K}_0$ and $\widetilde{\mathbb{T}}$ seem to be rather elusive objects. Thus, in the next section, we will investigate the local structure of range spaces of T and $\mathbb{T}$.

4. Local structure of range spaces

In this section, we need some facts from de Branges-Rovnyak space theory and Kreĭn space geometry. Let $\mathscr{H}$ and $\mathscr{G}$ be Hilbert spaces, and let A be any bounded linear operator from $\mathscr{H}$ to $\mathscr{G}$. The following theorem seems to be well known to specialists in Hilbert space operator theory.

THEOREM 4.1. *Let A be a bounded linear operator from $\mathscr{H}$ to $\mathscr{G}$ and let u be a vector in $\mathscr{G}$. Then u belongs to* $\operatorname{ran} A$ *if and only if*

$$\gamma := \sup_{A^*y \neq 0} \frac{|\langle y, u\rangle_{\mathscr{G}}|}{\|A^*y\|_{\mathscr{H}}}$$

is finite. Further, then $\|u\|_{\mathscr{M}(A)} = \gamma$.

The first half of Theorem 4.1 is known as Shmuly'an's theorem (see Corollary 2 of Theorem 2.1 in Fillmore-Williams [8]). For the second half (norm identity) and also the proof, we referred to Ando [2].

DEFINITION 4.1. Let $\mathscr{K}$ be a Kreĭn space. A subspace $\mathfrak{M}$ of $\mathscr{K}$ is said to be uniformly positive with lower bound $\delta > 0$ if

$$\langle \mathbf{x}, \mathbf{x}\rangle_{\mathscr{K}} \geqslant \delta \langle J\mathbf{x}, \mathbf{x}\rangle_{\mathscr{K}} \quad (\mathbf{x} \in \mathfrak{M}).$$

In particular, for the Kreĭn space defined in Section 3, if $\mathfrak{M}$ is uniformly positive with lower bound $\delta > 0$, then

$$\|x_1\|_{\mathscr{H}}^2 + \|x_2\|_{\mathscr{H}}^2 - \|x_3\|_{\mathscr{H}}^2 \geqslant \delta(\|x_1\|_{\mathscr{H}}^2 + \|x_2\|_{\mathscr{H}}^2 + \|x_3\|_{\mathscr{H}}^2) \quad ((x_1, x_2, x_3)^t \in \mathfrak{M}).$$

THEOREM 4.2. *Let $\mathscr{K}$ be a Kreĭn space. Every uniformly positive subspace of $\mathscr{K}$ with lower bound $\delta > 0$ is contained in a maximal uniformly positive subspace with lower bound $\delta > 0$, and every maximal uniformly positive subspace is a Hilbert space with the inner product of $\mathscr{K}$.*

For the details of Definition 4.1 and Theorem 4.2, see Dritschel-Rovnyak [7].

LEMMA 4.1. *Let (T_1, T_2, T_3) be any triplet in $\mathfrak{T}(\mathscr{H})$. Then, for any $\varepsilon > 0$, $\mathfrak{M}_\varepsilon = \{(T_1^*x, T_2^*x, T_3^*x)^t : x \in E_\varepsilon^\perp \mathscr{H}\}$ is uniformly positive.*

Proof. For any $\varepsilon > 0$ and any vector x in $E_\varepsilon^\perp \mathscr{H}$,

$$\begin{aligned}\|T_1^*x\|_{\mathscr{H}}^2 + \|T_2^*x\|_{\mathscr{H}}^2 - \|T_3^*x\|_{\mathscr{H}}^2 = \|Tx\|_{\mathscr{H}}^2 &= \int_\varepsilon^\infty |\lambda|^2 \, d\|E_\lambda x\|_{\mathscr{H}}^2 \geqslant \varepsilon^2 \|x\|_{\mathscr{H}}^2 \\ &\geqslant \frac{\varepsilon^2}{3\max_{1 \leqslant j \leqslant 3} \|T_j^*\|^2} (\|T_1^*x\|_{\mathscr{H}}^2 + \|T_2^*x\|_{\mathscr{H}}^2 + \|T_3^*x\|_{\mathscr{H}}^2).\end{aligned}$$

Hence $\mathfrak{M}_\varepsilon$ is uniformly positive. $\square$

Let $\widetilde{\mathfrak{M}_\varepsilon}$ be a maximal uniformly positive subspace containing $\mathfrak{M}_\varepsilon$. Then, $\widetilde{\mathfrak{M}_\varepsilon}$ is a Hilbert space with the inner product of $\mathscr{K}$ by Theorem 4.2, and hence so is $\overline{\mathfrak{M}_\varepsilon}$, the closure of $\mathfrak{M}_\varepsilon$ in $\widetilde{\mathfrak{M}_\varepsilon}$. We note that $\overline{\mathfrak{M}_\varepsilon}$ is uniformly positive, that is, there exists some $\delta > 0$ such that the following inequality holds:

$$\|x\|_{\mathscr{H}}^2 + \|y\|_{\mathscr{H}}^2 - \|z\|_{\mathscr{H}}^2 \geqslant \delta(\|x\|_{\mathscr{H}}^2 + \|y\|_{\mathscr{H}}^2 + \|z\|_{\mathscr{H}}^2) \quad ((x,y,z)^t \in \overline{\mathfrak{M}_\varepsilon}). \tag{4.1}$$

LEMMA 4.2. *Let (T_1,T_2,T_3) be any triplet in $\mathfrak{T}(\mathscr{H})$. Then, for any $\varepsilon > 0$, $\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}} : \overline{\mathfrak{M}_\varepsilon} \to \mathscr{H}$ is bounded as a Hilbert space operator.*

Proof. Since $\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}}$ is defined everywhere in $\overline{\mathfrak{M}_\varepsilon}$, by the closed graph theorem, it suffices to show that $\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}}$ is closed. Suppose that

$$\|x_n - x\|_{\mathscr{H}}^2 + \|y_n - y\|_{\mathscr{H}}^2 - \|z_n - z\|_{\mathscr{H}}^2 \to 0 \quad (n \to \infty)$$

and

$$\|T_1x_n + T_2y_n - T_3z_n - u\|_{\mathscr{H}}^2 \to 0 \quad (n \to \infty).$$

Then it follows from (4.1) that x_n, y_n and z_n converge to x, y and z in $\mathscr{H}$, respectively. Hence we have that $u = T_1x + T_2y - T_3z$. This concludes that $\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}}$ is closed. □

We will deal with $\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}}$ as a Hilbert space operator from $\overline{\mathfrak{M}_\varepsilon}$ to $\mathscr{H}$.

LEMMA 4.3. *Let (T_1,T_2,T_3) be any triplet in $\mathfrak{T}(\mathscr{H})$. Then, for any $\varepsilon > 0$, $\mathscr{M}(T|_{E_\varepsilon^\perp \mathscr{H}})$ is contractively embedded into $\mathscr{M}(\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}})$.*

Proof. Suppose that $u = Tx_\varepsilon$ where x_ε is in $E_\varepsilon^\perp \mathscr{H}$. In the proof of Theorem 3.1, we showed that

$$\mathbf{z}_\varepsilon = \mathbb{T}^\sharp y_\varepsilon = \begin{pmatrix} T_1^* y_\varepsilon \\ T_2^* y_\varepsilon \\ T_3^* y_\varepsilon \end{pmatrix}$$

satisfies $\mathbb{T}\mathbf{z}_\varepsilon = Tx_\varepsilon = u$ and

$$0 \leqslant \langle \mathbf{z}_\varepsilon, \mathbf{z}_\varepsilon \rangle_{\mathscr{K}} = \|z_1(\varepsilon)\|_{\mathscr{H}}^2 + \|z_2(\varepsilon)\|_{\mathscr{H}}^2 - \|z_3(\varepsilon)\|_{\mathscr{H}}^2 = \|x_\varepsilon\|_{\mathscr{H}}^2 = \|u\|_{\mathscr{M}(T|_{E_\varepsilon^\perp \mathscr{H}})}^2.$$

Moreover, this $\mathbf{z}_\varepsilon$ belongs to $\mathfrak{M}_\varepsilon$. □

The next theorem is a generalization of Corollary 3.1.

THEOREM 4.3. Let (T_1,T_2,T_3) be any triplet in $\mathfrak{T}(\mathscr{H})$. Then, for any $\varepsilon > 0$,

$$\mathscr{M}(T|_{E_\varepsilon^\perp \mathscr{H}}) = \mathscr{M}(\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}})$$

as Hilbert spaces.

Proof. By Lemma 4.3, it suffices to show that $\mathscr{M}(\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}})$ is contractively embedded into $\mathscr{M}(T|_{E_\varepsilon^\perp \mathscr{H}})$. For any $\mathbb{T}\mathbf{x}$ where $\mathbf{x} = (x_1, x_2, x_3)^t$ in $\overline{\mathfrak{M}_\varepsilon}$ and any x in $E_\varepsilon^\perp \mathscr{H}$, we have that

$$\begin{aligned}|\langle x, \mathbb{T}\mathbf{x}\rangle_{\mathscr{H}}|^2 &= |\langle \mathbb{T}^\sharp x, \mathbf{x}\rangle_{\mathscr{K}}|^2 \\ &\leqslant (\|T_1^* x\|_{\mathscr{H}}^2 + \|T_2^* x\|_{\mathscr{H}}^2 - \|T_3^* x\|_{\mathscr{H}}^2)(\|x_1\|_{\mathscr{H}}^2 + \|x_2\|_{\mathscr{H}}^2 - \|x_3\|_{\mathscr{H}}^2)\end{aligned}$$

because $T^\sharp \mathbf{x} = (T_1^* x, T_2^* x, T_3^* x)^t$ and $(x_1, x_2, x_3)^t$ belong to $\overline{\mathfrak{M}_\varepsilon}$. Hence

$$\sup_{x \in E_\varepsilon^\perp \mathscr{H} \setminus \{0\}} \frac{|\langle x, \mathbb{T}\mathbf{x}\rangle_{\mathscr{H}}|^2}{\|T_1^* x\|_{\mathscr{H}}^2 + \|T_2^* x\|_{\mathscr{H}}^2 - \|T_3^* x\|_{\mathscr{H}}^2} = \sup_{x \in E_\varepsilon^\perp \mathscr{H} \setminus \{0\}} \frac{|\langle x, \mathbb{T}\mathbf{x}\rangle_{\mathscr{H}}|^2}{\|Tx\|_{\mathscr{H}}^2}$$

is finite. By Theorem 4.1, $\mathbb{T}\mathbf{x}$ belongs to $\operatorname{ran} T|_{E_\varepsilon^\perp \mathscr{H}}$ and

$$\|\mathbb{T}\mathbf{x}\|_{\mathscr{M}(T|_{E_\varepsilon^\perp \mathscr{H}})}^2 \leqslant \|x_1\|_{\mathscr{H}}^2 + \|x_2\|_{\mathscr{H}}^2 - \|x_3\|_{\mathscr{H}}^2.$$

Thus we have that $\|\mathbb{T}\mathbf{x}\|_{\mathscr{M}(T|_{E_\varepsilon^\perp \mathscr{H}})}^2 \leqslant \|\mathbb{T}\mathbf{x}\|_{\mathscr{M}(\mathbb{T}|_{\overline{\mathfrak{M}_\varepsilon}})}^2$. $\square$

5. Application

We shall consider a Hilbert space $\mathscr{H}$ consisting of analytic functions on $\mathbb{D}$. Further, we assume that constant function 1 is in $\mathscr{H}$ and multiplication operators induced by functions in H^∞ are bounded on $\mathscr{H}$. For example, the Hardy space and the Bergman space belong to this class. T_φ will denote the multiplication operator on $\mathscr{H}$ induced by φ in H^∞. Now, we shall consider an indefinite Toeplitz corona problem. Let δ be a positive real number. Suppose

$$T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^* \geqslant \delta I_{\mathscr{H}}.$$

Then

$$T = (T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^*)^{1/2}$$

is invertible and $\|T^{-1}\| \leqslant 1/\sqrt{\delta}$. Hence there exists a function F in $\mathscr{H}$ such that

$$1 = T^2 F = \varphi_1 \psi_1 + \varphi_2 \psi_2 - \varphi_3 \psi_3 \quad (\psi_j := T_{\varphi_j}^* F)$$

and trivially

$$\|\psi_j\|_{\mathscr{H}}^2 \leqslant \frac{\|T_{\varphi_j}^*\|^2}{\delta} \|1\|_{\mathscr{H}}^2.$$

It will be nontrivial problem to improve this norm estimate. On the other hand, applying Theorem 3.1 to this setting, since $\mathscr{M}(T) = \mathscr{H}$ as vector spaces and

$$\|1\|_{\mathscr{M}(T)} = \|T^{-1} 1\|_{\mathscr{H}} \leqslant \frac{1}{\sqrt{\delta}} \|1\|_{\mathscr{H}},$$

there exist functions $\psi_{1,n}$, $\psi_{2,n}$ and $\psi_{3,n}$ in $\mathscr{H}$ such that:

(i) $\lim_{n\to\infty}(\varphi_1\psi_{1,n}+\varphi_2\psi_{2,n}-\varphi_3\psi_{3,n})=1$,

(ii) $0\leqslant \lim_{n\to\infty}(\|\psi_{1,n}\|^2_{\mathscr{H}}+\|\psi_{2,n}\|^2_{\mathscr{H}}-\|\psi_{3,n}\|^2_{\mathscr{H}})\leqslant \frac{1}{\delta}\|1\|^2_{\mathscr{H}}$.

Thus, we obtain a norm estimate of the approximate solution, and we note that estimate (ii) depends only on δ and $\mathscr{H}$.

REMARK 5.1. It is well known that Toeplitz corona theorem is based on the complete Pick property of the Szegö kernel. We should note that the Bergman kernel does not have the property, but our method can be applied. For indefinite operator theory on the Hardy space, see Ball-Helton [3, 4, 5, 6] and Helton-Ball-Johnson-Palmer [9].

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