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INEQUALITIES FOR THE λ–WEIGHTED MIXED ARITHMETIC–GEOMETRIC–HARMONIC MEANS OF SECTOR MATRICES

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Abstract. In this note, we first explain a minor error in the literature [3]. Secondly, we prove the λ-weighted mixed arithmetic-geometric-harmonic-mean inequalities of A and B which are the generalizations of the results already introduced in [3]. Finally, we extend our results to sums of n $(n \geqslant 2)$ sector matrices.

1. Introduction

For $A \in \mathbb{M}_n(\mathbb{C})$, we write $A \geqslant 0$ if A is positive semidefinite (i.e., $x^*Ax \geqslant 0$ for all $x \in \mathbb{C}^n$) and $A > 0$ if A is positive definite (i.e., $x^*Ax > 0$ for all nonzero $x \in \mathbb{C}^n$). For two Hermitian matrices A and B of the same size, we denote $A \geqslant B$ if $A - B \geqslant 0$. As is well known, every matrix $A \in \mathbb{M}_n(\mathbb{C})$ can be decomposed as $A = \Re A + i\Im A$, where the Hermitian matrices $\Re A = \frac{A+A^*}{2}$ and $\Im A = \frac{A-A^*}{2i}$ are called the real and imaginary parts of A, respectively. This is called the Cartesian decomposition of A. The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{x^*Ax | x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \pi/2)$, let

$$S_\alpha = \{z \in \mathbb{C} | \Re z \geqslant 0, |\Im z| \leqslant (\Re z)\tan(\alpha)\}$$

be a sector region on the complex plane. A matrix whose numerical ranges are contained in a sector region S_α is called a sector matrix [5]. Recent research interest in this class of matrices starts with a resolution of a problem from numerical analysis [1]. Some research results on sector matrices can be found in [1, 4, 5, 9]. Now we introduce the main object in this note. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called accretive if $\Re A$ is positive definite. If $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive definite, then the geometric mean

$$A \sharp B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}},$$

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is well studied, while the arithmetic mean is defined by

$$A\nabla B := \frac{A+B}{2},$$

and the harmonic mean is defined by

$$A!B := \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}.$$

If A,B are positive definite and $\lambda \in (0,1)$ is a real number, then the following quantities

$$A\nabla_\lambda B := (1-\lambda)A+\lambda B, \tag{1}$$
$$A!_\lambda B := ((1-\lambda)A^{-1}+\lambda B^{-1})^{-1},$$
$$A\sharp_\lambda B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}, \tag{2}$$

are known, in the literature, as the λ-weighted arithmetic, λ-weighted harmonic and λ-weighted geometric means of A and B, respectively. If $\lambda = \frac{1}{2}$, they are simply denoted by $A\nabla B$, $A!B$ and $A\sharp B$, respectively. The following inequalities are well known in the literature:

$$A!_\lambda B \leqslant A\sharp_\lambda B \leqslant A\nabla_\lambda B. \tag{3}$$

Now, let A,B be sector matrices and $\lambda \in (0,1)$. It is easy to see that the set of all sector matrices acting on $\mathbb{C}$ is a convex cone of $\mathbb{M}_n(\mathbb{C})$. Further, A^{-1} and B^{-1} are also sector matrices. Consequently, $A\nabla_\lambda B$ and $A!_\lambda B$ can be defined by the same formulas as previously when $A,B \in \mathbb{M}_n(\mathbb{C})$ are two sector matrices. However, by virtue of the presence of non-integer exponents for matrices in (2), the λ-weighted geometric mean $A\sharp_\lambda B$ for two sector matrices can not be defined by (2). Raïssouli et al. [8] defined the λ-weighted geometric mean $A\sharp_\lambda B$ for two sector matrices $A,B \in \mathbb{M}_n(\mathbb{C})$ via the following formula

$$\begin{aligned} A\sharp_\lambda B &:= \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}(A^{-1}+tB^{-1})^{-1}dt \\ &= \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}A(B+tA)^{-1}Bdt. \end{aligned} \tag{4}$$

When $\lambda = \frac{1}{2}$, Drury [2] defined the geometric mean for two sector matrices $A,B \in \mathbb{M}_n(\mathbb{C})$ via the formula

$$A\sharp B := \left(\frac{2}{\pi}\int_0^\infty (tA+t^{-1}B)^{-1}\frac{dt}{t}\right)^{-1}. \tag{5}$$

It is proved in [2] that $A\sharp B = B\sharp A$ and $A\sharp B = (A^{-1}\sharp B^{-1})^{-1}$ for any sector matrices $A,B \in \mathbb{M}_n(\mathbb{C})$. It follows that (5) is equivalent to:

$$A\sharp B = \frac{2}{\pi}\int_0^\infty (tA^{-1}+t^{-1}B^{-1})^{-1}\frac{dt}{t} = \frac{2}{\pi}\int_0^\infty A(tB+t^{-1}A)^{-1}B\frac{dt}{t}.$$

Mond and Pečarić (see [7, Theorem 2 and Theorem 3]) proved the following mixed arithmetic-geometric-mean inequality and mixed harmonic-geometric-mean inequality.

THEOREM 1.1. *(see* [7, Theorem 2]*) Let A and B be positive definite matrices. The mixed arithmetic-geometric-mean inequality is valid, i.e.,*

$$A\nabla(A\sharp B) \leqslant A\sharp(A\nabla B).$$

THEOREM 1.2. *(see* [7, Theorem 3]*) Let A and B be positive definite matrices. The mixed harmonic-geometric-mean inequality holds, i.e.,*

$$A\sharp(A!B) \leqslant A!(A\sharp B).$$

Recently, J. Liu et al. [3] presented analogous inequalities for two sector matrices $A,B \in \mathbb{M}_n(\mathbb{C})$ as follows.

THEOREM 1.3. *(see* [3, Theorem 1.2]*) If $A,B \in \mathbb{M}_n(\mathbb{C})$ with $W(A),W(B) \subseteq S_\alpha$, then*

$$\mathfrak{R}[A\nabla(\cos^2(\alpha)(A\sharp B))] \leqslant \mathfrak{R}[A\sharp(A\nabla B)], \tag{6}$$

and

$$\sec^2(\alpha)\mathfrak{R}[A!(\sec^2(\alpha)(A\sharp B))] \leqslant \mathfrak{R}[A\sharp(A!B)]. \tag{7}$$

However, a careful examination of the authors' proof in [3, Theorem 1.2] actually revealed (7) should be the following result

$$\mathfrak{R}[A\sharp(A!B)] \leqslant \sec^2(\alpha)\mathfrak{R}[A!(\sec^2(\alpha)(A\sharp B))]. \tag{8}$$

In this paper, we first extend the results (6) and (8) to inequalities for weighted mixed arithmetic-geometric-harmonic means of two sector matrices. After that, we generalize our results to sums of n $(n \geqslant 2)$ sector matrices.

2. Main results

In this section, we first prove mixed arithmetic-geometric-mean inequality with λ-weighted and mixed harmonic-geometric-mean inequality with λ-weighted for two sector matrices. To do this, we need the following several lemmas.

LEMMA 2.1. *(see* [5, Lemma 2.3 and Lemma 3.2]*) If $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$, then*

$$\begin{aligned}\mathfrak{R}(A^{-1}) &\leqslant (\mathfrak{R}A)^{-1} \\ &\leqslant \sec^2(\alpha)\mathfrak{R}(A^{-1}).\end{aligned} \tag{9}$$

LEMMA 2.2. *(see* [8, Theorem 2.4]*) If* $A,B \in \mathbb{M}_n(\mathbb{C})$ *with* $W(A),W(B) \subseteq S_\alpha$*, then*

$$\Re A \sharp_\lambda \Re B \leqslant \Re(A \sharp_\lambda B). \tag{10}$$

LEMMA 2.3. *Let* $A,B \in \mathbb{M}_n(\mathbb{C})$ *with* $W(A),W(B) \subseteq S_\alpha$*. Then*

$$\begin{aligned} \Re A !_\lambda \Re B &\leqslant \Re(A !_\lambda B) \\ &\leqslant \sec^2(\alpha)(\Re A !_\lambda \Re B). \end{aligned} \tag{11}$$

Proof. The first inequality is due to Raïssouli et al. [8, Lemma 2.3]. Now, we compute

$$\begin{aligned} \Re(A !_\lambda B) &= \Re(\lambda A^{-1} + (1-\lambda)B^{-1})^{-1} \\ &\leqslant \left[\Re\left(\lambda A^{-1} + (1-\lambda)B^{-1}\right)\right]^{-1} \quad \text{(by (9))} \\ &= \{\lambda[(\Re A^{-1})^{-1}]^{-1} + (1-\lambda)[(\Re B^{-1})^{-1}]^{-1}\}^{-1} \\ &= (\Re A^{-1})^{-1} !_\lambda (\Re B^{-1})^{-1} \\ &\leqslant (\sec^2(\alpha)\Re A) !_\lambda (\sec^2(\alpha)\Re B) \quad \text{(by (9))} \\ &= \sec^2(\alpha)(\Re A !_\lambda \Re B). \end{aligned}$$

Thus, the second inequality holds. □

LEMMA 2.4. *Let* $A,B \in \mathbb{M}_n(\mathbb{C})$ *with* $W(A),W(B) \subseteq S_\alpha$ *for some* $\alpha \in [0,\pi/2)$*. Then*

$$\Re(A \sharp_\lambda B) \leqslant \sec^2(\alpha)\Re(A \nabla_\lambda B), \ \ \lambda \in (0,1).$$

Proof. Compute

$$\begin{aligned} \Re(A \sharp_\lambda B) &= \Re\left[\frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}(A^{-1}+tB^{-1})^{-1}dt\right] \quad \text{(by (4))} \\ &= \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}\Re[(A^{-1}+tB^{-1})^{-1}]dt \\ &\leqslant \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}[\Re(A^{-1}+tB^{-1})]^{-1}dt \quad \text{(by (9))} \\ &= \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}[\Re(A^{-1})+t\Re(B^{-1})]^{-1}dt \\ &= (\Re(A^{-1}))^{-1} \sharp_\lambda (\Re(B^{-1}))^{-1} \\ &\leqslant \sec^2(\alpha)\Re A \sharp_\lambda \sec^2(\alpha)\Re B \quad \text{(by (9))} \\ &= \sec^2(\alpha)(\Re A \sharp_\lambda \Re B) \\ &\leqslant \sec^2(\alpha)(\Re A \nabla_\lambda \Re B) \quad \text{(by (3))} \\ &= \sec^2(\alpha)\Re(A \nabla_\lambda B), \end{aligned}$$

which completes the proof. □

LEMMA 2.5. *Let* $A,B \in \mathbb{M}_n(\mathbb{C})$ *with* $W(A),W(B) \subseteq S_\alpha$ *for some* $\alpha \in [0,\pi/2)$. *Then*

$$\Re(A!_\lambda B) \leqslant \sec^2(\alpha)\Re(A\sharp_\lambda B),\ \ \lambda \in (0,1).$$

Proof. Compute

$$\begin{aligned}
\Re(A!_\lambda B) &= \Re[(\lambda A^{-1} + (1-\lambda)B^{-1})^{-1}] \\
&\leqslant [\Re(\lambda A^{-1} + (1-\lambda)B^{-1})]^{-1} \quad \text{(by (9))} \\
&= [\lambda\Re(A^{-1}) + (1-\lambda)\Re(B^{-1})]^{-1} \\
&= [\Re(A^{-1})\nabla_\lambda\Re(B^{-1})]^{-1} \\
&\leqslant [\Re(A^{-1})\sharp_\lambda\Re(B^{-1})]^{-1} \quad \text{(by (3))} \\
&= [\Re(A^{-1})]^{-1}\sharp_\lambda[\Re(B^{-1})]^{-1} \\
&\leqslant [\sec^2(\alpha)\Re(A)]\sharp_\lambda[\sec^2(\alpha)\Re(B)] \quad \text{(by (9))} \\
&= \sec^2(\alpha)[\Re(A)\sharp_\lambda\Re(B)] \\
&\leqslant \sec^2(\alpha)\Re(A\sharp_\lambda B), \quad \text{(by (10))}
\end{aligned}$$

as claimed. □

REMARK 2.6. Set $\alpha = 0$ in Lemma 2.4 and Lemma 2.5, i.e., A and B are positive semidefinite matrices. Then our result is the inequality (3).

With the above preparation, let us first present the generalizations of the inequalities (6) and (8) in the next two theorems. The following result is a generalization of the result (6).

THEOREM 2.7. *Let* $A,B \in \mathbb{M}_n(\mathbb{C})$ *with* $W(A),W(B) \subseteq S_\alpha$ *for some* $\alpha \in [0,\pi/2)$. *Then*

$$\Re A\nabla_\lambda(\cos^2(\alpha)\Re(A\sharp_\lambda B)) \leqslant \Re[A\sharp_\lambda(A\nabla_\lambda B)],\ \ \lambda \in (0,1). \tag{12}$$

Proof. From the proof of Lemma 2.4, we know that,

$$\Re(A\sharp_\lambda B) \leqslant \sec^2(\alpha)(\Re A\sharp_\lambda\Re B),\ \ \lambda \in (0,1). \tag{13}$$

Since the function $f(t) = t^\lambda, \lambda \in (0,1)$ is operator concave, by Jensen's inequality we have

$$\lambda f(C) + (1-\lambda)f(D) \leqslant f(\lambda C + (1-\lambda)D).$$

Let $C = I$ and $D = (\Re A)^{-\frac{1}{2}}\Re B(\Re A)^{-\frac{1}{2}}$. Then we have

$$\lambda I + (1-\lambda)[(\Re A)^{-\frac{1}{2}}\Re B(\Re A)^{-\frac{1}{2}}]^\lambda \leqslant [\lambda I + (1-\lambda)(\Re A)^{-\frac{1}{2}}\Re B(\Re A)^{-\frac{1}{2}}]^\lambda. \tag{14}$$

From (14), it follows that

$$\lambda\Re A+(1-\lambda)\left[(\Re A)^{\frac{1}{2}}[(\Re A)^{-\frac{1}{2}}\Re B(\Re A)^{-\frac{1}{2}}]^{\lambda}(\Re A)^{\frac{1}{2}}\right]$$
$$\leqslant(\Re A)^{\frac{1}{2}}\left[(\Re A)^{-\frac{1}{2}}[\lambda\Re A+(1-\lambda)\Re B](\Re A)^{-\frac{1}{2}}\right]^{\lambda}(\Re A)^{\frac{1}{2}},$$

which is just

$$\Re A\nabla_{\lambda}(\Re A\sharp_{\lambda}\Re B)\leqslant\Re A\sharp_{\lambda}(\Re A\nabla_{\lambda}\Re B),\ \ \lambda\in(0,1). \tag{15}$$

Now,

$$\begin{aligned}\Re[A\sharp_{\lambda}(A\nabla_{\lambda}B)] &\geqslant \Re A\sharp_{\lambda}\Re(A\nabla_{\lambda}B) \quad \text{(by (10))}\\ &= \Re A\sharp_{\lambda}(\Re A\nabla_{\lambda}\Re B)\\ &\geqslant \Re A\nabla_{\lambda}(\Re A\sharp_{\lambda}\Re B) \quad \text{(by (15))}\\ &\geqslant \Re A\nabla_{\lambda}(\cos^2(\alpha)\Re(A\sharp_{\lambda}B)), \quad \text{(by (13) and monotonicity)}\end{aligned}$$

which proves the result. □

REMARK 2.8. When $\lambda=\frac{1}{2}$, our result (12) reduces to (6).

The next theorem is another generalization of the inequality (8).

THEOREM 2.9. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ with $W(A),W(B)\subseteq S_{\alpha}$ for some $\alpha\in[0,\pi/2)$. Then*

$$\Re[A\sharp_{\lambda}(A!_{\lambda}B)]\leqslant\sec^2(\alpha)\Re[A!_{\lambda}(A\sharp_{\lambda}B)],\ \ \lambda\in(0,1). \tag{16}$$

Proof. Taking the inverse of both sides in (15) and substituting $\Re A$ for $(\Re A)^{-1}$ and $\Re B$ for $(\Re B)^{-1}$ gives the following inequality:

$$\Re A\sharp_{\lambda}(\Re A!_{\lambda}\Re B)\leqslant\Re A!_{\lambda}(\Re A\sharp_{\lambda}\Re B),\ \ \lambda\in(0,1). \tag{17}$$

Compute

$$\begin{aligned}\Re[A\sharp_{\lambda}(A!_{\lambda}B)] &= \Re\left[A\sharp_{\lambda}\left(\lambda A^{-1}+(1-\lambda)B^{-1}\right)^{-1}\right]\\ &= \Re\left[\frac{\sin\lambda\pi}{\pi}\int_0^{\infty}t^{\lambda-1}\left(A^{-1}+t(\lambda A^{-1}+(1-\lambda)B^{-1})\right)^{-1}dt\right]\\ &= \frac{\sin\lambda\pi}{\pi}\int_0^{\infty}t^{\lambda-1}\Re\left[\left(A^{-1}+t(\lambda A^{-1}+(1-\lambda)B^{-1})\right)^{-1}\right]dt\\ &\leqslant \frac{\sin\lambda\pi}{\pi}\int_0^{\infty}t^{\lambda-1}\left[\Re\left(A^{-1}+t(\lambda A^{-1}+(1-\lambda)B^{-1})\right)\right]^{-1}dt\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (9))}\\ &= \frac{\sin\lambda\pi}{\pi}\int_0^{\infty}t^{\lambda-1}\{\Re(A^{-1})+t\Re[\lambda A^{-1}+(1-\lambda)B^{-1}]\}^{-1}dt\end{aligned}$$

$$
\begin{aligned}
&= [\Re(A^{-1})]^{-1}\sharp_\lambda \left[\lambda \Re(A^{-1}) + (1-\lambda)\Re(B^{-1})\right]^{-1} \\
&= [\Re(A^{-1})]^{-1}\sharp_\lambda \left[(\Re(A^{-1}))^{-1} !_\lambda (\Re(B^{-1}))^{-1}\right] \\
&\leqslant \sec^2(\alpha)\Re A \sharp_\lambda \sec^2(\alpha)(\Re A !_\lambda \Re B) \text{ (by (9) and monotonicity)} \\
&= \sec^2(\alpha)[\Re A \sharp_\lambda (\Re A !_\lambda \Re B)] \\
&\leqslant \sec^2(\alpha)[\Re A !_\lambda (\Re A \sharp_\lambda \Re B)] \quad \text{(by (17))} \\
&\leqslant \sec^2(\alpha)[\Re A !_\lambda \Re(A \sharp_\lambda B)] \quad \text{(by (10))} \\
&\leqslant \sec^2(\alpha)\Re[A !_\lambda (A \sharp_\lambda B)], \quad \text{(by (11))}
\end{aligned}
$$

as required. □

REMARK 2.10. When $\lambda = \frac{1}{2}$, our result (16) is the inequality (8).

Secondly, we end this section with mixed arithmetic-geometric-mean inequality with λ-weighted and mixed harmonic-geometric-mean inequality with λ-weighted for sums of n $(n \geqslant 2)$ sector matrices.

THEOREM 2.11. *Let A_i and B_i be sector matrices. The mixed harmonic-geometric-mean inequality with λ-weighted for the sum of A_i and B_i holds, i.e., for $\lambda \in (0,1)$*

$$
\begin{aligned}
&\Re\left[\left(\sum_{i=1}^n A_i\right)\nabla_\lambda\left(\left(\sum_{i=1}^n A_i\right)\sharp_\lambda\left(\sum_{i=1}^n B_i\right)\right)\right] \\
&\leqslant \sec^2(\alpha)\Re\left[\left(\sum_{i=1}^n A_i\right)\sharp_\lambda\left(\left(\sum_{i=1}^n A_i\right)\nabla_\lambda\left(\sum_{i=1}^n B_i\right)\right)\right].
\end{aligned} \tag{18}
$$

Proof. By (1), it is straightforward to observe that $\sum_{i=1}^n A_i$ and $\sum_{i=1}^n B_i$ are also sector matrices. Thus, the conclusion (18) follows immediately from Theorem 2.7. □

THEOREM 2.12. *Let A and B be sector matrices. The mixed harmonic-geometric-mean inequality with λ-weighted for the sum of A_i and B_i holds, i.e., for $\lambda \in (0,1)$*

$$
\begin{aligned}
&\Re\left(\sum_{i=1}^n A_i\right)\sharp_\lambda\left[\Re\left(\sum_{i=1}^n A_i\right)!_\lambda \Re\left(\sum_{i=1}^n B_i\right)\right] \\
&\leqslant \sec^2(\alpha)\Re\left(\sum_{i=1}^n A_i\right)!_\lambda\left[\Re\left(\sum_{i=1}^n A_i\right)\sharp_\lambda \Re\left(\sum_{i=1}^n B_i\right)\right].
\end{aligned}
$$

Proof. This follows from Theorem 2.9 as in the proof of Theorem 2.11. □

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