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ON THE k POINT DENSITY PROBLEM FOR BAND–DIAGONAL M–BASES

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Abstract. In the early 1990s the works of Larson, Wogen and Argyros, Lambrou, Longstaff disclosed an example of a strong tridiagonal M-basis that was not rank one dense. Later Katavolos, Lambrou and Papadakis studied k point density property of this example. In this paper we present new methods for the analysis of k point density and rank one density properties for band-diagonal M-bases.

1. Introduction

1.1. Density properties

Consider an infinite-dimensional real Hilbert space $\mathscr{H}$. Suppose that $\mathscr{H}$ has an orthonormal basis $\{e_j\}_{j=0}^{\infty}$. A sequence $\mathfrak{F} = \{f_n\}_{n=0}^{\infty}$ of vectors in $\mathscr{H}$ is said to be *complete* when it spans the whole space, that is, $\overline{span}\{f_k\} = \mathscr{H}$, where $\overline{span}$ denotes the closed linear span. The sequence $\mathfrak{F}$ is called *minimal* if none of its elements can be approximated by the linear combinations of the others: $f_n \notin \overline{span}\{f_k\}_{k\neq n}$ for any n. The system $\mathfrak{F}^* = \{f_l^*\}_{l=0}^{\infty}$ is *biorthogonal* to $\mathfrak{F}$ if for any $k,l \geqslant 0$ we have $\langle f_k, f_l^*\rangle = \delta_{kl}$, where δ is the Kronecker delta. By the Hahn–Banach theorem, $\mathfrak{F}$ is complete and minimal when and only when it possesses a unique biorthogonal system $\mathfrak{F}^*$. We call the minimal system $\mathfrak{F}$ *band-diagonal* if there exists $L \in \mathbb{N}$ such that $\langle f_t, e_l\rangle = \langle f_t^*, e_l\rangle = 0$ whenever $|t-l| > L$. We say that $\mathfrak{F}$ is an M-basis if $\mathfrak{F}$ is complete, minimal and $\mathfrak{F}^*$ is complete.

Consider the operator algebra $\mathscr{A} = \{T \in B(\mathscr{H}) : Tf_n = \lambda_n f_n, \lambda_n \in \mathbb{R}\}$ and the algebra $R_1(\mathscr{A})$ generated by rank one operators of $\mathscr{A}$. We are interested in the following properties of the algebra $\mathscr{A}$.

DEFINITION 1. (k point density property) We say that the algebra $R_1(\mathscr{A})$ is k *point dense* in $\mathscr{A}$ (or that the algebra $\mathscr{A}$ has k *point density property*) if for any $x_1, x_2, \dots x_k \in \mathscr{H}$ and $\varepsilon > 0$ there exists $R \in R_1(\mathscr{A})$ such that $||Rx_s - x_s|| < \varepsilon$ for any $1 \leqslant s \leqslant k$.

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The definition for $k=1$ is equivalent to $\mathfrak{F}$ being a *strong M-basis* (see [8]): the system $\mathfrak{F}$ is called a *strong M-basis* if for any $x \in \mathscr{H}$ we have $x \in \overline{span}\{\langle x, f_n^* \rangle f_n\}_{n=0}^{\infty}$.

DEFINITION 2. (rank one density property) We say that the algebra $\mathscr{A}$ has *rank one density property* if the unit ball of rank one subalgebra $R_1(\mathscr{A})$ is dense in the unit ball of $\mathscr{A}$ in the strong operator topology.

By abuse of notation, we say that $\mathfrak{F}$ is k point dense (rank one dense) when the corresponding algebra $R_1(\mathscr{A})$ is k point dense in $\mathscr{A}$ (rank one dense).

Notice that rank one density property implies k point density property for any k.

1.2. Motivation

Longstaff in [11] studied abstract subspace lattices and corresponding operator algebras. In that paper Longstaff raised an important question: does one point density property always imply rank one density property?

The solution remained unknown until Larson and Wogen showed [9] that the answer is negative. They constructed an example of a vector system $\mathfrak{F}$ such that it is one point dense but does not possess rank one density property.

EXAMPLE 1. (Larson–Wogen system $\mathfrak{F}_{LW}$ parameterized with real a_n) For any $j \geqslant 0$ we define

$$f_{2j+1} = -a_{2j+1}e_{2j} + e_{2j+1} + a_{2j+2}e_{2j+2}, \qquad f_{2j} = e_{2j},$$
$$f_{2j}^* = -a_{2j}e_{2j-1} + e_{2j} + a_{2j+1}e_{2j+1}, \qquad f_{2j+1}^* = e_{2j+1},$$

where a_n are nonzero real numbers for any $n > 0$ and $a_0 = 0$.

The construction presented by Larson and Wogen was remarkably simple and elementary, — notice that the matrices corresponding to the vectors $\{f_j\}_{j=0}^{\infty}$ and $\{f_j^*\}_{j=0}^{\infty}$ are both tridiagonal. Afterwards this example was also studied in [1] (see Addendum), by Azoff and Shehada in [2], in [13]. In 1993 Katavolos, Lambrou and Papadakis in [8] performed a deep analysis of the density properties of this vector system and deduced that for $\mathfrak{F}_{LW}$ one point density does not imply rank one density. Moreover, they showed that for such system rank one density is equivalent to two point density.

We are going to consider band-diagonal systems similar to the one regarded by Larson and Wogen and to determine the exact conditions for k point density property of such vector systems. In this paper we present a few new techniques for the analysis of k point density and rank one density of band-diagonal vector systems.

In the next section we will gather some basic facts and outline the main idea of the paper. In Section 3 we perform the analysis for the Larson–Wogen example, providing a simpler proof of Theorems 2.1 and 2.2 in [8]. In Section 4 we prove a similar theorem for a pentadiagonal system.

2. Preliminaries

Suppose that $\mathfrak{F} = \{f_n\}_{n=0}^{\infty}$ is an arbitrary band-diagonal M-basis and $\mathfrak{F}^* = \{f_n^*\}_{n=0}^{\infty}$ is its biorthogonal sequence. In this section we establish several facts about $\mathfrak{F}$.

PROPOSITION 2.1. *The system $\mathfrak{F}$ is rank one dense if and only if any trace class operator T, such that $\langle Tf_n, f_n^*\rangle = 0$ for any $n \geqslant 0$, has zero trace.*

Proof. It is well known that rank one density property is equivalent to $R_1(\mathscr{A})$ being dense in $\mathscr{A}$ in the ultraweak (or σ-weak) topology (see [8], Theorem 2.2). But the annihilator of $R_1(\mathscr{A})$ consists precisely of the trace class operators satisfying $\langle Tf_n, f_n^*\rangle = 0$ for all $n \geqslant 0$.

PROPOSITION 2.2. *The system $\mathfrak{F}$ is k point dense if and only if any k-dimensional operator T, such that $\langle Tf_n, f_n^*\rangle = 0$ for any $n \geqslant 0$, has zero trace.*

Proof. In the paper [8] authors proved the proposition for $k = 2$. For larger k's the same reasoning works.
For an arbitrary linear operator T we will be interested in the differences between the partial sums of the Fourier series using the system $\mathfrak{F}$ and partial sums of the canonical Fourier series (using the orthonormal basis $\{e_n\}_{n=0}^{\infty}$):

$$\Xi_n = \sum_{m=0}^{n} \langle Tf_m, f_m^*\rangle - \sum_{m=0}^{n} \langle Te_m, e_m\rangle, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathscr{H}$. It appears that Ξ_n takes a concise and compact form for the finite-band system $\mathfrak{F}$, and it is much easier to study Ξ_n than, for example, $\langle Tf_m, f_m^*\rangle$.

PROPOSITION 2.3. *The operator T is a trace class operator annihilating the subalgebra $R_1(\mathscr{A})$ if and only if for any $n \geqslant 0$ one has*

$$\Xi_n + \sum_{m=0}^{n} \langle Te_m, e_m\rangle = 0. \tag{2.2}$$

We will use this formulation in the following sections.

Now consider an operator T which has a finite rank. In this case we write T as a finite sum $T = \sum_{s=1}^{k} y^s \otimes x^s$, where $x^s, y^s \in \mathscr{H}$ and $y \otimes x$ denotes the rank one operator sending a vector $v \in \mathscr{H}$ to $\langle v, y\rangle x$.

Let us define vectors v_n and u_n in $\mathbb{R}^k$ as follows:

$$v_n = (x_n^1, x_n^2, \ldots, x_n^k) \qquad u_n = (y_n^1, y_n^2, \ldots, y_n^k),$$

where $x_n^s = \langle x^s, e_n\rangle$ and $y_n^s = \langle y^s, e_n\rangle$. Since $\langle Te_m, e_l\rangle = \langle u_m, v_l\rangle$ for any m and l, we can rewrite Ξ_n in terms of the scalar products of $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$. In turn it means that (2.2) can be rewritten in terms of the scalar products of $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$.

Hence, the existence of T would be reduced to the existence of the vectors u_n, v_n in $\mathbb{R}^k$ such that the sequences $\{|u_n|\}_{n=0}^{\infty}$, $\{|v_n|\}_{n=0}^{\infty}$ are both square summable and (2.2) is satisfied. Thus, instead of looking for k vectors x^s and y^s in $\mathscr{H}$, we would look for an infinite sequence of k-dimensional vectors v_n and u_n such that $\{v_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ belong to $\ell^2(\mathbb{R}^k)$. This is one of the key ideas in our method of analysing k point density property for $\mathfrak{F}$.

Thus, we have just found the following reformulation for k point density property.

PROPOSITION 2.4. *The following two statements are equivalent:*

1. *there exists a k-dimensional operator T which annihilates $R_1(\mathscr{A})$ such that $TrT \neq 0$,*
2. *there exist vectors $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ in $\ell^2(\mathbb{R}^k)$ such that for the operator $\widehat{T} = \sum_{t,l=0}^{\infty} \langle u_t, v_l\rangle e_t \otimes e_l$ we have*

$$\Xi_n + \sum_{m=0}^{n} \langle \widehat{T} e_m, e_m\rangle = 0 \tag{2.3}$$

for any $n \geqslant 0$.

As we already mentioned, the equation (2.3) can be expressed via u_n and v_n. Moreover, we can also write the trace of T in terms of u_n, v_n:

$$TrT = \sum_{s=1}^{k} \langle y^s, x^s\rangle = \sum_{s=1}^{k}\sum_{n=0}^{\infty} y_n^s x_n^s = \sum_{n=0}^{\infty}\sum_{s=1}^{k} y_n^s x_n^s = \sum_{n=0}^{\infty} \langle u_n, v_n\rangle. \tag{2.4}$$

Essentially, the k point density property can be viewed as a possibility of placing the sequence of vectors in $\mathbb{R}^k$ which are constrained with a series of relations (2.3) and (2.4).

3. Classification for the Larson–Wogen M-basis

In this section we study Larson–Wogen vector system $\mathfrak{F}_{LW}$ (Example 1). Namely, we prove a theorem similar to Theorem 2.2 of [8]. Up until now there existed two different techniques in studying k point density, one for $k = 1$ (strong M-bases) and a different one for $k \geqslant 2$. Here we demonstrate a universal method for the analysis of k point density property.

THEOREM 3.1. ([8], Theorem 2.2) *The sequences $\mathfrak{F}_{LW}$ and $\mathfrak{F}_{LW}^*$ are biorthogonal and both are complete in $\mathscr{H}$. Moreover, the following is true:*

1. *the system $\mathfrak{F}_{LW}$ is one point dense (a strong M-basis) if and only if the sequence*

$$\mu_n = \frac{a_{n-1}a_{n-3}\dots}{a_n a_{n-2}\dots} \tag{3.1}$$

does not belong to ℓ^2.

2. *the system* $\mathfrak{F}_{LW}$ *is* k *point dense (* $k>1$ *) if and only if the sequence* $\{1/a_n\}_{n=1}^{\infty}$ *does not belong to* ℓ^1.

Proof. Due to Proposition 2.4 we know that k point density of the system $\mathfrak{F}$ is equivalent to the existence of k-dimensional vectors u_n, v_n such that (2.3) holds for the corresponding operator T. For the given M-basis $\mathfrak{F}=\mathfrak{F}_{LW}$ we can calculate Ξ_n precisely:

$$\Xi_{2n-1}=a_{2n}T_{2n-1,2n},$$
$$\Xi_{2n}=a_{2n+1}T_{2n+1,2n},$$

where $T_{ij}=\langle Te_j,e_i\rangle$.

Since $T_{ij}=\langle u_j,v_i\rangle$, we have

$$\Xi_{2n-1}=a_{2n}\langle u_{2n},v_{2n-1}\rangle,$$
$$\Xi_{2n}=a_{2n+1}\langle u_{2n},v_{2n+1}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^k$.

For the convenience of the reader we will introduce the sequences of vectors w_n and w_n^*.

$$w_{2n}=u_{2n},\quad w_{2n}^*=v_{2n},$$
$$w_{2n+1}=v_{2n+1},\quad w_{2n+1}^*=u_{2n+1}.$$

In view of this notation $\Xi_n=a_{n+1}\langle w_n,w_{n+1}\rangle$ and due to Equation (2.4) $TrT=\sum_{m=0}^{\infty}\langle w_m,w_m^*\rangle$.

Thus, we get that $\mathfrak{F}$ is not k point dense if and only if there exist k-dimensional vectors $\{w_n\}_{n=0}^{\infty}$, $\{w_n^*\}_{n=0}^{\infty}$ lying in $\ell^2(\mathbb{R}^k)$ such that

$$a_{n+1}\langle w_n,w_{n+1}\rangle=-\sum_{m=0}^{n}\langle w_m,w_m^*\rangle, \tag{3.2}$$

for any $n\geqslant 0$, and $\sum_{m=0}^{\infty}\langle w_m,w_m^*\rangle\neq 0$.

In what follows we show that the latter can be simplified even more.

PROPOSITION 3.1. *The system* $\mathfrak{F}$ *is not* k *point dense if and only if there exists a sequence of vectors* $\{r_n\}_{n=0}^{\infty}$ *in* $\ell^2(\mathbb{R}^k)$ *such that*

$$a_{n+1}\langle r_n,r_{n+1}\rangle=1, \tag{3.3}$$

for any $n\geqslant 0$.

Proof. Suppose we found such r_n. Then we solve (3.2) by putting w_n^* to zero, w_n to r_n for any $n>0$ and choosing the vector w_0^* so that $\langle w_0,w_0^*\rangle=-1$.

Now we prove the converse. Suppose we found such w_n that (3.2) holds. Given that the vectors w_n lie in $\mathbb{R}^k$, we rewrite the scalar product as the product of the vector lengths and the cosine of the angle between the vectors. Namely, we define $W_n = |w_n|$ and real θ_n that $\langle w_n, w_{n+1}\rangle = W_n W_{n+1}\cos\theta_n$.

The sequence $\Xi_n = -\sum_0^n \langle w_m, w_m^*\rangle$ has a non-zero limit, so let us find the largest $N > 0$ such that $\Xi_N = 0$. Then we can modify the original sequence by setting w_n, w_n^* to zero for any $0 \leqslant n \leqslant N$ so that (3.2) still holds. Therefore, without loss of generality we can assume that $\Xi_n \neq 0$ for any $n \geqslant 0$. Setting $a_n' = a_n \cos\theta_n$ we see that the sequence

$$W_n = \frac{\Xi_{n-1}/a_n'}{\Xi_{n-2}/a_{n-1}'} \cdot \frac{\Xi_{n-3}/a_{n-2}'}{\Xi_{n-4}/a_{n-3}'} \cdots$$

belongs to ℓ^2. Now since $\Xi_n = -\sum_0^n \langle w_m, w_m^*\rangle$, we discover that

$$\frac{\Xi_n}{\Xi_{n-1}} = 1 + \eta_n,$$

where $\{\eta_n\}_{n=1}^\infty \in \ell^1$. Thus the product of such $(1+\eta_m)$ fractions is bounded by some constant above. It follows that the sequence

$$W_n^\# = \frac{1/a_n'}{1/a_{n-1}'} \cdot \frac{1/a_{n-2}'}{1/a_{n-3}'} \cdots$$

belongs to ℓ^2. Now we set r_n to $\frac{W_n^\#}{W_n} w_n$, and then (3.2) holds since

$$a_{n+1}\langle r_n, r_{n+1}\rangle = a_{n+1} \frac{1/a_{n+1}'}{\Xi_n/a_{n+1}'} \langle w_n, w_{n+1}\rangle = 1.$$

Since $|r_n| = |W_n^\#|$ and the sequence $\{|W_n^\#|\}_{n=1}^\infty$ belongs to ℓ^2, the sequence $\{r_n\}_{n=1}^\infty$ belongs to $\ell^2(\mathbb{R}^k)$ as well.

Now we are ready to prove the theorem for the case $k = 1$.

PROPOSITION 3.2. *The system $\mathfrak{F}_{LW}$ is one point dense if and only if $\{\mu_n\}_{n=1}^\infty$ does not belong to ℓ^2.*

Proof. It follows from Proposition 3.1.

The case $k = 1$ has all the vectors r_n, r_n^* lying on the same line ($\mathbb{R}^1$). Since all r_n are collinear, the lengths of the vectors r_n are precisely μ_n. Hence, Equation (3.3) can be satisfied if and only if $\{\mu_n\}_{n=1}^\infty$ is square summable.

After this we consider the case $k > 1$.

PROPOSITION 3.3. *The system $\mathfrak{F}_{LW}$ is k point dense ($k > 1$) if and only if the sequence $\{1/a_n\}_{n=1}^\infty$ does not belong to ℓ^1.*

Proof. According to Proposition 3.1, the system $\mathfrak{F}_{LW}$ is k point dense if and only if there is no such sequence $\{r_n\}_{n=0}^\infty$ in $\ell^2(\mathbb{R}^k)$ which satisfy $a_n\langle r_n, r_{n-1}\rangle = 1$. Obviously, if there are such vectors r_n, then $\{1/a_n\}_{n=1}^\infty$ belongs to ℓ^1.

Conversely, suppose $\{1/a_n\}_{n=1}^{\infty}$ belongs to ℓ^1. Then the sequence $R_n = \max(|a_n|^{-\frac{1}{2}}, |a_{n+1}|^{-\frac{1}{2}})$ is square summable. Observe that $R_nR_{n-1} \geqslant 1/|a_n|$, and so it is always possible to choose the angle θ_n so that

$$a_n\langle r_n, r_{n-1}\rangle = a_nR_nR_{n-1}\cos\theta_n = 1.$$

Now we have defined the lengths for r_n and the angles between each two consecutive vectors r_{n-1}, r_n. Obviously, for any $k \geqslant 2$ we are able to place the vectors r_n in $\mathbb{R}^k$. The last two propositions prove Theorem 3.1.

4. Pentadiagonal example

In this section we explore another vector system $\mathfrak{F}$ and its biorthogonal system $\mathfrak{F}^*$ defined as follows:

$$\mathbf{f_{4j}} = e_{4j}, \quad \mathbf{f^*_{4j}} = e_{4j} + d_{2j-1}e_{4j-2} - b_{2j-1}e_{4j-1} + a_{2j}e_{4j+1} + c_{2j}e_{4j+2}$$
$$\mathbf{f_{4j+1}} = -a_{2j}e_{4j} + e_{4j+1}, \quad \mathbf{f^*_{4j+1}} = e_{4j+1} + b_{2j}e_{4j+2},$$
$$\mathbf{f_{4j+2}} = e_{4j+2} + d_{2j}e_{4j} - b_{2j}e_{4j+1} + a_{2j+1}e_{4j+3} + c_{2j+1}e_{4j+4}, \quad \mathbf{f^*_{4j+2}} = e_{4j+2},$$
$$\mathbf{f_{4j+3}} = e_{4j+3} + b_{2j+1}e_{4j+4}, \quad \mathbf{f^*_{4j+3}} = -a_{2j+1}e_{4j+2} + e_{4j+3},$$

where the real coefficients a_n, b_n, c_n, d_n are equal to zero whenever $n < 0$, and satisfy the equality $c_n + d_n = a_nb_n$ for any $n \geqslant 0$.

PROPOSITION 4.1. *The given system is an M-basis.*

Proof. The equality $c_n + d_n = a_nb_n$ guarantees the biorthogonality, while the completeness of $\mathfrak{F}$ and $\mathfrak{F}^*$ is easy to check.

We prove a theorem similar to Theorem 3.1, though we do not investigate the case $k = 1$ in this section.

THEOREM 4.1. *The following statements are equivalent:*

1. *the given system is rank one dense,*

2. *the given system is k point dense for some (equivalently any) $k > 1$,*

3. *the sequence*
$$\mu_n = \min\left(\frac{1}{|a_n|} + \frac{1}{|b_n|}, \frac{1+|b_n|}{|d_n|}, \frac{1+|a_n|}{|c_n|}\right)$$
does not belong to ℓ^1.

Proof. In order to investigate the density properties we repeat the reasoning from Section 2. Presume that Ξ_n are defined by (2.1).

Thus, for any $j \geqslant 0$ we have

$$\begin{aligned} \Xi_{4j} &= a_{2j}T_{4j+1,4j} + c_{2j}T_{4j+2,4j}, \\ \Xi_{4j+1} &= -d_{2j}T_{4j+2,4j} + b_{2j}T_{4j+2,4j+1}, \\ \Xi_{4j+2} &= a_{2j+1}T_{4j+2,4j+3} + c_{2j+1}T_{4j+2,4j+4}, \\ \Xi_{4j+3} &= -d_{2j+1}T_{4j+2,4j+4} + b_{2j+1}T_{4j+3,4j+4}, \end{aligned} \tag{4.1}$$

where T_{ij} stands for $\langle Te_j, e_i \rangle$.

First of all we investigate the conditions of rank one density property for $\mathfrak{F}$.

PROPOSITION 4.2. *The following statements are equivalent:*

1. *the system $\mathfrak{F}$ is not rank one dense,*

2. *there exists an operator T such that $Tr\,T = -1$ and for any $n \geqslant 0$ one has*

$$\Xi_n + \sum_{m=0}^{n} \langle Te_m, e_m \rangle = 0,$$

3. *the sequence $\{\mu_n\}_{n=1}^{\infty}$ belongs to ℓ^1.*

Proof. The equivalence of the first two statements is due to Proposition 2.3. We are going to prove the equivalence between the last two statements.

Assume that $\{\mu_n\}_{n=1}^{\infty} \in \ell^1$; our purpose is to construct the required operator T. Let T_{00} be equal to -1, and T_{jj} be equal to zero for any $j > 0$. Next we consider three cases for each $n \geqslant 0$.

Case 1. Suppose $\mu_n = 1/|a_n| + 1/|b_n|$. For $n = 2j$ we set

$$T_{4j+1,4j} = 1/a_n, \quad T_{4j+2,4j} = 0, \quad T_{4j+2,4j+1} = 1/b_n.$$

That guarantees the equality $\Xi_{2n} = \Xi_{2n+1} = 1$. For $n = 2j+1$ we set

$$T_{4j+2,4j+3} = 1/a_n, \quad T_{4j+2,4j+4} = 0, \quad T_{4j+3,4j+4} = 1/b_n,$$

which provides the equality $\Xi_{2n} = \Xi_{2n+1} = 1$.

Case 2. Assume $\mu_n = (1+|b_n|)/|d_n|$. For $n = 2j$ we set

$$T_{4j+1,4j} = b_{2j}/d_{2j}, \quad T_{4j+2,4j} = -1/d_{2j}, \quad T_{4j+2,4j+1} = 0.$$

Again, we have $\Xi_{2n} = \Xi_{2n+1} = 1$. For $n = 2j+1$ we set

$$T_{4j+2,4j+3} = b_{2j+1}/d_{2j+1}, \quad T_{4j+2,4j+4} = -1/d_{2j+1}, \quad T_{4j+3,4j+4} = 0,$$

The third case $\mu_n = (1+|a_n|)/|c_n|$ is left to the reader.

All the other entries T_{ij} we set to zero. These equalities ensure that

$$\Xi_n = -\sum_{s=0}^{n} T_{ss} = 1$$

for any $n \geqslant 0$.

The constructed operator T belongs to the trace class since the non-zero operator matrix entries are summable due to the assumption that $\{\mu_n\}_{n=1}^{\infty} \in \ell^1$. Since the trace of T is equal to -1, the sufficiency is proved.

Conversely, assume that there exists a trace class operator T in the annihilator of $R_1(\mathscr{A})$ with the trace equal to -1.

First of all, we prove that $\{\mu_{2j}\}_{j=1}^{\infty}$ is a summable sequence. Since T is in the trace class, the sequence of vectors $v_n = |T_{nn}| + |T_{n,n+1}| + |T_{n,n+2}|$ belongs to ℓ^1. Obviously, $\{\Xi_n\}_{n=1}^{\infty}$ belongs ℓ^1 as well. As a consequence, we have $|\Xi_n| \geqslant 0.5$ for all n large enough; we will assume that it holds for any $n > 0$.

It can be easily checked that if for some n one of the numbers a_n, b_n, c_n, d_n is equal to zero, then $v_n \geqslant \mu_n/2$. From this point we will suppose that the coefficients are nonzero for any $n > 0$.

For any even $n = 2j$ consider the linear function

$$g_n(x) = \left|\frac{\Xi_{2n} - c_n x}{a_n}\right| + |x| + \left|\frac{\Xi_{2n+1} + d_n x}{b_n}\right|.$$

Obviously, we have $v_n = g_n(T_{2n+2,2n})$. The function g_n is piecewise linear, so its minimum is attained at the breakpoints. The breakpoints are zero, $y_n = \Xi_{2n}/c_n$ and $z_n = -\Xi_{2n+1}/d_n$. We have $g_n(0) \geqslant \mu_n/2$. Consider the set $N_1 \subseteq \mathbb{N}_{even}$, such that for any $n \in N_1$ the function g_n attains its minimum at the point Ξ_{2n}/c_n. Thus, for any $n \in N_1$ we have $v_n \geqslant g_n(y_n)$. We have

$$g_n(y_n) = \left|\Xi_{2n}/c_n\right| + \left|\frac{\Xi_{2n+1} + d_n \Xi_{2n}/c_n}{b_n}\right|.$$

Since v_n is summable and $v_n \geqslant g_n(y_n) \geqslant 0.5/|c_n|$ for any $n \in N_1$ we deduce that $\sum_{n \in N_1} |c_n|^{-1} < \infty$.

Clearly, $G_n = g_n(y_n) - \left|\Xi_{2n}/c_n\right|$ is summable. Let Δ_n stand for the difference $(\Xi_{2n+1} - \Xi_{2n})$. Then

$$G_n = \left|\frac{c_n \Xi_{2n+1} + d_n \Xi_{2n}}{c_n b_n}\right| = \left|\frac{c_n \Delta_n + (c_n + d_n)\Xi_{2n}}{c_n b_n}\right| = \left|\frac{c_n \Delta_n + a_n b_n \Xi_{2n}}{c_n b_n}\right|.$$

Hence, $\left|G_n - \Xi_{2n}|a_n/c_n|\right| \leqslant |\Delta_n/b_n|$.

Consider the sets

$$N_2 = \{n \in N_1 \mid 0.5 \leqslant |b_n|\}$$

and

$$N_3 = N_1 \setminus N_2.$$

Since Ξ_n has a finite limit, we have $\sum_{n \in N_2} |a_n/c_n| < \infty$. Hence, $\{\mu_n\}_{n \in N_2} \in \ell^1$.

Assume that $\sum_{n \in N_3} |a_n/c_n| = \infty$.

We have $\left||b_n| G_n - \Xi_{2n}|(c_n + d_n)/c_n|\right| \leqslant |\Delta_n|$. Since the sequences $\{b_n G_n\}_{n \in N_3}$ and $\{\Delta_n\}_{n \in N_3}$ are absolutely summable, the sequence $\{(c_n + d_n)/c_n\}_{n \in N_3}$ is absolutely summable as well. Consequently, $|d_n/c_n| \geqslant 0.5$ when n is large enough. We get

$\{1/c_n\}_{n\in N_3} \in \ell^1$, and thus $\{1/d_n\}_{n\in N_3} \in \ell^1$. Since for $n \in N_3$ one has $|b_n| \leqslant 0.5$, we have

$$\sum_{n\in N_3} \mu_n \leqslant \sum_{n\in N_3} \frac{1+|b_n|}{|d_n|} < \infty.$$

Repeating the reasoning for odd n, we get that $\{\mu_n\}_{n=1}^{\infty}$ is a summable sequence.

Now consider the case of the k-dimensional operator $T = \sum_{s=1}^{k} y^s \otimes x^s$, where $x^s, y^s \in \mathscr{H}$. This time we define the vectors $\{v_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ in $\mathbb{R}^k$ as follows:

$$\begin{aligned}
v_{2j} &= (y^1_{4j}, y^2_{4j}, \dots, y^k_{4j}) & v^*_{2j} &= (x^1_{4j}, x^2_{4j}, \dots, x^k_{4j})\\
v_{2j+1} &= (x^1_{4j+2}, x^2_{4j+2}, \dots, x^k_{4j+2}) & v^*_{2j+1} &= (y^1_{4j+2}, y^2_{4j+2}, \dots, y^k_{4j+2})\\
u_{2j} &= (x^1_{4j+1}, x^2_{4j+1}, \dots, x^k_{4j+1}) & u^*_{2j} &= (y^1_{4j+1}, y^2_{4j+1}, \dots, y^k_{4j+1})\\
u_{2j+1} &= (y^1_{4j+3}, y^2_{4j+3}, \dots, y^k_{4j+3}) & u^*_{2j+1} &= (x^1_{4j+3}, x^2_{4j+3}, \dots, x^k_{4j+3})
\end{aligned}$$

Note that the sequences $\{v_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ belong to $\ell^2(\mathbb{R}^k)$.

Now we can rewrite the equations (4.1) using the introduced vectors:

$$\begin{aligned}
\Xi_{4j} &= a_{2j}\langle u_{2j}, v_{2j}\rangle + c_{2j}\langle v_{2j+1}, v_{2j}\rangle,\\
\Xi_{4j+1} &= -d_{2j}\langle v_{2j+1}, v_{2j}\rangle + b_{2j}\langle v_{2j+1}, u_{2j}\rangle,\\
\Xi_{4j+2} &= a_{2j+1}\langle v_{2j+1}, u_{2j+1}\rangle + c_{2j+1}\langle v_{2j+1}, v_{2j+2}\rangle,\\
\Xi_{4j+3} &= -d_{2j+1}\langle v_{2j+1}, v_{2j+2}\rangle + b_{2j+1}\langle u_{2j+1}, v_{2j+2}\rangle,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^k$. These equations simplify to

$$\begin{aligned}
\Xi_{2j} &= a_j\langle u_j, v_j\rangle + c_j\langle v_{j+1}, v_j\rangle,\\
\Xi_{2j+1} &= -d_j\langle v_{j+1}, v_j\rangle + b_j\langle v_{j+1}, u_j\rangle.
\end{aligned} \tag{4.2}$$

Next we are going to analyze the necessary condition of k point density property for $\mathfrak{F}$.

PROPOSITION 4.3. *If $\{\mu_n\}_{n=1}^{\infty}$ belongs to ℓ^1 then it is possible to construct the vector sequences $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{R}^2)$ such that for any $n \geqslant 0$ we have $\Xi_n = 1$.*

COROLLARY 4.1. *If $\mathfrak{F}$ is k point dense for any $k \geqslant 2$ then $\sum_{n=1}^{\infty}|\mu_n| = \infty$.*

Proof of the Corollary. Assume the converse: $\{\mu_n\}_{n=1}^{\infty} \in \ell^1$.

We apply the proposition and get the vectors u_n and v_n. Now without loss of generality we can assume that $u_0 \neq 0$. Then consider u_0^* so that $\langle u_0, u_0^*\rangle = -1$ and set all u_n^* $(n > 0)$, v_n^* to zero. Since the trace of the resulting operator T is equal to $\sum_{n=0}^{\infty}(\langle u_n, u_n^*\rangle + \langle v_n, v_n^*\rangle) \neq 0$, Proposition 2.4 implies that $\mathfrak{F}$ is not two point dense. Trivially, when $\mathfrak{F}$ is not two point dense, it is also not k point dense for any $k \geqslant 2$.

Proof of Proposition 4.3. First we are going to present the vector lengths $V_n = |v_n|$ for each $n \geqslant 0$.

For this purpose we are going to define an auxiliary sequence $\{M_n\}_{n=1}^{\infty} \in \ell^2$ such that $V_n \geqslant M_n$ for any n. On each step n we will define V_n and M_{n+1}.

We start by setting $V_0 = M_1 = 1$.

For any $n > 0$ we have three choices for μ_n:

1. if $\mu_n = 1/|a_n| + 1/|b_n|$, we set

$$M_{n+1} = \frac{1}{\sqrt{|b_n|}}, \quad V_n = \max\left(M_n, \frac{1}{\sqrt{|a_n|}}\right),$$

2. whenever $\mu_n = (1+|a_n|)/|c_n|$, we set

$$M_{n+1} = \max\left(\frac{\sqrt{|a_n|}}{\sqrt{|c_n|}}, \frac{1}{\sqrt{|c_n|}}\right), \quad V_n = \max\left(M_n, \frac{2}{\sqrt{|c_n|}}\right),$$

3. if $\mu_n = (1+|b_n|)/|d_n|$, we set

$$M_{n+1} = \max\left(\frac{\sqrt{|b_n|}}{\sqrt{|d_n|}}, \frac{1}{\sqrt{|d_n|}}\right), \quad V_n = \max\left(M_n, \frac{2+\sqrt{|b_n|}}{\sqrt{|d_n|}}\right).$$

Now all V_n, M_n are set, and obviously $V_n \geqslant M_n$ for any $n > 0$.

Next we are going to present the vector lengths $U_n = |u_n|$, for each $n \geqslant 0$. Set U_0 to zero and for any $n > 0$ we have three cases again:

1. when $\mu_n = 1/|a_n| + 1/|b_n|$, we set $U_n = \sqrt{\frac{1}{a_n^2 V_n^2} + \frac{1}{b_n^2 V_{n+1}^2}}$,

2. whenever $\mu_n = (1+|a_n|)/|c_n|$, we set $U_n = \frac{a_n V_n}{\sqrt{c_n^2 V_{n+1}^2 V_n^2 - 1}}$,

3. if $\mu_n = (1+|b_n|)/|d_n|$, we set $U_n = \frac{b_n V_{n+1}}{\sqrt{d_n^2 V_{n+1}^2 V_n^2 - 1}}$.

Due to our choice of V_n, M_n the values U_n are well-defined for each $n > 0$.

LEMMA 4.1. *If for some nonzero real A, B, X, Y, Z we have $(AX)^{-2} + (BY)^{-2} = Z^2$, there exist such vectors x, y, z with lengths X, Y, Z correspondingly such that*

$$\begin{aligned} \langle x, y \rangle &= 0, \\ \langle x, z \rangle &= 1/A, \\ \langle y, z \rangle &= 1/B. \end{aligned} \tag{4.3}$$

Proof. Take α such that $\cos\alpha = 1/(AXZ)$ and $\sin\alpha = 1/(BYZ)$. Consider three vectors x, y, z in $\mathbb{R}^2$ with lengths X, Y, Z such that $\angle(y,z) = \pi/2 - \alpha$ and $\angle(x,z) = \alpha$. Clearly, the vectors x and y must be orthogonal now. The equations (4.3) are trivial to check.

PROPOSITION 4.4. *For any $n \geqslant 0$ there are vectors u_n, v_n with lengths U_n, V_n in $\mathbb{R}^2$ such that (4.2) are satisfied.*

Proof. We argue by induction. We start with $v_0 = (0,1)$ and $u_0 = 0$. Suppose that we have constructed a sequence of vectors v_m, u_m for all $m < n$ and v_n. We are going to build u_n and v_{n+1}. We consider three cases for μ_n.

In the first case the chosen U_n, $1/|a_nV_n|$ and $1/|b_nV_{n+1}|$ form a right triangle with hypotenuse U_n, and so here Lemma 4.1 can be applied. It follows that there are vectors u'_n, v'_n, v'_{n+1} in $\mathbb{R}^2$ with lengths U_n, V_n, V_{n+1} correspondingly such that

$$\begin{aligned} \langle u'_n, v'_n\rangle &= 1/a_n,\\ \langle u'_n, v'_{n+1}\rangle &= 1/b_n,\\ \langle v'_n, v'_{n+1}\rangle &= 0, \end{aligned} \tag{4.4}$$

which in turn yields the equations (4.2). Now we can simply rotate the triple (u'_n, v'_n, v'_{n+1}) so that v'_n coincides with v_n. We will set u_n and v_{n+1} to the rotated u'_n and v'_{n+1} accordingly. Since the rotation preserves the scalar product inside the triple, the equations (4.4) hold for u_n, v_n, v_{n+1} as well.

In the second case the chosen V_{n+1}, $a_n/(c_nU_n)$ and $1/(c_nV_n)$ also form a right triangle and Lemma 4.1 applies here as well. It implies that there are vectors u'_n, v'_n, v'_{n+1} in $\mathbb{R}^2$ with lengths U_n, V_n, V_{n+1} correspondingly such that

$$\begin{aligned} \langle u'_n, v'_n\rangle &= 0,\\ \langle u'_n, v'_{n+1}\rangle &= a_n/c_n,\\ \langle v'_n, v'_{n+1}\rangle &= 1/c_n, \end{aligned} \tag{4.5}$$

and the equations (4.2) follow from that. Using rotation again, we receive u_n, v_n, v_{n+1}.

In the third case the chosen V_n, $b_n/(d_nU_n)$ and $1/(d_nV_{n+1})$ also form a right triangle and Lemma 4.1 applies here as well. It implies that there are vectors u'_n, v'_n, v'_{n+1} in $\mathbb{R}^2$ with lengths U_n, V_n, V_{n+1} correspondingly such that

$$\begin{aligned} \langle u'_n, v'_n\rangle &= b_n/d_n,\\ \langle u'_n, v'_{n+1}\rangle &= 0,\\ \langle v'_n, v'_{n+1}\rangle &= -1/d_n, \end{aligned} \tag{4.6}$$

and the equations (4.2) are also true. One more time we do the rotation, and we get u_n, v_n, v_{n+1}.

PROPOSITION 4.5. *The following inequalities are true.*

$$\begin{aligned} M_{n+1} &\leqslant \sqrt{\mu_n},\\ V_n &\leqslant \max(2\sqrt{\mu_n}, M_n),\\ U_n &\leqslant 2\sqrt{\mu_n}. \end{aligned}$$

Proof. First two statements are trivial. For the last statement we consider the same three cases.

In the first case we have $|a_nV_n| \geqslant \sqrt{|a_n|}$ and $|b_nV_{n+1}| \geqslant |b_nM_{n+1}| = \sqrt{|b_n|}$. It follows that $U_n \leqslant \sqrt{\mu_n}$.

In the second case we have $c_nV_{n+1}V_n \geqslant c_nM_{n+1}V_n \geqslant 2$ and so

$$U_n \leqslant \frac{|a_n|V_n}{\sqrt{\frac{3}{4}c_n^2V_{n+1}^2V_n^2}} \leqslant 2\frac{|a_n|}{|c_n|V_{n+1}} \leqslant 2\frac{|a_n|}{|c_n|M_{n+1}} \leqslant 2\sqrt{\frac{|a_n|}{|c_n|}} < 2\sqrt{\mu_n}.$$

In the third case we have $d_nV_{n+1}V_n \geqslant 2$ and hence

$$U_n \leqslant \frac{|b_n|V_{n+1}}{\sqrt{\frac{3}{4}d_n^2V_{n+1}^2V_n^2}} \leqslant 2\frac{|b_n|}{|d_n|V_n} \leqslant 2\sqrt{\frac{|b_n|}{|d_n|}} < 2\sqrt{\mu_n}.$$

The last proposition implies that V_n is bounded up to some constant by $\max(\sqrt{\mu_{n-1}}, \sqrt{\mu_n})$, and $U_n \leqslant \sqrt{\mu_n}$ for any $n > 0$. Hence, the constructed sequences V_n and U_n belong to ℓ^2. That finishes the proof of Proposition 4.3.

Now due to Corollary 4.1 we get that two point density of $\mathfrak{F}$ implies the divergence of $\sum_{n=1}^{\infty}|\mu_n|$, which in turn is equivalent to rank one density property of $\mathfrak{F}$ (see Proposition 4.2). Since rank one density implies k point density for any k, Theorem 4.1 is proved.

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