Operators
and
Matrices
Volume 14, Number 2 (2020), 469–490
doi:10.7153/oam-2020-14-34

ON NONCOMMUTATIVE JOININGS III

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(*Communicated by D. R. Farenick*)

Abstract. This paper is a continuation of our investigation on noncommutative joinings, containing a study of disjointness of induced representations, topology on the space of noncommutative (relative) joinings, a semitopological semigroup structure on (relative) self-joinings, and new examples.

1. Introduction

This paper is a continuation of [2, 3], in which we studied noncommutative joinings of W^*-dynamical systems, pursuing the theory from a correspondence point of view, and motivated by its compatibility with the mixing phenomena essential to the structure theory of von Neumann algebras. We continue to justify further in this paper that our point of view is appropriate for developing a theory of noncommutative joinings parallel to the classical theory. All Hilbert spaces in this paper are separable and all von Neumann algebras have separable preduals and all embeddings or inclusions of von Neumann algebras are unital. All groups appearing in this paper are locally compact and separable.

A W^*-dynamical system (or simply a system) is a tuple $\mathfrak{N} = (N, \rho, \alpha, G)$, with N a von Neumann algebra, ρ a faithful normal state on N and α a strongly continuous action of a group G on N by ρ-preserving automorphisms. The most commonly studied dynamical system is the one in which the group is $\mathbb{R}$ and the action is via modular automorphisms (σ_t^ρ) because of its relations to quantum physics. We suppose that N is acting in standard form on the GNS space $L^2(N, \rho)$ and Ω_ρ will denote the standard vacuum vector. The associated inner product and norm on $L^2(N, \rho)$ are denoted by $\langle \cdot, \cdot \rangle_\rho$ and $\|\cdot\|_\rho$ respectively. Further, J_ρ, Δ_ρ will have their standard meanings. For details on modular theory, we refer the reader to [22, 20].

Throughout the paper, $\mathfrak{N} = (N, \rho, \alpha, G)$, $\mathfrak{M} = (M, \varphi, \beta, G)$ and $\mathfrak{B} = (B, \mu, \gamma, G)$ will denote W^*-dynamical systems. Whenever two or more systems are involved, the underlying acting group will remain the same.

Mathematics subject classification (2010): 46L55, 37A55.
Keywords and phrases: Von Neumann Algebras, ergodic theory, joinings.

Paper OaM-14-34

DEFINITION 1.1. [2] A **joining** of the W^*-dynamical systems $\mathfrak{N}$ and $\mathfrak{M}$ is a state ω on the algebraic tensor product $N \odot M^{op}$ satisfying

$$\omega(x \otimes 1_M^{op}) = \rho(x), \qquad (1)$$
$$\omega(1_N \otimes y^{op}) = \varphi(y),$$

and

$$\omega \circ (\alpha_g \otimes \beta_g^{op}) = \omega, \qquad (2)$$
$$\omega \circ (\sigma_t^{\rho} \otimes (\sigma_t^{\varphi})^{op}) = \omega,$$

for all $x \in N$, $y \in M$, $g \in G$ and $t \in \mathbb{R}$. We denote by $J_s(\mathfrak{N}, \mathfrak{M})$ the set of all joinings of $\mathfrak{N}$ and $\mathfrak{M}$.

A unital completely positive (u.c.p. henceforth) map $\Phi : N \to M$ that satisfies $\varphi \circ \Phi = \rho$ and $\sigma_t^{\varphi} \circ \Phi = \Phi \circ \sigma_t^{\rho}$ for $t \in \mathbb{R}$ is called a (ρ, φ)-**Markov map**. A (ρ, φ)-Markov map Φ that satisfies $\beta_g \circ \Phi = \Phi \circ \alpha_g$ for all $g \in G$, will be called a G-**equivariant Markov map**, or simply an equivariant Markov map. In [2], we proved that joinings can be alternatively viewed (via a bijective association) as pointed correspondences with extra structure, and as equivariant Markov maps (cf. Theorem 4.6 of [2]). We denote by $J_m(\mathfrak{N}, \mathfrak{M})$ the collection of all equivariant Markov maps from N to M. Note that by [2, Corollary 4.5], $J_m(\mathfrak{N}, \mathfrak{M})$ consists of normal maps. Under the above association $\Phi \in J_m(\mathfrak{N}, \mathfrak{M})$ corresponds to $\omega_\Phi \in J_s(\mathfrak{N}, \mathfrak{M})$, where

$$\omega_\Phi(x \otimes y^{op}) = \langle x\xi_\Phi y, \xi_\Phi \rangle_\Phi, \qquad (3)$$
$$= \langle \Phi(x)\Omega_\varphi y, \Omega_\varphi \rangle_\varphi, \ x \in N \text{ and } y \in M,$$

where ξ_Φ is the distinguished cyclic vector in the N-M Hilbert bimodule $\mathscr{H}_\Phi$ associated to Φ (see [2, §4] for details). Note that ω_Φ is normal separately in each variable.

The main result of Høegh-Krohn, Landstad and Størmer in [11] was leveraged in [2] to obtain full characterization of weak mixing with respect to the above notion of joining (cf. [2, Theorem 6.10, 6.15, 6.16]), futher motivating our approach to joinings.

In [3], we considered the notion of joining of two systems over a common subsystem from the point of view of correspondences. The modular symmetry appearing in [4] of the associated u.c.p. (Markov) maps was essential for obtaining full generalizations of classical results on relative independence of systems over a common subsystem. In fact, by using the noncommutative analogue of disintegration of measures (bimodules in our setting), we demonstrated that the relatively independent joining of two systems over a common subsystem with respect to fixed embeddings corresponds to an equivariant Markov map whose L^2-extension is a partial isometry. We also discussed relative ergodicity and primeness of systems in our framework.

In the present paper, we record further results on noncommutative joinings and the unitary representations they induce. We also study an appropriate topology on the space of joinings, as well as a natural semitopological semigroup structure on noncommutative (relative) joinings that generalizes the classical theory. This leads to a kind of functional calculus on joinings, which yields new examples of noncommutative (relative) joinings. We summarize in the next few paragraphs the results obtained in this paper.

Much of this paper is concerned with noncommutative joinings over subsystems. So, in §2 we collect all the necessary ingredients on joinings over subsystems from [3] that we will need. As state-preserving actions naturally induce unitary representations of the underlying group, it is natural to try to relate disjointness of these unitary representations to disjointness of the original systems. The former notion is the stronger of the two, as seen in Theorem 3.1. From this, we deduce that a mixing system and a rigid system admit no nontrivial joining (Corollary 3.4).

The analogue of the topology on the space of joinings in the classical case, defined by convergence on measurable rectangles, is the BW-topology on completely positive maps (Proposition 4.2). This shift in point of view reveals that the space of joinings (when viewed as equivariant Markov maps) is compact and metrizable, which is not immediately obvious if we work with a set of states on the algebraic tensor product, due to lack of Banach-Alaoglu type theorem in this context (Theorem 4.4). Thus, given systems $\mathfrak{N},\mathfrak{M}$ one can now deduce that $J_s(\mathfrak{N},\mathfrak{M})$ is compact and metrizable in the topology of pointwise convergence on $N \odot M^{op}$ (Theorem 4.5). Further, all of these topological statements pass to joinings of two systems over a common subsystem.

The fact that equivariant Markov maps have adjoints and the adjoint operation is BW-continuous guarantees that the self-joinings of a system (and self-joinings over a subsystem) form an affine semitopological semigroup with continuous involution, unique identity and unique zero such that its idempotents are precisely the conditional expectations onto subsystems (Theorem 5.1, 5.5). We emphasize here that, if we had not considered states invariant with respect to the doubled modular action in Definition 1.1, these results would remain inaccessible.

In Theorem 6.3, we generalize the notion of joinings over a common subsystem to provide new examples of joinings of two W^*-dynamical systems. We consider an analytic functional calculus on the semigroup of operators induced by joinings in Theorem 6.9, which provides a machine for producing many examples of noncommutative joinings.

2. Preliminaries

We continue to adopt the notations and hypotheses of [2, 3]. A cautious reader can look at §2 of [2] for detailed notations and conventions, which are repeated in [3]. The material presented in this section has some overlap with [3].

In this section, we recall facts on joinings over subsystems from [3] and record a criterion that characterizes joinings of two systems $\mathfrak{N}$ and $\mathfrak{M}$ over a common subsystem $\mathfrak{B}$ as elements of $J_s(\mathfrak{N},\mathfrak{M})$ and $J_m(\mathfrak{N},\mathfrak{M})$. Following [3] we have:

DEFINITION 2.1. Let $\mathfrak{B}$ and $\mathfrak{N}$ be W^*-dynamical systems. We say that $\mathfrak{B}$ is a subsystem of $\mathfrak{N}$ if there is an injective $*$-homomorphism $\iota \in J_m(\mathfrak{B},\mathfrak{N})$, and call such a map ι an embedding of the system $\mathfrak{B}$ into $\mathfrak{N}$. If $\mathfrak{M}$ is another W^*-dynamical system, we say that $\mathfrak{B}$ is a common subsystem of $\mathfrak{N}$ and $\mathfrak{M}$ if there are embeddings $\iota_N \in J_m(\mathfrak{B},\mathfrak{N})$ and $\iota_M \in J_m(\mathfrak{B},\mathfrak{M})$.

We note that if $\mathfrak{N}$ is a system, and B is an α-invariant von Neumann subalgebra of N which is the image of a normal, ρ-preserving conditional expectation $\mathbb{E}_B^N$, then $\sigma_t^\rho \circ \mathbb{E}_B^N = \mathbb{E}_B^N \circ \sigma_t^\rho$ for all $t \in \mathbb{R}$. Since such an expectation is necessarily unique [19], it also commutes with the automorphisms α_g for all $g \in G$. Therefore, $\mathfrak{D} = (B, \rho_{|B}, \alpha_{|B}, G)$ defines a subsystem of $\mathfrak{N}$ via the inclusion map of B into N. On the other hand, if $\mathfrak{B}$ is a subsystem of $\mathfrak{N}$, then the associated embedding $\iota : B \to N$ necessarily satisfies $\iota \circ \sigma_t^\mu = \sigma_t^\rho \circ \iota$ for all $t \in \mathbb{R}$, and it follows that there is a unique normal, ρ-preserving conditional expectation $\mathbb{E}_{\iota(B)}^N : N \to \iota(B)$ (cf. Theorem 5.4 of [2]). The uniqueness of $\mathbb{E}_{\iota(B)}^N$ implies, furthermore, that $\alpha_g \circ \mathbb{E}_{\iota(B)}^N = \mathbb{E}_{\iota(B)}^N \circ \alpha_g$ for all $g \in G$, so that G acts on the von Neumann subalgebra $\iota(B)$ of N. Finally, by definition ι satisfies $\mu = \rho \circ \iota$, so we can without loss of any dynamical information identify B with its image $\iota(B)$ inside N. This identification simplifies arguments in many situations. The proof of the following result can be found in [3, Proposition 2.5].

PROPOSITION 2.2. *Let $\mathfrak{N}$ and $\mathfrak{M}$ be systems, and let $\mathfrak{B}$ be a common subsystem of $\mathfrak{N}$ and $\mathfrak{M}$. Denote by ι_N and ι_M the respective embeddings of $\mathfrak{B}$ into $\mathfrak{N}$ and $\mathfrak{M}$, and ψ the state on $B \odot B^{op}$ defined by*

$$\psi(b_1 \otimes b_2^{op}) = \left\langle b_1 \Omega_\mu b_2, \Omega_\mu \right\rangle_\mu, \; b_1, b_2 \in B.$$

Let $\Phi \in J_m(\mathfrak{N}, \mathfrak{M})$. Then, the following conditions are equivalent:

(i) The restriction of Φ to $\iota_N(B)$ is the injective $*$-homomorphism $\iota_N(b) \mapsto \iota_M(b)$, $b \in B$.

(ii) The state $\omega = \omega_\Phi$ satisfies $\omega \circ (\iota_N \otimes \iota_M^{op}) = \psi$ on the $*$-algebra $B \odot B^{op}$, where ι_M^{op} is the natural map $J_\mu b^* J_\mu \mapsto J_\varphi \iota_M(b)^* J_\varphi$, $b \in B$, from $B' \cap \mathbf{B}(L^2(B,\mu)) \to M' \cap \mathbf{B}(L^2(M,\varphi))$.

Let (N,ρ) and (M,φ) be as above. Let $\Phi : N \to M$ be a normal u.c.p. map. Then there exists a normal u.c.p. map $\Phi^* : M \to N$ satisfying

$$\rho(\Phi^*(y)x) = \varphi(y\Phi(x)), \; y \in M \text{ and } x \in N, \tag{4}$$

if and only if $\varphi \circ \Phi = \rho$ and $\Phi \circ \sigma_t^\rho = \sigma_t^\varphi \circ \Phi$ for all $t \in \mathbb{R}$ [1]. The u.c.p. map Φ^* is said to be the Accardi-Cecchini adjoint of Φ.

DEFINITION 2.3. Let $\mathfrak{N}$ and $\mathfrak{M}$ be systems, and let $\mathfrak{B}$ be a common subsystem of $\mathfrak{N}$ and $\mathfrak{M}$ with embeddings $\iota_N \in J_m(\mathfrak{B}, \mathfrak{N})$ and $\iota_M \in J_m(\mathfrak{B}, \mathfrak{M})$. We say that $\Phi \in J_m(\mathfrak{N}, \mathfrak{M})$ is a joining of $\mathfrak{N}$ and $\mathfrak{M}$ over the common subsystem $\mathfrak{B}$ if the restriction of Φ to $\iota_N(B)$ is the $*$-isomorphism $\iota_M \circ \iota_N^{-1}$ of $\iota_N(B)$ with $\iota_M(B)$ (i.e., Φ satisfies the equivalent conditions of Proposition 2.2). Denote by $J_{\mathfrak{B},m}(\mathfrak{N}, \mathfrak{M})$ the collection of all joinings of $\mathfrak{N}$ and $\mathfrak{M}$ over the common subsystem $\mathfrak{B}$.

Note that $J_{\mathfrak{B},m}(\mathfrak{N}, \mathfrak{M})$ is nonempty and $\iota_M \circ \iota_N^* \in J_{\mathfrak{B},m}(\mathfrak{N}, \mathfrak{M})$. The joining $\iota_M \circ \iota_N^* \in J_{\mathfrak{B},m}(\mathfrak{N}, \mathfrak{M})$ is called the *relatively independent* joining of $\mathfrak{N}$ and $\mathfrak{M}$ over $\mathfrak{B}$. Though 'relative disjointness' will not be studied in this paper, we state its definition for the sake of completeness.

DEFINITION 2.4. Let $\mathfrak{N}$ and $\mathfrak{M}$ be systems, and let $\mathfrak{B}$ be a common subsystem of $\mathfrak{N}$ and $\mathfrak{M}$ with embeddings $\iota_N \in J_m(\mathfrak{B},\mathfrak{N})$ and $\iota_M \in J_m(\mathfrak{B},\mathfrak{M})$. We say that the systems $\mathfrak{N}$ and $\mathfrak{M}$ are relatively independent over $\mathfrak{B}$ (or disjoint over $\mathfrak{B}$) if $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M}) = \{\iota_M \circ \iota_N^*\}$.

For an algebraic characterization of *relatively independent* joinings, check [3, Theorem 3.3].

PROPOSITION 2.5. *Let $\mathfrak{B}$ be a common subsystem of two systems $\mathfrak{N}$ and $\mathfrak{M}$ with respect to embeddings ι_N and ι_M respectively. Let $\Phi \in J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})$. Then, $\Phi^* \in J_{\mathfrak{B},m}(\mathfrak{M},\mathfrak{N})$.*

For a proof of Proposition 2.5, see [3, Proposition 2.8].

Thus, we have the following result which generalizes disintegration of measures to obtain joinings over subsystems and this result will be useful throughout the paper.

THEOREM 2.6. (*Disintegration of bimodules*) *Let $\mathfrak{N}$ and $\mathfrak{M}$ be systems, and let $\mathfrak{B}$ be a common subsystem of $\mathfrak{N}$ and $\mathfrak{M}$ with embeddings $\iota_N \in J_m(\mathfrak{B},\mathfrak{N})$ and $\iota_M \in J_m(\mathfrak{B},\mathfrak{M})$. Let $\mathbb{E}^N_{\iota_N(B)}$ and $\mathbb{E}^M_{\iota_M(B)}$ respectively denote the ρ and φ preserving faithful normal conditional expectations onto $\iota_N(B)$ and $\iota_M(B)$. Then*

$$J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M}) = \{\Phi \in J_m(\mathfrak{N},\mathfrak{M}) : \Phi\mathbb{E}^N_{\iota_N(B)} = \mathbb{E}^M_{\iota_M(B)}\Phi\mathbb{E}^N_{\iota_N(B)} = \iota_M \circ \iota_N^{-1} \circ \iota_N^*,$$
$$\Phi^*\mathbb{E}^M_{\iota_M(B)} = \mathbb{E}^N_{\iota_N(B)}\Phi^*\mathbb{E}^M_{\iota_M(B)} = \iota_N \circ \iota_M^{-1} \circ \iota_M^*\}.$$

Moreover, the bijective correspondence $J_m(\mathfrak{N},\mathfrak{M}) \ni \Phi \overset{j}{\mapsto} \omega_\Phi \in J_s(\mathfrak{N},\mathfrak{M})$ (in Theorem 4.6 [2]) restricted to $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})$ establishes a bijective correspondence between $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})$ and

$$J_{\mathfrak{B},s}(\mathfrak{N},\mathfrak{M}) = \{\omega \in J_s(\mathfrak{N},\mathfrak{M}) : \omega_{j^{-1}\omega}(\iota_N(b_1) \otimes \iota_M^{op}(b_2^{op})) = \langle b_1\Omega_\mu b_2, \Omega_\mu\rangle_\mu,$$
$$b_1, b_2 \in B\}.$$

Proof. The proof follows easily from Propositions 2.2, 2.5 and Definition 2.3. □

REMARK 2.7. (i) Observe that $J_{\mathfrak{B},s}$ generalizes the classical notion of joinings over subsystems as measures. In the classical case, joinings over subsystems are measures obtained via disintegration of measures over the common factor [9, pp. 130].
(ii) Note that, when $B{=}\mathbb{C}$, then $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M}){=}J_m(\mathfrak{N},\mathfrak{M})$ and $J_{\mathfrak{B},s}(\mathfrak{N},\mathfrak{M}){=}J_s(\mathfrak{N},\mathfrak{M})$.
(iii) We will denote $j^{-1}\omega$ for $\omega \in J_s(\mathfrak{N},\mathfrak{M})$ (appearing in Theorem 2.6) by Φ_ω. Thus, combining with Eq. (3), it follows that $\omega_{\Phi_\omega} = \omega$.

The following result is a direct consequence of Theorem 2.6 and will be useful in §5.

THEOREM 2.8. *Let $\mathfrak{B}$ be a subsystem of a system $\mathfrak{N}$ with embedding ι. Then*

$$J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N}) = \{\Phi \in J_m(\mathfrak{N},\mathfrak{N}) : \Phi\mathbb{E}^N_{\iota(B)} = \mathbb{E}^N_{\iota(B)}\Phi = \mathbb{E}^N_{\iota(B)}\}.$$

3. Associated unitary representations of groups

In this section, we relate the disjointness of representations of groups to the disjointness of W^*-dynamical systems. This leads us to consider connections of the representation theory of full group C^*-algebras with our joining theory. Thus, the results of this section further show that noncommutative joining theory can be approached from several interconnected viewpoints.

Let $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ be two unitary representations of a group G. Recall that $(\pi_1, \mathscr{H}_1)$ is (unitarily) equivalent (resp. subequivalent) to $(\pi_2, \mathscr{H}_2)$ written $\pi_1 \sim \pi_2$ (resp. $\pi_1 \preceq \pi_2$), if there exists a unitary (resp. isometry) $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that $U\pi_1(\cdot)U^* = \pi_2(\cdot)$ (resp. $U\pi_1(\cdot)U^* = \pi_2(\cdot)UU^*$). They are said to be *disjoint*, if no subrepresentation of π_1 is equivalent to a subrepresentation of π_2. Note that disjointness of unitary representations generalizes the notion of *singularity of measures*.

Recall that any unitary representation of a group G on a Hilbert space induces a canonical representation of $C^*(G)$ by the universal property. Thus, the study of G-equivariant quantum channels is naturally tied to the study of intertwiners of pairs of representations of $C^*(G)$. For detailed information on quantum channels see [10].

Let N and M be von Neumann algebras equipped with faithful normal states ρ and φ respectively. Let $\Phi : N \to M$ be a u.c.p. map such that $\varphi \circ \Phi = \rho$. Define the L^2-extension of Φ as $T_\Phi : L^2(N,\rho) \to L^2(M,\varphi)$ by $T_\Phi(x\Omega_\rho) = \Phi(x)\Omega_\varphi$ for all $x \in N$. By Kadison-Schwarz inequality T_Φ is bounded and $\|T_\Phi\| = 1$.

Let $\mathfrak{N}$ and $\mathfrak{M}$ be systems, and let $\Phi \in J_m(\mathfrak{N},\mathfrak{M})$. Then $\Phi \circ \alpha_g = \beta_g \circ \Phi$ and $\Phi \circ \sigma_t^\rho = \sigma_t^\varphi \circ \Phi$ for all $g \in G$ and $t \in \mathbb{R}$. Fix $g \in G$. Since $\alpha_g \in Aut(N,\rho)$ (resp. $\beta_g \in Aut(M,\varphi)$), so α_g (resp. β_g) is implemented on $L^2(N,\rho)$ (resp. $L^2(M,\varphi)$) by an unique unitary U_g (resp. V_g) satisfying $U_g(x\Omega_\rho) = \alpha_g(x)\Omega_\rho$, $x \in N$ (resp. $V_g(y\Omega_\varphi) = \beta_g(y)\Omega_\varphi$, $y \in M$). It is easy to check that $\pi_U : g \mapsto U_g$ (resp. $\pi_V : g \mapsto V_g$) is a strongly continuous unitary representation of G on $L^2(N,\rho)$ (resp. $L^2(M,\varphi)$). Similarly, when the action of G is replaced by the action of $\mathbb{R}$ by modular automorphisms, the associated unitary representation on $L^2(N,\rho)$ (resp. $L^2(M,\varphi)$) is $\pi_{\Delta_\rho} : t \mapsto \Delta_\rho^{it}$ (resp. $\pi_{\Delta_\varphi} : t \mapsto \Delta_\varphi^{it}$) for $t \in \mathbb{R}$. Thus, for $g \in G$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} T_\Phi U_g &= V_g T_\Phi; \\ T_\Phi \Delta_\rho^{it} &= \Delta_\varphi^{it} T_\Phi. \end{aligned} \tag{5}$$

We have the following analogue of a classical theorem related to spectral measures.

THEOREM 3.1. *Let $\mathfrak{N}$ and $\mathfrak{M}$ be two W^*-dynamical systems such that the associated representations $\pi_U^0 : g \mapsto U_{g|\mathbb{C}\Omega_\rho{}^\perp}$ and $\pi_V^0 : g \mapsto V_{g|\mathbb{C}\Omega_\varphi{}^\perp}$ of G on $L^2(N,\rho) \ominus \mathbb{C}\Omega_\rho$ and $L^2(M,\varphi) \ominus \mathbb{C}\Omega_\varphi$ respectively are disjoint. Then $J_m(\mathfrak{N},\mathfrak{M})$ is trivial, i.e. the systems are disjoint.*

Proof. Let $\mathscr{H}_0 = L^2(N,\rho) \ominus \mathbb{C}\Omega_\rho$ and $\mathscr{H}_0' = L^2(M,\varphi) \ominus \mathbb{C}\Omega_\varphi$. Then, $\mathscr{H}_0$ and $\mathscr{H}_0'$ are respectively reducing subspaces of π_U and π_V. Let $P_{\mathscr{H}_0}$ and $P_{\mathscr{H}_0'}$ denote the orthogonal projections onto $\mathscr{H}_0$ and $\mathscr{H}_0'$ respectively, and let $\Phi \in J_m(\mathfrak{N},\mathfrak{M})$. Hence, from Eq. (5), we have $T_\Phi U_g = V_g T_\Phi$, for all $g \in G$. Thus,

$$T_\Phi P_{\mathscr{H}_0} U_g = T_\Phi U_g P_{\mathscr{H}_0} = V_g T_\Phi P_{\mathscr{H}_0},\ g \in G.$$

With $T_0 = T_\Phi P_{\mathscr{H}_0}$, we have $T_0 U_g = V_g T_0$ for all $g \in G$. By a standard result in unitary representation theory (cf. Proposition A.1.4 [5]), there exists a partial isometry W with initial space $Ker(T_0)^\perp$ and final space $\overline{Ran(T_0)}$ such that $WU_g = V_g W$, for all $g \in G$. Note that $Ker(T_0)^\perp$ and $\overline{Ran(T_0)}$ are both invariant with respect to π_U^0 and π_V^0 respectively. However, the hypothesis forces that $Ker(T_0) = \mathscr{H}_0$. Since $\varphi \circ \Phi = \rho$, it follows that $\Phi(x) = \rho(x) 1_M$. □

In order to avoid confusion regarding the term "disjoint" arising from two contexts, in this section we will say that $J_m(\mathfrak{N},\mathfrak{M})$ is trivial if $J_m(\mathfrak{N},\mathfrak{M}) = \{\rho(\cdot)1_M\}$.

COROLLARY 3.2. *If $J_m(\mathfrak{N},\mathfrak{M})$ is nontrivial, then π_U and π_V have nontrivial equivalent subrepresentations.*

REMARK 3.3. We remark that the converse of Corollary 3.2 is false. There exists a Gaussian automorphism with countable Lebesgue spectrum and zero entropy [15]. Note that K-automorphisms have countable Lebesgue spectrum as well but are of completely positive entropy (equivalently, they have trivial Pinsker σ-algebra) [17]. However, it is known that zero entropy systems and K-systems are disjoint [13].

For a countable discrete group G, a W^*-dynamical system $\mathfrak{M}$ is *rigid* if $\overline{\{V_g : g \in G\}}^{s.o.t.}$ is not a discrete subgroup of $\mathscr{U}(L^2(M,\varphi))$. In the next result (Corollary 3.4), G is assumed to be countable and discrete. Further, $\mathfrak{M}$ is said to be *mixing* if

$$\varphi(y\beta_g(x)) \to \varphi(y)\varphi(x), \text{ as } g \to \infty, \text{ for all } x,y \in M.$$

COROLLARY 3.4. *If $\mathfrak{N}$ is a mixing system and $\mathfrak{M}$ is a rigid system, then $J_m(\mathfrak{N},\mathfrak{M})$ is trivial.*

Proof. By Theorem 3.1, it is enough to show that the representations π_U^0 and π_V^0 are disjoint. Let $p \in \left(\pi_U(C^*(G))\right)'$ and $q \in \left(\pi_V(C^*(G))\right)'$ be projections that satisfy $p\Omega_\rho = 0$, $q\Omega_\varphi = 0$ and $\pi_{U|p(L^2(N,\rho))} \sim \pi_{V|q(L^2(M,\varphi))}$. By the hypothesis, $\pi_U(g)p \overset{w.o.t.}{\to} 0$ as $g \to \infty$, i.e., given $\zeta_1, \zeta_2 \in L^2(N,\rho) \ominus \mathbb{C}\Omega_\rho$ and $\varepsilon > 0$, there exists a finite set $F \subseteq G$ such that $|\langle \pi_U(g)p\zeta_1, \zeta_2\rangle| < \varepsilon$ for all $g \in G \setminus F$. On the other hand, there exists a unitary $W \in \mathbf{B}(L^2(M,\varphi))$ and a sequence $\{g_n\}_n$ in G such that $W \neq \pi_V(g_n) \overset{s.o.t.}{\to} W$ as $n \to \infty$. Thus, $\pi_V(g_n)q \overset{s.o.t.}{\to} Wq$ as $n \to \infty$. This contradicts the fact that $\pi_{U|p(L^2(N,\rho))} \sim \pi_{V|q(L^2(M,\varphi))}$. Thus, $p = 0$ and $q = 0$. Consequently, $J_m(\mathfrak{N},\mathfrak{M})$ is trivial by Theorem 3.1. □

Letting $\pi_{\Delta_\rho}^0$ and $\pi_{\Delta_\varphi}^0$ denote the obvious subrepresentations of $\mathbb{R}$ on $L^2(N,\rho) \ominus \mathbb{C}\Omega_\rho$ and $L^2(M,\varphi) \ominus \mathbb{C}\Omega_\varphi$ associated respectively with π_{Δ_ρ} and π_{Δ_φ}, the following result is evident. We leave the proof to the reader.

THEOREM 3.5. *If $\pi^0_{\Delta_\rho}$ and $\pi^0_{\Delta_\varphi}$ are disjoint, then $J_m(\mathfrak{N},\mathfrak{M})$ is trivial.*

REMARK 3.6. (i) Eq. (5) directly implies that whenever $\Phi \in J_m(\mathfrak{N},\mathfrak{M})$, it follows that $\Phi \circ \mathbb{E}_{N^\rho} = \mathbb{E}_{M^\varphi} \circ \Phi$, i.e., joinings respect the centralizers of the states. Thus, if ρ is a trace and φ is such that $M^\varphi = \mathbb{C}1_M$ (forcing M is a type III_1 factor unless $M = \mathbb{C}$), then the systems $\mathfrak{N}$ and $\mathfrak{M}$ are disjoint regardless of the group and the action.
(ii) The notion of *spectral state* of an unitary representation of a locally compact separable group G was defined in [14] to be a state on $C^*(G)$ which encodes the spectral properties of the representation. It is obvious that two representations of G are disjoint when the associated states on $C^*(G)$ yield GNS representations which have no nonzero equivalent sub-representations. Thus, disjointness of representations can also be discussed under the light of spectral states.
(iii) Other basic disjointness results of systems have been covered in [2]. Note that the induced representations of G on the orthocomplement of the vacuum vectors corresponding to compact and weakly mixing systems are disjoint, and this can be argued via spectral states (see for instance Theorem 5.17 [14]). This provides a different viewpoint to the fact that there is no nontrivial joining between compact and weakly mixing systems.

4. Topology on joinings

In [2, 3] and so far in this paper, we have studied single joining at a time. From this section onward, we study the collection of all joinings of $\mathfrak{N}$ and $\mathfrak{M}$, viewed either as states in $J_s(\mathfrak{N},\mathfrak{M})$ or as u.c.p. maps in $J_m(\mathfrak{N},\mathfrak{M})$. We will see that the topology on joinings in the classical case (given by convergence on measurable rectangles) is recovered via the BW-topology on $J_m(\mathfrak{N},\mathfrak{M})$ and so the BW-topology is the appropriate generalization of the natural topology on classical joinings. To keep the results as general as possible, we will topologize the collection of joinings of two systems over a common subsystem with fixed embeddings.

Let us first recall some basic facts about the BW-topology. Let X be a Banach space, and let $\mathscr{H}$ be a Hilbert space. Then a bounded net $\{L_\lambda\}_\lambda$ in $\mathbf{B}(X,\mathbf{B}(\mathscr{H}))$ converges in the BW-topology to $L \in \mathbf{B}(X,\mathbf{B}(\mathscr{H}))$ if and only if, $\langle L_\lambda(x)\xi,\eta\rangle_{\mathscr{H}}$ converges to $\langle L(x)\xi,\eta\rangle_{\mathscr{H}}$ for all $\xi,\eta \in \mathscr{H}$ and $x \in X$ (cf. Proposition 7.3 [16]). The BW-topology is Hausdorff. If N and M are von Neumann algebras with $M \subseteq \mathbf{B}(\mathscr{H})$, then by Proposition 7.4 of [16], $CP(N,M) = \{\Phi : N \to M \mid \Phi \text{ is u.c.p.}\}$ is compact in the BW-topology. It is a standard fact that the BW-topology on $CP(N,M)$ is independent of the Hilbert space on which M acts faithfully.

PROPOSITION 4.1. *Let N and M be von Neumann algebras and let ρ and φ be faithful normal states on N and M respectively. Let*

$$CP(N,M,\rho,\varphi) = \{\Phi : N \to M \mid \Phi \textit{ is u.c.p. and } \varphi\circ\Phi = \rho\}.$$

Then $CP(N,M,\rho,\varphi)$ is compact and metrizable in the BW-topology.

Proof. First note that $\Phi \in CP(N,M,\rho,\varphi)$ is normal by the proof of Corollary 4.5 of [2]. Let $\{\Phi_\lambda\}_\lambda$ be a net in $CP(N,M,\rho,\varphi)$ such that $\Phi_\lambda \overset{BW}{\to} \Phi \in CP(N,M)$. Then for each $x \in N$, the net $\{\Phi_\lambda(x)\}_\lambda$ in M is bounded and converges to $\Phi(x)$ in the weak operator topology $($*w.o.t.* in the sequel$)$, and consequently $\Phi_\lambda(x) \overset{w^*}{\to} \Phi(x)$. Since φ is normal, $\varphi(\Phi(x)) = \varphi(\lim_\lambda \Phi_\lambda(x)) = \lim_\lambda \varphi(\Phi_\lambda(x)) = \rho(x)$. This proves that $CP(N,M,\rho,\varphi)$ is a closed subset of a compact Hausdorff space and is thus compact.

Since N and M have separable preduals, $CP(N,M,\rho,\varphi)$ is metrizable in the BW-topology. In fact, note that N contains a w^*-dense norm separable C^*-subalgebra B $($Lemma 14.1.17, KRII$)$. Given any dense sequence $\{\xi_n\}$ in the unit ball of $L^2(M,\varphi)$ and any w^*-dense sequence $\{x_n\}$ in the unit ball of B $($by Kaplansky density theorem$)$, define a metric d on $CP(N,M,\rho,\varphi)$ by

$$d(\Phi,\Psi) = \sum_{n,k,l\in\mathbb{N}} \frac{1}{2^{n+k+l}} \left|\langle \Phi(x_n)\xi_k,\xi_l\rangle_\varphi - \langle \Psi(x_n)\xi_k,\xi_l\rangle_\varphi\right|,$$

for all $\Phi,\Psi \in CP(N,M,\rho,\varphi)$. The identity map $i : (CP(N,M,\rho,\varphi),BW) \to (CP(N,M,\rho,\varphi),d)$ is a continuous and bijective map from a compact space to a metrizable space, and is therefore a homeomorphism. □

Therefore, in view of Proposition 4.1, we will work with sequences of u.c.p. maps while dealing with convergence in the BW-topology.

We will now justify that the BW-topology generalizes the topology on joinings in the classical case. If N and M are abelian with faithful normal states ρ and φ respectively, then there are Borel probability spaces (X,μ) and (Y,ν), with X and Y being compact and metrizable spaces such that $N \cong L^\infty(X,\mu)$, $M \cong L^\infty(Y,\nu)$, $\rho = \int_X \cdot\, d\mu$ and $\varphi = \int_Y \cdot\, d\nu$. If $\Phi : L^\infty(X,\mu) \to L^\infty(Y,\nu)$ is unital and positive, then it is u.c.p., and thus there exists a unique Borel probability measure η_Φ on $X \times Y$ such that

$$\begin{aligned}&(\pi_X)_*\eta_\Phi \ll \mu, &(6)\\ &\int_Y g\Phi(f)d\nu = \int_{X\times Y} (f\otimes g)d\eta_\Phi,\ f \in L^\infty(X,\mu), g \in L^\infty(Y,\nu).\end{aligned}$$

Note that in this case, it is automatic that $(\pi_Y)_*\eta_\Phi = \nu$ and $(\pi_X)_*\eta_\Phi = \nu \circ \Phi$.

PROPOSITION 4.2. *If $\Phi_n,\Phi \in CP(L^\infty(X,\mu),L^\infty(Y,\nu),\mu,\nu)$, for $n \geqslant 1$, then $\Phi_n \to \Phi$ in the BW-topology if and only if $\eta_{\Phi_n}(A\times B) \to \eta_\Phi(A\times B)$ for all measurable subsets $A \subseteq X$ and $B \subseteq Y$.*

Proof. If $A \subseteq X$ and $B \subseteq Y$ are measurable, then $\Phi_n \to \Phi$ in the BW-topology implies $\Phi_n(\chi_A) \overset{w^*}{\to} \Phi(\chi_A)$ and therefore $\nu(\chi_B\Phi_n(\chi_A)) \to \nu(\chi_B\Phi(\chi_A))$. By Eq. (6), it follows that $\eta_{\Phi_n}(A\times B) \to \eta_\Phi(A\times B)$.

For the converse, fix a measurable set $B \subseteq Y$ and $f \in L^\infty(X,\mu)$. Given $\varepsilon > 0$, there exists a simple function $s \in L^\infty(X,\mu)$ such that $\|f-s\|_\infty < \varepsilon$. By the hypothesis and Eq. (6) choose n_0 such that

$$|\nu(\chi_B\Phi_n(s)) - \nu(\chi_B\Phi(s))| < \varepsilon$$

for all $n \geqslant n_0$. Then Eq. (6) yields that for all $n \geqslant n_0$,

$$\begin{aligned}
&|\nu(\chi_B\Phi_n(f)) - \nu(\chi_B\Phi(f))| \\
&\leqslant |\nu(\chi_B\Phi_n(f-s))| + |\nu(\chi_B(\Phi_n(s) - \Phi(s)))| + |\nu(\chi_B\Phi(s-f))| \\
&< 3\varepsilon.
\end{aligned}$$

Thus, $\nu(\chi_B\Phi_n(f)) \to \nu(\chi_B\Phi(f))$ for all $f \in L^\infty(X,\mu)$ and for all measurable subsets $B \subseteq Y$. Using the fact that $\{\Phi_n(f)\}$ is norm-bounded in $L^\infty(Y,\nu)$ and that simple functions are dense in $L^1(Y,\nu)$, a similar triangle inequality argument as the above shows that $\nu(g\Phi_n(f)) \to \nu(g\Phi(f))$ for all $f \in L^\infty(X,\mu)$ and $g \in L^1(Y,\nu)$. Thus, $\Phi_n \to \Phi$ in the BW-topology. □

Thus, in view of Proposition 4.2 and Theorem 6.2 of Chap. 6 [9], the BW-topology generalizes the topology on joinings in the classical case.

The following proposition is an interesting fact in its own right and will be of use in this section.

PROPOSITION 4.3. *Let $\mathbb{M}(N,M,\rho,\varphi)$ denote the collection of all (ρ,φ)-Markov maps from N to M. If $\Phi_n \in \mathbb{M}(N,M,\rho,\varphi)$ and $\Phi_n \to \Phi \in CP(N,M,\rho,\varphi)$ in the BW-topology, then $\Phi \in \mathbb{M}(N,M,\rho,\varphi)$ and $\Phi_n^* \to \Phi^*$ in the BW-topology of $CP(M,N,\varphi,\rho)$.*

Proof. First, let $\Psi_n \in CP(N,M,\rho,\varphi)$ be such that $\Psi_n \to \Psi \in CP(N,M,\rho,\varphi)$ in the BW-topology. Then, $\Psi_n(x) \xrightarrow{w^*} \Psi(x)$ for each $x \in N$. So, $\langle \Psi_n(x)\xi,\eta\rangle_\varphi \to \langle \Psi(x)\xi,\eta\rangle_\varphi$ for all $\xi,\eta \in L^2(M,\varphi)$. Consider the L^2-extension of u.c.p. maps defined in §3. Consequently,

$$\langle T_{\Psi_n}(x\Omega_\rho),\eta\rangle_\varphi = \langle \Psi_n(x)\Omega_\varphi,\eta\rangle_\varphi \to \langle \Psi(x)\Omega_\varphi,\eta\rangle_\varphi = \langle T_\Psi(x\Omega_\rho),\eta\rangle_\varphi,$$

for all $x \in N$ and $\eta \in L^2(M,\varphi)$. Since $\sup_n \|T_{\Psi_n}\| < \infty$, it follows that $T_{\Psi_n} \to T_\Psi$ in the *w.o.t*.

It is routine to check that $\Phi \in \mathbb{M}(N,M,\rho,\varphi)$. Note that by Eq. (4) and the discussion following it, Φ_n^* and Φ^* exist and they are (φ,ρ)-Markov maps from M to N. Further, from Eq. (4) it follows that

$$T_\Phi^* = T_{\Phi^*}. \tag{7}$$

Fix $x,z \in N$ and $y \in M$. Therefore,

$$\begin{aligned}
\langle \Phi_n^*(y)\Omega_\rho x,\Omega_\rho z\rangle_\rho &= \langle \Omega_\rho x,\Phi_n^*(y^*)\Omega_\rho z\rangle_\rho \\
&= \langle \Omega_\rho x,\Phi_n^*(y^*)J_\rho z^* J_\rho \Omega_\rho\rangle_\rho \\
&= \langle \Omega_\rho xz^*,\Phi_n^*(y^*)\Omega_\rho\rangle_\rho \\
&= \langle \Omega_\rho xz^*,T_{\Phi_n^*}(y^*\Omega_\varphi)\rangle_\rho \\
&= \langle T_{\Phi_n}(\Omega_\rho xz^*),y^*\Omega_\varphi\rangle_\varphi \text{ (by Eq. (7))} \\
&\to \langle T_\Phi(\Omega_\rho xz^*),y^*\Omega_\varphi\rangle_\varphi \\
&= \langle \Omega_\rho xz^*,T_{\Phi^*}(y^*\Omega_\varphi)\rangle_\rho \text{ (by Eq. (7))}
\end{aligned}$$

$$= \langle J_\rho z J_\rho \Omega_\rho x, \Phi^*(y^*)\Omega_\rho \rangle_\rho$$
$$= \langle \Phi^*(y)\Omega_\rho x, \Omega_\rho z \rangle_\rho .$$

Since $\sup_n \|\Phi_n^*(y)\| < \infty$, conclude that $\Phi_n^*(y) \to \Phi^*(y)$ in *w.o.t.* and hence in the w^*-topology. Thus, $\Phi_n^* \to \Phi^*$ in the BW-topology. □

THEOREM 4.4. *Let $\mathfrak{N}$ and $\mathfrak{M}$ be systems, and let $\mathfrak{B}$ be a common subsystem of $\mathfrak{N}$ and $\mathfrak{M}$ with embeddings $\iota_N \in J_m(\mathfrak{B},\mathfrak{N})$ and $\iota_M \in J_m(\mathfrak{B},\mathfrak{M})$. Then $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})$ is a compact metrizable space in the BW-topology.*

Proof. Note that by Theorem 2.6 and Proposition 4.3, it follows that the subset $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})$ of $CP(N,M,\rho,\varphi)$ is BW-closed, and the latter is compact and metrizable in the BW-topology by Proposition 4.1. □

The BW-topology on $J_m(\mathfrak{N},\mathfrak{M})$ induces a topology on the space $J_s(\mathfrak{N},\mathfrak{M})$. We explicitly describe a basis for this topology which will be of use in this paper. Let $\omega \in J_s(\mathfrak{N},\mathfrak{M})$. By Eq. (3), for all $x \in N, y \in M$,

$$\omega(x \otimes y^{op}) = \langle \Phi_\omega(x)\Omega_\varphi y, \Omega_\varphi \rangle_\varphi . \tag{8}$$

We see that Eq. (8) generalizes Eq. (6) using [2, Eq. (5)]. Thus for $\omega_0 \in J_s(\mathfrak{N},\mathfrak{M})$ and finite subsets $E \subseteq N$ and $F \subseteq M$ and $\varepsilon > 0$, define a basic open neighborhood $N(\omega_0,E,F,\varepsilon)$ of ω_0 by $N(\omega_0,E,F,\varepsilon) = \{\omega \in J_s(\mathfrak{N},\mathfrak{M}) : |\omega(x \otimes y^{op}) - \omega_0(x \otimes y^{op})| < \varepsilon, \forall x \in E,\ \forall\, y \in F\}$. Generate a topology $\mathfrak{T}$ on $J_s(\mathfrak{N},\mathfrak{M})$ with this family of basic open neighborhoods. It is evident that $\mathfrak{T}$ is Hausdorff. The convergence of elements with respect to $\mathfrak{T}$ is clear.

THEOREM 4.5. *Let $\mathfrak{N}$ and $\mathfrak{M}$ be systems, and let $\mathfrak{B}$ be a common subsystem of $\mathfrak{N}$ and $\mathfrak{M}$ with embeddings $\iota_N \in J_m(\mathfrak{B},\mathfrak{N})$ and $\iota_M \in J_m(\mathfrak{B},\mathfrak{M})$. The map $\Lambda : (J_m(\mathfrak{N},\mathfrak{M}),BW) \to (J_s(\mathfrak{N},\mathfrak{M}),\mathfrak{T})$ defined by $\Lambda(\Phi) = \omega_\Phi$ (cf. Eq. (3)) is an affine homeomorphism, i.e., a homeomorphism that preserves convex combinations. Consequently, $(J_s(\mathfrak{N},\mathfrak{M}),\mathfrak{T})$ is compact and metrizable with a $\mathfrak{T}$-compatible metric $\tilde{d}$ defined for $\omega_1,\omega_2 \in J_s(\mathfrak{N},\mathfrak{M})$, by $\tilde{d}(\omega_1,\omega_2) = d(\Lambda^{-1}(\omega_1),\Lambda^{-1}(\omega_2))$, where d is the metric defined in Proposition 4.1. Further, Λ restricted to $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})$ is a homeomorphism between $(J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M}),BW)$ and $(J_{\mathfrak{B},s}(\mathfrak{N},\mathfrak{M}),\mathfrak{T})$. Consequently, $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})$ and $J_{\mathfrak{B},s}(\mathfrak{N},\mathfrak{M})$ are also respectively compact and metrizable in the BW-topology and $\mathfrak{T}$.*

Proof. Note that both $J_m(\mathfrak{N},\mathfrak{M})$ and $J_s(\mathfrak{N},\mathfrak{M})$ are convex sets. That Λ is bijective follows from [2, Theorem 4.6].

We now proceed to show that Λ is continuous. Indeed, let $J_m(\mathfrak{N},\mathfrak{M}) \ni \Phi_n \overset{BW}{\to} \Phi \in J_m(\mathfrak{N},\mathfrak{M})$ and let $x \in N$ and $y \in M$. Then,

$$\omega_{\Phi_n}(x \otimes y^{op}) = \langle \Phi_n(x)\Omega_\varphi y, \Omega_\varphi \rangle_\varphi \to \langle \Phi(x)\Omega_\varphi y, \Omega_\varphi \rangle_\varphi = \omega_\Phi(x \otimes y^{op}),$$

i.e., $\omega_{\Phi_n} \to \omega_\Phi$ in $\mathfrak{T}$. Thus, Λ is continuous. As $J_m(\mathfrak{N},\mathfrak{M})$ is compact (see Theorem 4.4) and $\mathfrak{T}$ is Hausdorff, it follows that Λ is a homeomorphism. This homeomorphism is clearly an affine map.

The statements regarding $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})$ and $J_{\mathfrak{B},s}(\mathfrak{N},\mathfrak{M})$ are direct consequences of Theorem 2.6 and Theorem 4.4. We omit the details. $\square$

5. Semigroup structure on joinings

In this section, we study algebraic and topological structures on $J_m(\mathfrak{N},\mathfrak{M})$ and $J_s(\mathfrak{N},\mathfrak{M})$ and their relative counterparts. From the operator algebra point of view, it is natural to take advantage of the rich theory of u.c.p. maps. To simplify notations, we use $CP(N,\rho),\mathbb{M}(N,\rho)$ to denote $CP(N,N,\rho,\rho)$ and $\mathbb{M}(N,N,\rho,\rho)$, respectively.

A compact semitopological semigroup is a compact Hausdorff space S equipped with a semigroup structure, i.e., an associative multiplication $S\times S\ni(t,s)\mapsto ts\in S$, such that the left and right multiplications on S are **separately** [1] continuous. Then, S has *idempotents*, i.e., $E(S)=\{s\in S:s^2=s\}\neq\emptyset$ and has a minimal ideal $M(S)$ which is a paragroup (cf. pp. 46 [6] for definitions). If S has a unique zero (both right and left), then $M(S)$ is unique and is the zero element, thus the paragroup above collapses to a point.

An affine semigroup is a semigroup structure on a convex set S such that the maps $t\mapsto ts$ and $t\mapsto st$ from S to itself are affine for all $s\in S$. If S is a compact semitopological affine semigroup, then every unit (invertible element) is an extreme point of S. For a gentle introduction to affine semigroups see Chapter II [6].

THEOREM 5.1. *Let $\mathfrak{B}$ be a subsystem of $\mathfrak{N}$. There is a BW-continuous involution $*:\mathbb{M}(N,\rho)\to\mathbb{M}(N,\rho)$ which leaves $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ invariant such that - (i) $(\Phi\Psi)^*=\Psi^*\Phi^*$, (ii) $(\Phi^*)^*=\Phi$ for $\Phi,\Psi\in\mathbb{M}(N,\rho)$, (iii) $*$ distributes over convex combinations. Regarding the composition of u.c.p. maps as multiplication, the sets $CP(N,\rho)$ (resp. $\mathbb{M}(N,\rho)$ and $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$) are BW-compact semitopological affine (resp. BW-compact semitopological affine continuously involutive) semigroups with unique identity and unique zero.*

Proof. First note that $CP(N,\rho),\mathbb{M}(N,\rho)$ and $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ are convex sets (use Theorem 2.6). The Accardi-Cecchini adjoint is an involution on $\mathbb{M}(N,\rho)$ (see Eq. (4)). From Theorem 4.7 of [2], it follows that the involution leaves $J_m(\mathfrak{N},\mathfrak{N})$ invariant. Thus, $J_m(\mathfrak{N},\mathfrak{N})$ inherits the involution from $\mathbb{M}(N,\rho)$. The continuity of $*$ follows from Proposition 4.3.

Let $\Phi,\Psi\in\mathbb{M}(N,\rho)$. To establish (i), note that Eq. (4) yields

$$\begin{aligned}\rho((\Phi\Psi)^*(y)x)=\rho(y(\Phi\Psi)(x))=\rho(y\Phi(\Psi(x)))=\rho(\Phi^*(y)\Psi(x))\\=\rho(\Psi^*(\Phi^*(y))x)=\rho((\Psi^*\Phi^*)(y)x),\ x,y\in N.\end{aligned}$$

Thus (i) follows. Similar arguments prove (ii) and (iii).

Let ι be the embedding associated to the subsystem $\mathfrak{B}$ of $\mathfrak{N}$ and let $\mathbb{E}^N_{\iota(B)}$ denote the ρ-preserving conditional expectation from N onto $\iota(B)$. By Theorem 2.6, it

[1] Note: We use the term 'semitopological semigroup' to indicate a semigroup in which the multiplication is separately continuous, rather than being jointly continuous. Semigroups satisfying the latter condition are often called 'topological semigroups'. There are occasions when separate continuity implies joint continuity, but as far as we know, it is an open problem to characterize this phenomenon.

follows that $\mathbb{E}^N_{\iota(B)} \in J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ and by Theorem 2.8, it follows that $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ is closed with respect to $*$.

Clearly, $CP(N,\rho)$ and $\mathbb{M}(N,\rho)$ are closed with respect to multiplication. Let $\Phi,\Psi \in J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$. Then,

$$\Phi\Psi\mathbb{E}^N_{\iota(B)} = \Phi\mathbb{E}^N_{\iota(B)}\Psi = \mathbb{E}^N_{\iota(B)}\Phi\Psi = \mathbb{E}^N_{\iota(B)}.$$

This proves that $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ is closed with respect to the multiplication. The associativity of the multiplication is obvious. The fact that the left and right multiplications are affine are straightforward observations.

It is obvious that $\mathbb{M}(N,\rho)$ is a closed subset of $CP(N,\rho)$. Thus, compactness of the convex sets in the statement follow from Propositions 4.1 and 4.4.

The identity map on N is clearly the identity in each case. Again, the map $\Phi : N \to N$ defined by $\Phi(x) = \rho(x)1_N$ is a two-sided zero of both $CP(N,\rho)$ and $\mathbb{M}(N,\rho)$. Further, note that $\mathbb{E}^N_{\iota(B)}$ is a two-sided zero of $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$. Uniqueness of the zero and identity are routine checks.

We now establish separate continuity of multiplication. By Proposition 4.1, it is enough to work with sequences. Let $CP(N,\rho) \ni \Phi_n \to \Phi \in CP(N,\rho)$ in the BW-topology. Fix $x \in N$. Then, $\Phi_n(x) \xrightarrow{w^*} \Phi(x)$. Since $\Psi \in CP(N,\rho)$ is normal [2, Corollary 4.5], $\Psi(\Phi_n(x)) \xrightarrow{w^*} \Psi(\Phi(x))$, i.e., $\Psi_n\Phi \to \Psi\Phi$ in the BW-topology. The continuity of the right multiplication follows similarly. This completes the proof. □

REMARK 5.2. The semigroup structure on $J_m(\mathfrak{N},\mathfrak{N})$ induces a semigroup structure on $J_s(\mathfrak{N},\mathfrak{N})$ and

$$J_h(\mathfrak{N},\mathfrak{N}) = \{{}_N\mathscr{H}_{\Phi N} \text{ is cyclic bimodule arising from } \Phi : \Phi \in J_m(\mathfrak{N},\mathfrak{N})\},$$

which may be regarded as some sort of convolution of states or bimodules as the case may be. Under this association, the product state given for all $x,y \in N$ by $\omega(x\otimes y^{op}) = \rho(x)\rho(y)$ (whose bimodule is the coarse correspondence) is the *zero* and the *identity* is given by $id(x\otimes y^{op}) = \langle xJ_\rho y^* J_\rho \Omega_\rho, \Omega_\rho\rangle_\rho$ for all $x,y \in N$ (whose bimodule is the trivial correspondence). Also note that the Accardi-Cecchini adjoint in Eq. (4) enables one to define an adjoint operation on the associated state spaces using Theorems 4.3, 4.4 and 4.7 of [2], which will be denoted by $*$ as well (a slight abuse of notation).

For $\omega \in J_s(\mathfrak{N},\mathfrak{N})$, to establish a formula for ω^*, we need elements of modular theory.

Thus, for $\omega \in J_s(\mathfrak{N},\mathfrak{N})$ using [2, Theorem 4.1] and following Eq. (3) one has

$$\begin{aligned}
\omega^*(x\otimes y^{op}) &= \langle \Phi^*_\omega(x)\Omega_\rho y, \Omega_\rho\rangle_\rho \\
&= \rho(\Phi^*_\omega(x)\sigma^\rho_{-\frac{i}{2}}(y)) \text{ [8, Eq. (6)]} \\
&= \rho(x\Phi_\omega(\sigma^\rho_{-\frac{i}{2}}(y))) \text{ (by Eq. (4))} \\
&= \rho(x\sigma^\rho_{-\frac{i}{2}}(\Phi_\omega(y))) \text{ (by Lemma 2.2 [3])} \\
&= \langle \Delta_\rho^{\frac{1}{2}}\Phi_\omega(y)\Omega_\rho, x^*\Omega_\rho\rangle_\rho
\end{aligned}$$

$$
\begin{aligned}
&= \langle \Phi_\omega(y)\Omega_\rho, \Delta_\rho^{\frac{1}{2}} x^* \Omega_\rho \rangle_\rho \\
&= \langle \Phi_\omega(y)\Omega_\rho, J_\rho x \Omega_\rho \rangle_\rho \\
&= \langle \Phi_\omega(y)\Omega_\rho x, \Omega_\rho \rangle_\rho \\
&= \omega(y \otimes x^{op}), \text{ for all } x \in N, \ y \in \cap_{z\in\mathbb{C}} \mathfrak{D}(\sigma_z^\rho).
\end{aligned}
$$

By density of $\cap_{z\in\mathbb{C}}\mathfrak{D}(\sigma_z^\rho)$ (see [23, §9.24]) in N and the fact that ω and ω^* are normal in each variable, it follows that

$$\omega^*(x \otimes y^{op}) = \omega(y \otimes x^{op}), \text{ for all } x, y \in N. \tag{9}$$

The following result may be known to experts, but we include it for completeness, since it exposes a fundamental link with the origin of the theory of joinings. Namely, the result asserts that idempotent elements of $CP(N,\rho)$ correspond to subsystems with the canonical embedding. Particularly this asserts that, in some sense, the difference between *disjointness* and *lack of common subsystem* is encoded in the failure of agreement of the Choi-Effros product and the product in N. We also remark that this result asserts that the essential hypothesis of state-preservation for a self-joining forces one to consider modular invariant u.c.p. maps, and not simply stationary couplings as considered by Sauvageot and Thouvenot [18]. Analogues of the following result using generalized conditional expectations appear to be inaccessible.

THEOREM 5.3. *Let N be a von Neumann algebra with a faithful normal state ρ. Let $\Phi \in CP(N,\rho)$ be such that $\Phi = \Phi^2$. Then Φ is the ρ-preserving conditional expectation onto $\Phi(N) = N \cap \{T_\Phi\}'$. Consequently, $\Phi \circ \sigma_t^\rho = \sigma_t^\rho \circ \Phi$ for all $t \in \mathbb{R}$ and $\Phi = \Phi^*$.*

Proof. Note that by Lemma 6.4 of [2], the algebra of harmonic elements (cf. [12])

$$
\begin{aligned}
B &= \{x \in N : \Phi(x) = x\} \qquad (10)\\
&= \{x \in N : \Phi(xy) = x\Phi(y), \ \Phi(yx) = \Phi(y)x \ \forall y \in N\}
\end{aligned}
$$

is a von Neumann subalgebra of N. Moreover, $\Phi^2 = \Phi$ implies that $\Phi(N) = B$. So $\Phi : N \to N$ is a projection of norm one whose image is B and Φ is also a B-bimodule map. By a well known theorem of Tomiyama [21], Φ is a conditional expectation onto B. But since $\rho \circ \Phi = \rho$, by [19] it follows that $\Phi = \mathbb{E}_B$, where $\mathbb{E}_B$ is the unique normal ρ-preserving conditional expectation onto B. Note that $xT_\Phi = T_\Phi x$ for all $x \in B$, and hence $B \subseteq \{T_\Phi\}' \cap N$. Conversely, $\{T_\Phi\}' \cap N \subseteq B$ follows from Eq. (10).

By [19], it follows that $\mathbb{E}_B$ commutes with σ_t^ρ for all $t \in \mathbb{R}$. Finally, it is clear that $\mathbb{E}_B = \mathbb{E}_B^*$. □

REMARK 5.4. Note that by Proposition 2.5, we have $J_{\mathfrak{B},m}(\mathfrak{M},\mathfrak{N}) = \{\Phi^* : \Phi \in J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M})\}$. The $*$ operation between $J_m(\mathfrak{N},\mathfrak{M})$ and $J_m(\mathfrak{M},\mathfrak{N})$ induces (as above) a $*$ operation between $J_s(\mathfrak{N},\mathfrak{M})$ and $J_s(\mathfrak{M},\mathfrak{N})$, namely, for $\omega \in J_s(\mathfrak{N},\mathfrak{M})$, define

$$\omega^*(y \otimes x^{op}) = \langle \Phi_\omega^*(y)\Omega_\rho x, \Omega_\rho \rangle_\rho, \ x \in N, y \in M. \tag{11}$$

Like Eq. (9), it can be shown that

$$\omega^*(y\otimes x^{op}) = \omega(x\otimes y^{op}),\ x\in N, y\in M. \tag{12}$$

Consequently, there is a topology $\mathfrak{T}^*$ on $J_s(\mathfrak{M},\mathfrak{N})$ analogous to $\mathfrak{T}$ on $J_s(\mathfrak{N},\mathfrak{M})$ which is compatible with the BW-convergence on $J_m(\mathfrak{M},\mathfrak{N})$.

Further, if $\mathfrak{B}$ is a common subsystem of $\mathfrak{N}$ and $\mathfrak{M}$, it is routine to check that the map $\Lambda : J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{M}) \to J_{\mathfrak{B},s}(\mathfrak{N},\mathfrak{M})$ defined in Theorem 4.5 preserves $*$ and by Theorem 4.3 and Eq. (11), Eq. (12) it is obvious that $* : J_{\mathfrak{B},s}(\mathfrak{N},\mathfrak{M}) \to J_{\mathfrak{B},s}(\mathfrak{M},\mathfrak{N})$ is $(\mathfrak{T},\mathfrak{T}^*)$ continuous.

The following result, along with those in §4 and Theorem 5.1, generalize Theorem 6.13 of [9]. This together with other results in this section and those in [2, 3] reinforces that our way of defining joinings is appropriate. The reader should notice that because of Theorem 5.3, we can safely drop the term 'self-adjoint' in statement 2 of Theorem 6.13 of [9].

THEOREM 5.5. *Let $\mathfrak{B}$ be a subsystem of $\mathfrak{N}$ with embedding ι. The map $\Lambda :$ $(J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N}),BW) \to (J_{\mathfrak{B},s}(\mathfrak{N},\mathfrak{N}),\mathfrak{T})$ defined by $\Phi \mapsto \omega_\Phi$ is a $*$-preserving homeomorphism of compact semitopological affine continuously involutive semigroups with unique identity and zero. If $\Phi \in J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ is an idempotent, then there exists a unique von Neumann subalgebra $B_\Phi \subseteq N$ such that:*

- $\iota(B) \subseteq B_\Phi$;
- $\alpha_g(B_\Phi) = B_\Phi$*, for all* $g\in G$;
- *there exists a faithful normal ρ-preserving conditional expectation* $\Phi = \mathbb{E}^N_{B_\Phi} :$ $N \to B_\Phi$.

Further, every α and (σ^ρ_t) invariant subalgebra of N containing $\iota(B)$ corresponds to a unique idempotent of $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$. Every idempotent in $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ is self-adjoint.

When $G=\{1\}$ or the actions are trivial, there are analgous statements for the space of cyclic N-N correspondences and associated states.

Proof. By Theorem 5.1, it follows that $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ is $*$-closed. Fix $\Phi\in J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$. Thus, by Remark 5.4 one has,

$$\begin{aligned}(\Lambda(\Phi))^*(y\otimes x^{op}) &= (\omega_\Phi)^*(y\otimes x^{op})\\ &= \langle \Phi^*(y)\Omega_\rho x, \Omega_\rho\rangle_\rho\\ &= \omega_{\Phi^*}(y\otimes x^{op})\\ &= \Lambda(\Phi^*)(y\otimes x^{op}),\ x,y\in N.\end{aligned}$$

Thus, Λ is $*$-preserving. Thus, the first part of the statement follows from Theorems 4.5 and 5.1.

Note that the collection of idempotents in $J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ is not empty as id and $\mathbb{E}^N_{\iota(B)}$ are idempotents. Let $\Phi\in J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ be such that $\Phi=\Phi^2$. By Theorem 5.3, it follows that $\Phi = \mathbb{E}^N_{\{T_\Phi\}'\cap N}$. Note that $\mathbb{E}^N_{\{T_\Phi\}'\cap N}$ commutes with α and (σ_t^ρ), and therefore Φ corresponds to the subsystem $(\{T_\Phi\}'\cap N, \rho_{|\{T_\Phi\}'\cap N}, \alpha_{|\{T_\Phi\}'\cap N}, G)$ with the obvious embedding. Put $B_\Phi = \{T_\Phi\}'\cap N$. Note that Φ is identity on $\iota(B)$, thus $\iota(B) \subseteq B_\Phi$. Then, B_Φ satisfies all the desired properties.

Conversely, let $D \subseteq N$ be a α and (σ_t^ρ) invariant subalgebra of N containing $\iota(B)$. Clearly, $\mathbb{E}^N_D \in J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ is idempotent. Uniqueness is obvious.

The remaining statements are routine, and we leave the details to the reader. □

A face of a convex set A is a non-empty subset F of A with the property that if $p,q \in A$ and $\theta \in (0,1)$, and $\theta p + (1-\theta)q \in F$, then $p,q \in F$.

COROLLARY 5.6. *Let $\mathfrak{B}$ be a subsystem of $\mathfrak{N}$ with embedding ι. Then $J^+_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ is a face of $J^+_m(\mathfrak{N},\mathfrak{N})$, where $J^+_m(\mathfrak{N},\mathfrak{N}) = \{\Phi \in J_m(\mathfrak{N},\mathfrak{N}) : T_\Phi \geqslant 0\}$ and $J^+_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N}) = J^+_m(\mathfrak{N},\mathfrak{N}) \cap J_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$.*

Proof. Note that $J^+_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N}) \neq \emptyset$, as $\mathbb{E}^N_{\iota(B)} \in J^+_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$. By Theorem 5.1, both $J^+_m(\mathfrak{N},\mathfrak{N})$ and $J^+_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ are compact convex sets and we know $J^+_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N}) \subseteq J^+_m(\mathfrak{N},\mathfrak{N})$.

Let $\Phi,\Psi \in J^+_m(\mathfrak{N},\mathfrak{N})$ be such that $\lambda\Phi + (1-\lambda)\Psi \in J^+_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$ for some $0 < \lambda < 1$. Since projections are extreme points of the ball of positives in $\mathbf{B}(L^2(N,\rho))$, by Theorems 2.8 and 5.3, it follows that $\Phi,\Psi \in J^+_{\mathfrak{B},m}(\mathfrak{N},\mathfrak{N})$. □

6. Examples

In this section, we provide examples of noncommutative joinings of W^*-dynamical systems, which expand on the examples from [2], and incorporate the bimodule machinery of [3]. Classically, in the 'state' or 'measure' point of view, joinings over subsystems are usually viewed as a kind of 'amalgamation' of two Hilbert spaces over a common subspace. The next result shows that the Hilbert space picture is not the complete story; rather, a joining of two W^*-dynamical systems over a common subsystem may be thought of as an 'amalgamation' of two bimodules over a common subbimodule arising out of the common subsystem. In this sense, Theorem 6.2 generalizes the notion of joinings over subsystems. The following lemma is a necessary first step.

LEMMA 6.1. *Let M and B be von Neumann algebras with faithful normal states φ and μ respectively. Let $\iota_M : B \to M$ be a unital injective $*$-homomorphism such that ι_M is (μ,φ)-Markov map. Then for all $x \in \cap_{z\in\mathbb{C}} \mathfrak{D}(\sigma_z^\mu)$, the map ι_M satisfies $\sigma_z^\varphi \circ \iota_M(x) = \iota_M \circ \sigma_z^\mu(x)$ for every $z \in \mathbb{C}$, and also*

$$\langle b_1\Omega_\mu b_2, \Omega_\mu\rangle_\mu = \langle \iota_M(b_1)\Omega_\varphi \iota_M(b_2), \Omega_\varphi\rangle_\varphi$$

for all $b_1, b_2 \in B$.

Proof. By the hypothesis $\sigma_t^\varphi \circ \iota_M = \iota_M \circ \sigma_t^\mu$ for all $t \in \mathbb{R}$. Let $B^\mu_\infty = \cap_{z\in\mathbb{C}} \mathfrak{D}(\sigma_z^\mu)$ (see [23, §9.24]). Then by uniqueness of analytic extension it follows that for all

$x \in B^{\mu}_{\infty}$, the map ι_M satisfies $\sigma^{\varphi}_{z} \circ \iota_M(x) = \iota_M \circ \sigma^{\mu}_{z}(x)$ for every $z \in \mathbb{C}$. Consequently, for all $b_1 \in B$ and $b_2 \in B^{\mu}_{\infty}$ we have,

$$\begin{aligned}\langle b_1\Omega_\mu b_2, \Omega_\mu\rangle_\mu &= \langle b_1\sigma^{\mu}_{-i/2}(b_2)\Omega_\mu, \Omega_\mu\rangle_\mu \\ &= \mu(b_1\sigma^{\mu}_{-i/2}(b_2)) \\ &= \varphi\Big(\iota_M(b_1)\iota_M(\sigma^{\mu}_{-i/2}(b_2))\Big) \\ &= \varphi\Big(\iota_M(b_1)\sigma^{\varphi}_{-i/2}(\iota_M(b_2))\Big) \\ &= \langle \iota_M(b_1)\sigma^{\varphi}_{-i/2}(\iota_M(b_2))\Omega_\varphi, \Omega_\varphi\rangle_\varphi \\ &= \langle \iota_M(b_1)\Omega_\varphi\iota_M(b_2), \Omega_\varphi\rangle_\varphi\end{aligned}$$

The rest follows by density of the Tomita algebra associated to μ. □

We now prove a "disintegration theorem" for joinings. The proof of this theorem makes extensive use of the theory of bimodules, for which we refer the reader to §3 and §4 of [2]. Note that the next result extends Proposition 2.2 in the case G is trivial.

THEOREM 6.2. *(Disintegration of joinings) Let N, M and B be von Neumann algebras with respective faithful normal states ρ, φ and μ, and let ι_N and ι_M be injective $*$-homomorphisms of B into N and M respectively. Further, suppose that ι_N and ι_M are respectively (μ,ρ)-Markov and (μ,φ)-Markov maps. Let $\Psi : B \to B$ be a (μ,μ)-Markov map, and $\psi : B \odot B^{op} \to \mathbb{C}$ be the associated state satisfying $\psi(b_1 \otimes b_2^{op}) = \langle \Psi(b_1)\Omega_\mu b_2, \Omega_\mu\rangle_\mu$ for all $b_1, b_2 \in B$. Then for a $(\sigma^{\rho}_t \otimes (\sigma^{\varphi}_t)^{op})$-invariant state $\omega : N \odot M^{op} \to \mathbb{C}$, the following are equivalent:*

(i) $\omega \circ (\iota_N \otimes (\iota_M)^{op}) = \psi$, where ι^{op}_M is the natural map $J_\mu b^* J_\mu \mapsto J_\varphi \iota_M(b)^* J_\varphi$, $b \in B$, from $B' \cap \mathbf{B}(L^2(B,\mu)) \to M' \cap \mathbf{B}(L^2(M,\varphi))$.

(ii) $(\Phi_\omega)_{|\iota_N(B)} = (\iota_M \circ \Psi \circ \iota^*_N)_{|\iota_N(B)}$, where $\Phi_\omega : N \to M$ is the unique (ρ,φ)-Markov map such that $\omega(x \otimes y^{op}) = \langle \Phi_\omega(x)\Omega_\varphi y, \Omega_\varphi\rangle_\varphi$, for all $x \in N$ and $y \in M$.

Proof. First we prove $(i) \implies (ii)$. Let ω be the state as above and let $\mathscr{H}_\omega$ be the associated GNS Hilbert space with cyclic vector ξ_ω. Then, by Theorem 3.3 and Theorem 4.1 of [2], the map $R(\Omega_\varphi z) = \xi_\omega z$ defined for $z \in M$ extends to a right Hilbert M-module isomorphism of $L^2(M,\varphi)$ onto $\overline{\xi_\omega M}$, and $\Phi_\omega(x) = R^*\pi_N(x)R$ for all $x \in N$, where π_N here denotes the left action of N on $\mathscr{H}_\omega$. Then for all $b_1, b_2, b_3 \in B$, we have

$$\begin{aligned}\langle \Phi_\omega(\iota_N(b_1))\Omega_\varphi\iota_M(b_2), \Omega_\varphi\iota_M(b_3)\rangle_\varphi &= \langle R^*\pi_N(\iota_N(b_1))R\,\Omega_\varphi\iota_M(b_2), \Omega_\varphi\iota_M(b_3)\rangle_\varphi \\ &= \langle \iota_N(b_1)\xi_\omega\iota_M(b_2), \xi_\omega\iota_M(b_3)\rangle_\omega \\ &= \langle \iota_N(b_1)\xi_\omega\iota_M(b_2b_3^*), \xi_\omega\rangle_\omega \\ &= \omega(\iota_N(b_1) \otimes (\iota_M(b_2b_3^*))^{op}) \qquad (13) \\ &= \omega \circ (\iota_N \otimes \iota^{op}_M)(b_1 \otimes (b_2b_3^*)^{op}) \\ &= \psi(b_1 \otimes (b_2b_3^*)^{op})\end{aligned}$$

$$= \langle \Psi(b_1)\Omega_\mu b_2, \Omega_\mu b_3\rangle_\mu$$
$$= \langle \iota_M(\Psi(b_1))\Omega_\varphi \iota_M(b_2), \Omega_\varphi \iota_M(b_3)\rangle_\varphi,$$

where the last equality is obtained via Lemma 6.1.

The fact that $\Phi_\omega(\iota_N(b_1)) = \iota_M(\Psi(b_1))$ for all $b_1 \in B$ follows by density.

We now establish $(ii) \implies (i)$. For $b_1, b_2 \in B$, we compute

$$\begin{aligned}(\omega \circ (\iota_N \otimes \iota_M^{op}))(b_1 \otimes b_2^{op}) &= \omega(\iota_N(b_1) \otimes (\iota_M(b_2))^{op}) \\ &= \langle \iota_N(b_1)\xi_\omega \iota_M(b_2), \xi_\omega\rangle_\omega \\ &= \langle \iota_N(b_1)\xi_{\Phi_\omega} \iota_M(b_2), \xi_{\Phi_\omega}\rangle_\omega \\ &= \langle \Phi_\omega(\iota_N(b_1))\Omega_\varphi \iota_M(b_2), \Omega_\varphi\rangle_\varphi \\ &= \langle \iota_M(\Psi(b_1))\Omega_\varphi \iota_M(b_2), \Omega_\varphi\rangle_\varphi \\ &= \langle \Psi(b_1)\Omega_\mu b_2, \Omega_\mu\rangle_\mu \\ &= \psi(b_1 \otimes b_2^{op}).\end{aligned}$$

The third equality above was established in the proof of Theorem 4.4 of [2], and the penultimate equality holds by Lemma 6.1. □

Thus, we have the following theorem which provides a new class of examples of joinings.

THEOREM 6.3. *Let $\mathfrak{B}$ be a common subsystem of two W^*-dynamical systems $\mathfrak{N}$ and $\mathfrak{M}$ with respect to embeddings ι_N and ι_M respectively. For $\Psi \in J_m(\mathfrak{B}, \mathfrak{B})$, let*

$$J_{\mathfrak{B},\Psi,m}(\mathfrak{N}, \mathfrak{M}) = \{\Phi \in J_m(\mathfrak{N}, \mathfrak{M}) : \Phi_{|\iota_N(B)} = (\iota_M \circ \Psi \circ \iota_N^*)_{|\iota_N(B)}\}.$$

Then, $J_{\mathfrak{B},\Psi,m}(\mathfrak{N}, \mathfrak{M})$ is a nonempty convex set which is compact in the BW-topology.

Proof. Note that $\iota_M \circ \Psi \circ \iota_N^* \in J_{\mathfrak{B},\Psi,m}(\mathfrak{N}, \mathfrak{M})$ (cf. Theorem 6.2). Consequently, $J_{\mathfrak{B},\Psi,m}(\mathfrak{N}, \mathfrak{M}) \neq \emptyset$. The rest follows from Theorem 4.4. □

REMARK 6.4. Elements of $J_{\mathfrak{B},\Psi,m}(\mathfrak{N}, \mathfrak{M})$ are joinings of $\mathfrak{N}$ and $\mathfrak{M}$ *extending* $\Psi \in J_m(\mathfrak{B}, \mathfrak{B})$ and such joinings generalize the notion of joinings over subsystems.

REMARK 6.5. To emphasize the relevance of Theorem 6.2, consider the following example. Following [9, §6], let $\mathbf{X} = (X, \mathscr{X}, \mu, G)$ and $\mathbf{Y} = (Y, \mathscr{Y}, \nu, G)$ be classical ergodic systems with a common factor $\mathbf{Z} = (Z, \mathscr{Z}, \eta, G)$. Let $\pi : \mathbf{X} \to \mathbf{Z}$ and $\sigma : \mathbf{Y} \to \mathbf{Z}$ be the factor maps, which are homomorphisms of ergodic systems. These factor maps induce obvious unital embeddings of $L^\infty(Z, \eta) \hookrightarrow L^\infty(X, \mu)$ and $L^\infty(Z, \eta) \hookrightarrow L^\infty(Y, \nu)$ which we denote with π and σ respectively with slight abuse of notation.

Let $\tau \in J(\mathbf{Z})$, where $J(\mathbf{Z})$ denotes the collection of self-joinings of $\mathbf{Z}$. Then, τ is a probability measure on $Z \times Z$ invariant under the diagonal action of G on $Z \times Z$ such that $L^2(Z \times Z, \tau)$ is a $L^\infty(Z, \eta)$-$L^\infty(Z, \eta)$ bimodule with the obvious G-equivariance structure.

Following the setup of Theorem 6.2, if $\omega \in J(\mathbf{X},\mathbf{Y})$ is such that $\omega\circ(\pi\otimes\sigma^{op}) = \tau$, then ω admits a $(\pi\times\sigma,\tau)$-disintegration $Z\times Z \ni (z,z') \mapsto \omega_{z,z'} \in M(X\times Y)$. Therefore,

$$\omega = \int_{Z\times Z} \omega_{z,z'}\, d\tau(z,z').$$

By uniqueness of the disintegration, it follows that $\omega_{z,z'}$ is invariant under the diagonal action of G for τ almost every (z,z'). Consequently, we are disintegrating the G-equivariant $L^\infty(X,\mu)$-$L^\infty(Y,\nu)$ bimodule $L^2(X\times Y,\omega)$ over the G-equivariant $L^\infty(Z,\eta)$-$L^\infty(Z,\eta)$ bimodule $L^2(Z\times Z,\tau)$.

In this picture, a joining of $\mathbf{X}$ and $\mathbf{Y}$ over the subsystem $\mathbf{Z}$ corresponds to the case, when $\tau = D_*\eta$, where $D: Z\to Z\times Z$ is the map $D(z) = (z,z)$, which corresponds to the trivial $L^\infty(Z,\eta)$-$L^\infty(Z,\eta)$ bimodule $L^2(Z,\eta)$.

In [2, 3] and so far in this paper, we have studied joinings of two systems $\mathfrak{N}$ and $\mathfrak{M}$ from three points of view: first, as certain invariant states on the algebraic tensor product of one von Neumann algebra with the opposite of the other; second as pointed correspondences with extra structure, and third, as equivariant Markov maps between the von Neumann algebras involved. A fourth point of view is also available, which seems to carry abundant analytical information.

As before, let $\mathfrak{N}$ be a system. For each $\Phi \in J_m(\mathfrak{N},\mathfrak{N})$ its L^2-extension T_Φ is bounded (cf. §6 of [2] or §3) and $\|T_\Phi\| = 1$. Also if $B\subset N$ is a von Neumann subalgebra with a ρ-preserving conditional expectation, then $T_\Phi \in B'\cap(J_\rho B J_\rho)'$ if and only if Φ is B-bimodular. Clearly, $\{T_\Phi : \Phi\in CP(N,\rho)\}$ and $\{T_\Phi : \Phi\in J_m(\mathfrak{N},\mathfrak{N})\}$ are convex subsets of the unit sphere of $\mathbf{B}(L^2(N,\rho))$. Also as seen in §4 and §5, note that if $\{\Phi_n\}_n$ is a sequence in $CP(N,\rho)$ such that $\Phi_n \to \Phi \in CP(N,\rho)$ in the BW-topology, then $T_{\Phi_n} \to T_\Phi$ in *w.o.t.* and hence in the w^*-topology.

THEOREM 6.6. *The map $\Phi\to T_\Phi$ from $\mathbb{M}(N,\rho)$ to $A(T) = \{T_\Phi : \Phi\in\mathbb{M}(N,\rho)\}$ (resp. from $J_m(\mathfrak{N},\mathfrak{N})$ to $A(T,G) = \{T_\Phi : \Phi\in J_m(\mathfrak{N},\mathfrak{N})\}$ is (BW, w^*)-continuous; thus is a homeomorphism and consequently an isomorphism between two compact semitopological involutive affine semigroups.*

Proof. Note that $A(T)$ (resp. $A(T,G)$) is *w.o.t.* compact, as it is the continuous image of a compact space. The rest is routine. □

COROLLARY 6.7. *Let $\Phi\in\mathbb{M}(N,\rho)$. Then,*

(i) T_Φ is normal (resp. unitary) if and only if $\Phi^*\Phi = \Phi\Phi^*$ (resp. if Φ is an automorphism);[2]

(ii) T_Φ is self-adjoint if and only if $\Phi^* = \Phi$;

(iii) T_Φ is projection if and only if $\Phi = \Phi^2 = \mathbb{E}^N_{N\cap\{T_\Phi\}'}$;[3]

[2] This is a classical result of Wigner when $N = B(\mathscr{H})$.

[3] The assumption that T_Φ is self-adjoint is redundant.

(iv) $T_{\Phi^*\Phi}$ *is positive*;

(v) T_Φ *is a partial isometry if and only if* $\Phi = \Phi\Phi^*\Phi$ *and there exists a von Neumann algebra* B *equipped with a faithful normal state* μ *and embeddings* $\iota_1, \iota_2 : B \to N$ *with* ι_1, ι_2 *being both* (μ, ρ)*-Markov maps such that* $\Phi^*\Phi = \mathbb{E}^N_{\iota_1(B)}$ *and* $\Phi\Phi^* = \mathbb{E}^N_{\iota_2(B)}$;

(vi) 1 *is an eigenvalue of* T_Φ. *If* Φ *is* B*-bimodular where* B *is infinite dimensional, then the eigenspace corresponding to* 1 *is infinite dimensional and thus* T_Φ *is not compact.*

Proof. (i) The statement relating to T_Φ being normal is obvious. If Φ is an automorphism, then $\Phi^* = \Phi^{-1}$. Thus, $\Phi^*\Phi = \Phi\Phi^* = id_N$ and therefore T_Φ is unitary. Conversely, if $T_\Phi^* T_\Phi = T_\Phi T_\Phi^* = 1$ then $\Phi^*\Phi = \Phi\Phi^* = id_N$. Thus for $x \in N$, Eq. (4) and Kadison-Schwarz inequality yields

$$\rho(x^*x) = \rho((\Phi^*\Phi)(x^*)x) = \rho(\Phi(x)^*\Phi(x)) \leqslant \rho(\Phi(x^*x)) = \rho(x^*x).$$

Thus, $\Phi(x^*x) = \Phi(x)^*\Phi(x)$ as ρ is faithful. By Theorem 3.1 [7], it follows that N is contained in the multiplicative domain of Φ, which forces Φ to be an automorphism.
(iii) The statement follows directly from Theorem 5.3.
(v) This is a direct consequence of [3, Theorem 3.3].
$(ii), (iv)$ and (vi) are obvious. □

Note that if $\Phi \in \mathbb{M}(N, \rho)$ is such that $T_\Phi \geqslant 0$, then T_Φ is the unique square root of $T_{\Phi^*\Phi}$. In this case, we can write $\sqrt{\Phi^*\Phi} := \Phi$. Also, we define $\Phi^0 = id$. Via the homeomorphism established in Theorem 5.5 and Corollary 6.7, one can make analogous definitions on the associated state space. While it is not possible to define the square root of every element in $\mathbb{M}(N, \rho)$ as an element of $\mathbb{M}(N, \rho)$, some elements in $\mathbb{M}(N, \rho)$ do have square roots in the prior sense.

The following lemma is classical and well known, so we omit its proof.

LEMMA 6.8. *For* $n > 0$ *define* $g_n : [0,1] \to [0,1]$ *by* $g_n(x) = x^n$ *and* $h_n : [-1,1] \to [-1,1]$ *by* $h_n(x) = x^n$. *Also let* $g_0 = 1$ *on* $[0,1]$ *and* $h_0 = 1$ *on* $[-1,1]$. *Then,*

$$\mathfrak{R}_+ = \overline{co}^{\|\cdot\|_\infty}\{g_n : n \geqslant 0\} = \{f \in C[0,1] : \ f \text{ real analytic}, \quad f^{(n)}(0) \geqslant 0\ \forall n, \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = 1\}.$$

$$\mathfrak{R}_{s.a} = \overline{co}^{\|\cdot\|_\infty}\{h_n : n \geqslant 0\} = \{f \in C[-1,1] : \ f \text{ real analytic}, \quad f^{(n)}(0) \geqslant 0\ \forall n, \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = 1\}.$$

Note that for $f \in \mathfrak{R}_+$ (resp. $f \in \mathfrak{R}_{s.a}$) as defined in Lemma 6.8 above, $f(1) = 1$. Thus, the functions in $\mathfrak{R}_+$ and $\mathfrak{R}_{s.a}$ are of norm one. In general, it is difficult to measure the size of $J_m(\mathfrak{N},\mathfrak{N})$ (it may collapse to a set with just two extreme points). The result below measures the size of $J_m(\mathfrak{N},\mathfrak{N})$. It is not immediate that the self-joinings obtained via the functional calculus below might be obtained by analogy with classical constructions as above. In the following result, let

$$\mathfrak{W} = \{f \in C(\mathbb{T}) : f(z) = \sum_{n\in\mathbb{Z}} \lambda_n z^n,\ 0 \leqslant \lambda_n \leqslant 1\ \forall\, n,\ \sum_{n\in\mathbb{Z}} \lambda_n = 1\}.$$

The following theorem is immediate.

THEOREM 6.9. (*Analytic fuctional calculus*) *Let* $\Phi \in \mathbb{M}(N,\rho)$ *be such that* T_Φ *is positive (resp. self-adjoint, unitary). Then, there exists a map* $\mathfrak{R}_+ \ni f \to f(\Phi) \in \mathbb{M}(N,\rho)$ *(resp.* $\mathfrak{R}_{s.a} \ni f \to f(\Phi) \in \mathbb{M}(N,\rho)$, $\mathfrak{W} \ni f \to f(\Phi) \in \mathbb{M}(N,\rho)$*) such that*

$$\rho(y^* f(\Phi)(x)) = \int_{[0,1],(resp.\ [-1,1],\mathbb{T})} f(t)\,d\mu_{x,y}(t),\ x,y \in N,$$

where $\mu_{x,y}$ *denotes the elementary spectral measure of* T_Φ *associated to* x,y*. Moreover, if* $\Phi \in J_m(\mathfrak{N},\mathfrak{N})$ *then* $f(\Phi) \in J_m(\mathfrak{N},\mathfrak{N})$*. For* $f,g \in \mathfrak{R}_+$ *(resp.* $\mathfrak{R}_{s.a}$, $\mathfrak{W}$*),* $f(\Phi) = g(\Phi)$ *if and only if* $f = g$ *on* $\sigma(T_\Phi)$.

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(Received July 31, 2019)

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