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EIGENVALUES OF DISCRETE STURM–LIOUVILLE PROBLEMS WITH SIGN–CHANGING WEIGHT AND COUPLED BOUNDARY CONDITIONS

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Abstract. In this paper, we study the eigenvalues of discrete Sturm-Liouville problems with sign-changing weight and coupled boundary conditions. The exact number (including multiplicity) of the real eigenvalues is obtained. The number of positive eigenvalues is equal to the number of positive elements in the weight function, and the number of negative eigenvalues is equal to the number of negative elements in the weight function. Meanwhile, the interlacing properties of these eigenvalues are also obtained as the parameter varies. These results extend the relevant existing results of discrete left-definite and right-definite Sturm-Liouville problems with coupled boundary conditions.

1. Introduction

Let $[a,b]_{\mathbb{Z}} = \{a, a+1, \cdots, b\}$, where a, b are two integers with $a < b$. In this paper, we consider the spectrum of the following second-order difference equation

$$-\nabla[p(t)\Delta y(t)] + q(t)y(t) = \lambda\omega(t)y(t), \qquad t \in [0, N-1]_{\mathbb{Z}} \tag{1.1}$$

with the coupled boundary condition

$$\begin{pmatrix} y(N-1) \\ \Delta y(N-1) \end{pmatrix} = e^{i\alpha} K \begin{pmatrix} y(-1) \\ \Delta y(-1) \end{pmatrix}. \tag{1.2}$$

Here, $N \geqslant 3$ is an integer, Δ is the forward difference operator with $\Delta y(t) = y(t+1) - y(t)$, ∇ is the backward difference operator with $\nabla y(t) = y(t) - y(t-1)$ and λ is the spectral parameter; $p : [-1, N-1]_{\mathbb{Z}} \to (0, \infty)$ with $p(-1) = p(N-1)$, $q : [0, N-1]_{\mathbb{Z}} \to [0, \infty)$ and $\omega(t) \neq 0$ changes its signs on $[0, N-1]_{\mathbb{Z}}$; the parameter α satisfies: $-\pi < \alpha \leqslant \pi$ and $i^2 = -1$. Moreover,

$$K = \begin{pmatrix} k_1 & 0 \\ k_2 & k_3 \end{pmatrix}, k_j \in R, j = 1, 2, 3; \;\; k_1 k_3 = 1.$$

The study of the spectrum of the Sturm-Liouville problems has been discussed more than 100 years before the time of Sturm and Liouville. Until now, these kind of

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problems have been studied in a variety of contexts. In 1914, Bôcher [1] studied the spectrum of the left-definite Sturm-Liouville problem

$$\frac{d}{dt}(ku') + (\lambda m - l)u = 0, \qquad t \in [0,1], \tag{1.3}$$

$$\alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0, \tag{1.4}$$

where $l \geqslant 0$ and m changes its sign on $[0,1]$. He obtained the eigenvalue problem (1.3), (1.4) has two sequences of real eigenvalues, $\lambda_k^{\pm}$, with

$$0 < \lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+ < \cdots \to +\infty$$

and

$$0 > \lambda_1^- > \lambda_2^- > \cdots > \lambda_k^- > \cdots \to -\infty.$$

This results could also be found in the book of Ince [2] and Zettl [3]. After this classical result, a variety of rich and excellent spectral results have been obtained by several authors, see, for instance, [4, 5, 6, 7, 8, 9, 10, 11] and the references therein.

For the discrete eigenvalue problem, several excellent spectral results for the classical discrete Sturm-Liouville problems have also been obtained by several authors, see, for instance, [12, 13, 14, 22, 23, 24, 25, 26, 27, 28, 31, 30, 29, 15, 17, 18, 19, 21, 20, 16, 32, 33, 34, 35, 36, 37, 38] and the references therein. In 1964, when the weight function $\omega(t) > 0$, Atkinson [12] studied a kind of discrete right-definite Sturm-Liouville problem

$$c(t)y(t+1) = (\omega(t)\lambda + b(t))y(t) - c(t-1)y(t-1), \qquad n \in [0, m-1]_{\mathbb{Z}}, \tag{1.5}$$

$$y(-1) = 0, \qquad y(m) + hy(m-1) = 0. \tag{1.6}$$

He obtained that (1.5), (1.6) has m real eigenvalues, which can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_m$. Later, a series of excellent spectral results, including the existence, simplicity, interlacing and dependency properties of the eigenvalues and the oscillation properties of the eigenfunctions, were obtained for the discrete eigenvalue problems with positive weight function $\omega(t) > 0$, see, [13, 14, 20, 16, 15, 17, 18, 19]. In particular, based on the properties of the characteristic polynomials and the spectral properties, constructed by Shi and Chen [17], of a discrete difference operator with coupled boundary condition, Sun and Shi [19] discussed the reality and the interlacing properties of the eigenvalue problems (1.1), (1.2) under the case that $\omega(t) > 0$ on $[0, N-1]_{\mathbb{Z}}$. According to the assumption that N is an odd number or an even number, Sun and Shi obtained some beautiful interlacing results for this kind of problems. Moreover, for the more general case, i.e., the discrete linear Hamiltonian systems, there are also several excellent spectral results on this kind of problems, see, for instance, [21,22,23,24,25,26,27,28,31,30,29] and the references therein. More precisely, the essential spectrum for the linear disrcete Hamilton system has been discussed by Sun [21], the oscillation properties of the eigenfunctions for the difference systems with separated conditions have been discussed by Bohner et al. [22, 23], Kratz [25], Došlý and Kratz [26, 27], Šepitka and Šimon Hilscher [29], Elyseeva and Šimon Hilscher [30] and Šimon Hilscher [31]. Meanwhile, the Sturmian type comparison theorems are also obtained for some kinds of difference system, see, for instance, Bohner et.al [24] and Došlý and Kratz [28].

For the case that the weight function $\omega(t)$ changes its sign on $[0,N-1]_{\mathbb{Z}}$, Ji and Yang [32] gave some comparison results of the eigenvalues of the discrete Neumann eigenvalue problems. Meanwhile, they also gave some comparison results of the eigenvalues of (1.1) with the periodic boundary condition in [33]. Later, the existence of real eigenvalues, the simplicity of the real eigenvalues and the oscillation properties of the corresponding eigenfunctions for the discrete Sturm-Liouville problems with Neumann and Dirichlet BCs were also obtained by [34, 35, 36]. Moreover, based on the spectral results in [34], the interlacing properties for the discrete Sturm-Liouville problems with periodic and antiperiodic boundary conditions were obtained by [37].

Now, the questions are: (i) Could we obtain the existence of the real eigenvalues of (1.1) with the coupled boundary condition (1.2)? (ii) How do the real eigenvalues of (1.1), (1.2) distribute? Based on the spectral results of the discrete Sturm-Liouville problems in [35], including the simplicity of the eigenvalues and the oscillation properties of the corresponding eigenfunctions, we try to answer the above two questions for the eigenvalue problem (1.1), (1.2). Actually, by discussing the properties of the corresponding characteristic polynomial of (1.1), (1.2), we obtain the existence, the distribution and the interlacing properties of the real eigenvalues as the parameter varies. Finally, it is worth to note that the boundary condition (1.2) will reduce to the periodic boundary condition if we take $\alpha=0$ and $K=I$ and will reduce to the antiperiodic boundary condition if we take $\alpha=0$ and $K=-I$. Moreover, the problems we discuss in this paper is more general than the left-definite problems, see Remark 4.11. Furthermore, if $\omega(t)>0$ on $[0,N-1]_{\mathbb{Z}}$, then our problem becomes a right-definite problem, and our results will reduce to the results of [18] and [19].

2. Preliminaries

In the rest of this paper, we shall make the following assumptions:

(H1) $p(t)>0$ on $[0,N-1]_{\mathbb{Z}}$ and $q(t)\geqslant 0$ for $t\in[0,N-1]_{\mathbb{Z}}$;

(H2) $\omega(t)$ changes its sign on $[0,N-1]_{\mathbb{Z}}$, i.e.,there are n points in $[0,N-1]_{\mathbb{Z}}$ such that $\omega(t)>0$ while $\omega(t)<0$ on other $N-n$ points;

DEFINITION 2.1. Let $y:\mathbb{Z}\to\mathbb{R}$ be a real function. If $y(t_0)=0$ and $y(t_0-1)y(t_0+1)<0$, then t_0 is a simple zero of $y(t)$. If $y(t_0)y(t_0+1)<0$, then

$$s=\frac{t_0y(t_0+1)-(t_0+1)y(t_0)}{y(t_0+1)-y(t_0)}\in(t_0,t_0+1)$$

is called a nodal point of $y(t)$. The simple zero and the nodal point are called the simple generalized zero of $y(t)$.

Now, as a direct consequence of [35, Theorem 1], we could get the following lemma.

LEMMA 2.2. *Suppose that (H1) and (H2) hold. Then the following eigenvalue problem*

$$-\nabla[p(t)\Delta y(t)]+q(t)y(t)=\lambda\omega(t)y(t), \qquad t\in[0,N-2]_{\mathbb{Z}}, \tag{2.1}$$
$$y(-1)=y(N-1)=0 \tag{2.2}$$

has exactly $N-1$ *real eigenvalues* $\mu_{k,\pm}$ *with*

$$\mu_{N-n,-}<\mu_{N-n-1,-}<\cdots<\mu_{1,-}<0<\mu_{1,+}<\cdots<\mu_{n-1,+}, \qquad \text{if } \omega(N-1)>0, \tag{2.3}$$

or

$$\mu_{N-n-1,-}<\cdots<\mu_{1,-}<0<\mu_{1,+}<\cdots<\mu_{n-1,+}<\mu_{n,+}, \qquad \text{if } \omega(N-1)<0. \tag{2.4}$$

Moreover, the eigenfunction $y(n,\mu_{k,\nu})$*, corresponding to the eigenvalue* $\mu_{k,\nu}$*, has exactly* $k-1$ *simple generalized zeros in* $[0,N-2]$.

Proof. If $\omega(N-1)>0$, then by (H2), there exist exactly $n-1$ points in $[0,N-2]_{\mathbb{Z}}$ such that $\omega(t)>0$ and $N-n$ points in $[0,N-2]_{\mathbb{Z}}$ such that $\omega(t)<0$. Therefore, the similar discussion of [35, Theorem 1] guarantees that (2.3) holds. If $\omega(N-1)<0$, then by (H2), there are exactly n points in $[0,N-2]_{\mathbb{Z}}$ such that $\omega(t)>0$ and $N-n-1$ points in $[0,N-2]_{\mathbb{Z}}$ such that $\omega(t)<0$. Therefore, (2.4) holds. $\square$

3. Characteristic polynomial and its properties

In this section, we try to look for the characteristic polynomial of the eigenvalue problem (1.1), (1.2) and also discuss its properties which are essential to our main results.

Let $\varphi(t,\lambda)$ be the solution of the initial value problem

$$-\nabla[p(t)\Delta\varphi(t)]+q(t)\varphi(t)-\lambda\omega(t)\varphi(t)=0, \qquad t\in[0,N-1]_{\mathbb{Z}}, \tag{3.1}$$
$$\varphi(-1,\lambda)=1, \quad \varphi(0,\lambda)=1 \tag{3.2}$$

and $\psi(t,\lambda)$ the solution of the initial value problem

$$-\nabla[p(t)\Delta\psi(t)]+q(t)\psi(t)-\lambda\omega(t)\psi(t)=0, \qquad t\in[0,N-1]_{\mathbb{Z}}, \tag{3.3}$$
$$\psi(-1,\lambda)=0, \quad \psi(0,\lambda)=1. \tag{3.4}$$

Then $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are two independent solutions of (1.1). Now, multiplying both sides of (3.1) by $\psi(t,\lambda)$ and multiplying both sides of (3.3) by $\varphi(t,\lambda)$, and summing from $t=0$ to $t=N-1$, then subtracting these two equations, we get

$$\psi(N,\lambda)\varphi(N-1,\lambda)-\psi(N-1,\lambda)\varphi(N,\lambda)=1. \tag{3.5}$$

Let $y(t,\lambda)=C_1\varphi(t,\lambda)+C_2\psi(t,\lambda)$. Then $y(t,\lambda)$ is the general solutions of Eq (1.1). By the boundary condition (1.2), we get

$$\begin{cases} C_1[\varphi(N-1,\lambda)-e^{i\alpha}k_1]+C_2\psi(N-1,\lambda)=0, \\ C_1[\Delta\varphi(N-1,\lambda)-k_2e^{i\alpha}]+C_2[\Delta\psi(N-1,\lambda)-e^{i\alpha}k_3]=0. \end{cases}$$

It is well-known that $y(t,\lambda)$ is a nontrivial solutions of (1.1), (1.2) if and only if

$$\begin{vmatrix} \varphi(N-1,\lambda)-e^{i\alpha}k_1 & \psi(N-1,\lambda) \\ \Delta\varphi(N-1,\lambda)-k_2e^{i\alpha} & \Delta\psi(N-1,\lambda)-k_3e^{i\alpha} \end{vmatrix} = 0. \tag{3.6}$$

Setting

$$f(\lambda) = k_3\varphi(N-1,\lambda)+k_1\Delta\psi(N-1,\lambda)-k_2\psi(N-1,\lambda). \tag{3.7}$$

Then, by virtue of (3.5), (3.6) and (3.7), we get

$$1+e^{2i\alpha}-e^{i\alpha}f(\lambda)=0.$$

Therefore, we get the characteristic polynomial of (1.1) and (1.2) as follows,

$$F(\lambda)=f(\lambda)-2\cos\alpha=0. \tag{3.8}$$

For the sake of convenience, in the rest of this paper, we always suppose that if λ_* is a double zero of $F(\lambda)$, then we count λ_* twice. Now, let us discuss some properties of the characteristic polynomial $F(\lambda)$.

LEMMA 3.1. *Suppose that $k_3>0$. Then:*

(i) if k is an even number and $\alpha=\pi$, then $F(\mu_{k,\nu})>0$;

(ii) if k is an even number and $\alpha\neq 0,\pi$, then $F(\mu_{k,\nu})>0$;

(iii) if k is an even number, $\alpha=0$ and $\psi(N,\mu_{k,\nu})\neq k_3$, then $F(\mu_{k,\nu})>0$;

(iv) if k is an odd number and $\alpha=0$, then $F(\mu_{k,\nu})<0$;

(v) if k is an odd number and $\alpha\neq 0,\pi$, then $F(\mu_{k,\nu})<0$;

(vi) if k is an odd number, $\alpha=\pi$ and $\psi(N,\mu_{k,\nu})\neq -k_3$, then $F(\mu_{k,\nu})<0$;

(vii) if k is an even number, $\alpha=0$ and $\psi(N,\mu_{k,\nu})=k_3$, then $F(\mu_{k,\nu})=0$;

(viii) if k is an odd number, $\alpha=\pi$ and $\psi(N,\mu_{k,\nu})=-k_3$, then $F(\mu_{k,\nu})=0$.

Proof. Obviously, $\psi(N-1,\mu_{k,\nu})=0$. Then (3.5) converts to

$$\psi(N,\mu_{k,\nu})\varphi(N-1,\mu_{k,\nu})=1. \tag{3.9}$$

Then, by (3.8) and (3.9), we get

$$\begin{aligned} F(\mu_{k,\nu}) =& f(\mu_{k,\nu})-2\cos\alpha \\ =& \frac{k_3}{\psi(N,\mu_{k,\nu})}+k_1\psi(N,\mu_{k,\nu})-2\cos\alpha \\ =& \frac{k_1\psi^2(N,\mu_{k,\nu})-2\psi(N,\mu_{k,\nu})\cos\alpha+k_3}{\psi(N,\mu_{k,\nu})} \\ =& \frac{k_3[k_1\psi(N,\mu_{k,\nu})-\cos\alpha]^2+k_3(1-\cos^2\alpha)}{\psi(N,\mu_{k,\nu})}. \end{aligned} \tag{3.10}$$

Then, in virtue of $k_3 > 0$ and $k_1 k_3 = 1$, we have

$$\begin{aligned}
&F(\mu_{k,\nu}) > 0, &&\text{if } \psi(N,\mu_{k,\nu}) > 0 \text{ and } \alpha \neq 0,\pi;\\
&F(\mu_{k,\nu}) > 0, &&\text{if } \psi(N,\mu_{k,\nu}) > 0 \text{ and } \alpha = \pi;\\
&F(\mu_{k,\nu}) > 0, &&\text{if } \psi(N,\mu_{k,\nu}) > 0, \psi(N,\mu_{k,\nu}) \neq k_3 \text{ and } \alpha = 0;\\
&F(\mu_{k,\nu}) < 0, &&\text{if } \psi(N,\mu_{k,\nu}) < 0 \text{ and } \alpha \neq 0,\pi;\\
&F(\mu_{k,\nu}) < 0, &&\text{if } \psi(N,\mu_{k,\nu}) < 0 \text{ and } \alpha = 0;\\
&F(\mu_{k,\nu}) < 0, &&\text{if } \psi(N,\mu_{k,\nu}) < 0, \psi(N,\mu_{k,\nu}) \neq -k_3 \text{ and } \alpha = \pi;\\
&F(\mu_{k,\nu}) = 0, &&\text{if } \psi(N,\mu_{k,\nu}) = k_3 \text{ and } \alpha = 0;\\
&F(\mu_{k,\nu}) = 0, &&\text{if } \psi(N,\mu_{k,\nu}) = -k_3 \text{ and } \alpha = \pi.
\end{aligned}$$

Since $\psi(0,\mu_{k,\nu}) = 1$ and $\psi(-1,\mu_{k,\nu}) = \psi(N-1,\mu_{k,\nu}) = 0$, $\psi(N,\mu_{k,\nu})$ is positive or negative according to whether $\psi(t,\mu_{k,\nu})$ has an even or an odd number of simple generalized zeros in the interval $[0,N-1)$. Therefore, when k is an odd number, $F(\mu_{k,\nu}) \leqslant 0$, and when k is an even number, $F(\mu_{k,\nu}) \geqslant 0$. □

REMARK 3.2. Since similar results can be obtained by substituting K for $-K$ under the case that $k_3 < 0$, in the rest of this paper, we always suppose that

(H3) $k_3 > 0$.

In fact, $e^{i\alpha}K = e^{i(\pi+\alpha)}(-K)$ for $\alpha \in (-\pi,0)$ and $e^{i\alpha}K = e^{i(-\pi+\alpha)}(-K)$ for $\alpha \in (0,\pi)$. Hence, the boundary condition (1.2) in the case of $k_3 < 0$ and $\alpha \neq 0, -\pi < \alpha < \pi$, can be written as condition (1.2), where α is replaced by $\pi+\alpha$ for $\alpha \in (-\pi,0)$ and $-\pi+\alpha$ for $\alpha \in (0,\pi)$, and K is replaced by $-K$.

LEMMA 3.3. *Suppose that $q(t) \equiv 0$ and (H1)-(H3) hold. Then:*

(i) $F(0) > 0$ *if and only if*

$$k_3 + k_1 - 2\cos\alpha > k_2 \sum_{s=0}^{N-1} \frac{p(-1)}{p(s)};$$

(ii) $F(0) = 0$ *if and only if*

$$k_3 + k_1 - 2\cos\alpha = k_2 \sum_{s=0}^{N-1} \frac{p(-1)}{p(s)};$$

(iii) $F(0) < 0$ *if and only if*

$$k_3 + k_1 - 2\cos\alpha < k_2 \sum_{s=0}^{N-1} \frac{p(-1)}{p(s)}.$$

Proof. Let $q(t) \equiv 0$. By $(3.1)-(3.4)$, $\varphi(N-1,0) = \Delta\psi(N-1,0) = 1$ and $\psi(N-1,0) = \sum_{s=0}^{N-1} \frac{p(-1)}{p(s)}$. This implies that

$$F(0) = k_3 + k_1 - 2\cos\alpha - k_2 \sum_{s=0}^{N-1} \frac{p(-1)}{p(s)}.$$

Therefore, the assertions (i)-(iii) hold. □

LEMMA 3.4. *Suppose that (H1)-(H3) hold. If* $q(t) \geqslant 0$ *and* $q(t) \not\equiv 0$ *on* $[0,N-1]_{\mathbb{Z}}$*, then for each* $k_2 \leqslant 0$*,* $F(0) > 0$.

Proof. First, let us prove that $\varphi(N-1,0) > 1$. We claim $\Delta\varphi(t) \geqslant 0$, $t \in \{-1,0,\cdots,N-1\}$. Suppose on the contrary, then there exists t_0 such that

$$t_0 = \min\{t | \Delta\varphi(t) < 0\}.$$

Obviously, $\Delta\varphi(t_0) < 0$. On the other hand, by (3.1),

$$\begin{aligned}
\varphi(t_0+1) &= \left(1 + \frac{p(t_0-1)+q(t_0)}{p(t_0)}\right)\varphi(t_0) - \frac{p(t_0-1)}{p(t_0)}\varphi(t_0-1) \\
&\geqslant \left(1 + \frac{p(t_0-1)+q(t_0)}{p(t_0)}\right)\varphi(t_0) - \frac{p(t_0-1)}{p(t_0)}\varphi(t_0) \\
&= \left(1 + \frac{q(t_0)}{p(t_0)}\right)\varphi(t_0).
\end{aligned}$$

Since $q(t_0) \geqslant 0$ and $p(t_0) > 0$, we get that $\Delta\varphi(t_0) \geqslant 0$, which is a contradiction.

Now, we prove that $\varphi(N-1,0) > 1$. In fact, by the conditions $q(t_0) > 0$, $\Delta\varphi(t) \geqslant 0$ and (3.2), we get that there exists at least one point $t_* \in \{0,1,\cdots,N-1\}$ such that $\Delta\varphi(t_*) > 0$. Furthermore, by $\varphi(-1,0) = 1$, we get $\varphi(N-1,0) > 1$.

Second, let us prove that $\Delta\psi(N-1,0) \geqslant 1$. In fact, by (3.4), $\psi(-1,0) = 0$ and $\Delta\psi(-1,0) = 1$, we obtain

$$\Delta\psi(0,0) = \frac{p(-1)}{p(0)}\Delta\psi(-1,0) + \frac{q(0)}{p(0)}\psi(0,0) \geqslant \frac{p(-1)}{p(0)}.$$

This implies that

$$\Delta\psi(1,0) = \frac{p(0)}{p(1)}\Delta\psi(0,0) + \frac{q(1)}{p(1)}\psi(1,0) \geqslant \frac{p(-1)}{p(1)}.$$

Similar, we get

$$\Delta\psi(N-1,0) \geqslant \frac{p(-1)}{p(N-1)}.$$

Since $p(-1) = p(N-1)$, $\Delta\psi(N-1,0) \geqslant 1$.

Thus,

$$F(0) = k_1\varphi(N-1,0) + k_3\Delta\psi(N-1,0) - 2\cos\alpha - k_2\psi(N-1,0) > 0. \quad \square$$

REMARK 3.5. Actually, under the assumptions of Lemma 3.4, we know from the proof of Lemma 3.4 that even if $k_2 > 0$, $F(0)$ may also be greater than 0 as long as

$$k_1\varphi(N-1,0)+k_3\Delta\psi(N-1,0)-2\cos\alpha-k_2\psi(N-1,0)>0.$$

Moreover, for any $k_2 \in \mathbb{R}$, it could be concluded that the following conclusions hold for $q(t) \geqslant 0$ and $q(t) \not\equiv 0$ and $k_3 > 0$.

(i) $F(0)>0 \Leftrightarrow k_1\varphi(N-1,0)+k_3\Delta\psi(N-1,0)-2\cos\alpha-k_2\psi(N-1,0)>0$;

(ii) $F(0)=0 \Leftrightarrow k_1\varphi(N-1,0)+k_3\Delta\psi(N-1,0)-2\cos\alpha-k_2\psi(N-1,0)=0$;

(iii) $F(0)<0 \Leftrightarrow k_1\varphi(N-1,0)+k_3\Delta\psi(N-1,0)-2\cos\alpha-k_2\psi(N-1,0)<0$.

However, these conditions on k_1, k_2 and k_3 are more difficult to verify. So, we use a simple condition $k_2 < 0$ in Lemma 3.4 to guarantee $F(0) > 0$. It is worth to note that the results of the existence of the eigenvalue are the same. On the other hand, since the case that $q(t) \equiv 0$ is more difficult than the case that $q(t) \not\equiv 0$, we always suppose that $q(t) \equiv 0$ in the rest of this paper. More precisely, we suppose that the following assumption holds for the rest context,

(H4) $q(t) \equiv 0$.

From Lemma 3.3, we know that $\lambda = 0$ will always be a zero of $F(\lambda)$ if and only if

(H5) $k_3 + k_1 - 2\cos\alpha = k_2 \sum_{s=0}^{N-1} \frac{p(-1)}{p(s)}$

holds. So, we will discuss that whether $\lambda = 0$ is a simple zero or a multiple zero in this case.

LEMMA 3.6. *Suppose (H1)-(H5) hold. Then:*

(i) if

$$k_3\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}+k_1\sum_{s=0}^{N-1}\omega(s)\sum_{t=-1}^{s-1}\frac{1}{p(t)}+k_2\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}\sum_{t=-1}^{s-1}\frac{p(-1)}{p(t)}>0, \tag{3.11}$$

then $F'(0)<0$;

(ii) if

$$k_3\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}+k_1\sum_{s=0}^{N-1}\omega(s)\sum_{t=-1}^{s-1}\frac{1}{p(t)}+k_2\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}\sum_{t=-1}^{s-1}\frac{p(-1)}{p(t)}<0, \tag{3.12}$$

then $F'(0)>0$;

(iii) if

$$k_3\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}+k_1\sum_{s=0}^{N-1}\omega(s)\sum_{t=-1}^{s-1}\frac{1}{p(t)}+k_2\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}\sum_{t=-1}^{s-1}\frac{p(-1)}{p(t)}=0, \tag{3.13}$$

then $F'(0)=0$.

Proof. Firstly, we claim that

$$\varphi(n,\lambda)=Q_n(\lambda)-\lambda\sum_{s=0}^{n-1}\omega(s)\sum_{t=s}^{n-1}\frac{1}{p(t)}+1, \tag{3.14}$$

where, $Q_n(\lambda)$ satisfies: for $0\leqslant n\leqslant 1$, $Q_n(\lambda)=0$ and for $n>1$, $Q_n(\lambda)$ is a n degree polynomial of λ and its lowest degree is 2.

Now, we use the induction method to prove (3.14).

Step 1. Let $n=1$. Then by (3.1) and (3.2), we obtain that

$$\varphi(1,\lambda)=-\lambda\frac{\omega(0)}{p(0)}+1,$$

this implies (3.14) holds for $n=1$.

Step 2. Let $n=2$. Then by (3.2), we get that

$$\varphi(2,\lambda)=\frac{\omega(0)\omega(1)\lambda^2}{p(0)p(1)}-\lambda\sum_{s=0}^{1}\omega(s)\sum_{t=s}^{1}\frac{1}{p(t)}+1,$$

this implies (3.14) holds for $n=2$.

Step 3. Suppose (3.14) holds for $2<n\leqslant k$. Then by (3.1) and (3.2), we get that

$$\begin{aligned}
&\varphi(k+1,\lambda)\\
=&\left(1+\frac{p(k-1)}{p(k)}-\lambda\frac{\omega(k)}{p(k)}\right)\varphi(k,\lambda)-\frac{p(k-1)}{p(k)}\varphi(k-1,\lambda)\\
=&\left(1+\frac{p(k-1)}{p(k)}-\lambda\frac{\omega(k)}{p(k)}\right)\left(Q_n(\lambda)-\lambda\sum_{s=0}^{k-1}\omega(s)\sum_{t=s}^{k-1}\frac{1}{p(t)}+1\right)\\
&-\frac{p(k-1)}{p(k)}\left(Q_{k-1}(\lambda)-\lambda\sum_{s=0}^{k-2}\omega(s)\sum_{t=s}^{k-2}\frac{1}{p(t)}+1\right)\\
=&Q_{k+1}(\lambda)-\lambda\left(1+\frac{p(k-1)}{p(k)}\right)\sum_{s=0}^{k-1}\omega(s)\sum_{t=s}^{k-1}\frac{1}{p(t)}\\
&-\lambda\frac{\omega(k)}{p(k)}+\lambda\frac{p(k-1)}{p(k)}\sum_{s=0}^{k-2}\omega(s)\sum_{t=s}^{k-2}\frac{1}{p(t)}+1\\
=&Q_{k+1}(\lambda)-\lambda\sum_{s=0}^{k}\omega(s)\sum_{t=s}^{k}\frac{1}{p(t)}+1.
\end{aligned}$$

Here, $Q_{k+1}(\lambda)$ is a $k+1$ degree polynomial of λ whose lowest degree is 2. Therefore, (3.14) holds.

Secondly, similar to the above proof, we get for

$$\begin{aligned}
\psi(n,\lambda)=&P_n(\lambda)-\lambda\sum_{s=0}^{n-1}\left(\frac{\omega(s)}{p(s)}+\frac{\omega(s)}{p(s+1)}+\cdots+\frac{\omega(s)}{p(n-1)}\right)\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right)\\
&+p(-1)\left(\frac{1}{p(0)}+\cdots+\frac{1}{p(n-1)}\right)+1,\qquad n\in\{1,2,\cdots\},
\end{aligned} \tag{3.15}$$

where, $P_n(\lambda)$ satisfies: for $0 \leqslant n \leqslant 1$, $Q_n(\lambda)=0$ and for $n>1$, $P_n(\lambda)$ is a n degree polynomial of λ and its lowest degree is 2.

Now, we use the induction method to prove (3.15).

Step 1. Let $n=1$. Then by (3.3) and (3.4), we obtain that

$$\psi(1,\lambda)=-\lambda\frac{\omega(0)}{p(0)}+1+\frac{p(-1)}{p(0)},$$

this implies (3.15) holds for $n=1$.

Step 2. Let $n=2$. Then by (3.4), we get

$$\begin{aligned}\psi(2,\lambda)=&\frac{\omega(0)\omega(1)\lambda^2}{p(0)p(1)}-\lambda\left(\frac{\omega(0)}{p(0)}+\frac{\omega(0)}{p(1)}+\frac{\omega(1)}{p(1)}+\frac{\omega(1)p(-1)}{p(1)p(0)}\right)\\&+p(-1)\left(\frac{1}{p(0)}+\frac{1}{p(1)}\right)+1\\=&\frac{\omega(0)\omega(1)\lambda^2}{p(0)p(1)}\\&-\lambda\sum_{s=0}^{1}\left(\frac{\omega(s)}{p(s)}+\frac{\omega(s)}{p(s+1)}+\cdots+\frac{\omega(s)}{p(1)}\right)\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right)\\&+p(-1)\left(\frac{1}{p(0)}+\frac{1}{p(1)}\right)+1.\end{aligned}$$

This implies (3.15) holds for $n=2$.

Step 3. Suppose (3.15) holds for $2<n\leqslant k$. Then by (3.3) and (3.4), we get that

$$\begin{aligned}&\psi(k+1,\lambda)\\=&\left(1+\frac{p(k-1)}{p(k)}-\lambda\frac{\omega(k)}{p(k)}\right)\psi(k,\lambda)-\frac{p(k-1)}{p(k)}\psi(k-1,\lambda)\\=&\left(1+\frac{p(k-1)}{p(k)}-\lambda\frac{\omega(k)}{p(k)}\right)\left\{P_k(\lambda)+p(-1)\left(\frac{1}{p(0)}+\cdots+\frac{1}{p(k-1)}\right)+1\right.\\&\left.-\lambda\sum_{s=0}^{k-1}\left(\frac{\omega(s)}{p(s)}+\frac{\omega(s)}{p(s+1)}+\cdots+\frac{\omega(s)}{p(k-1)}\right)\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right)\right\}\\&-\frac{p(k-1)}{p(k)}\left\{P_{k-1}(\lambda)+p(-1)\left(\frac{1}{p(0)}+\cdots+\frac{1}{p(k-2)}\right)+1\right.\\&\left.-\lambda\sum_{s=0}^{k-2}\left(\frac{\omega(s)}{p(s)}+\frac{\omega(s)}{p(s+1)}+\cdots+\frac{\omega(s)}{p(k-2)}\right)\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right)\right\}\\=&P_{k+1}(\lambda)-\lambda\sum_{s=0}^{k}\left(\frac{\omega(s)}{p(s)}+\frac{\omega(s)}{p(s+1)}+\cdots+\frac{\omega(s)}{p(k)}\right)\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right)\\&+p(-1)\left(\frac{1}{p(0)}+\cdots+\frac{1}{p(k)}\right)+1,\end{aligned}$$

where $P_{k+1}(\lambda)$ is a polynomial of λ and its lowest degree is 2. Therefore, (3.15) holds.

Finally, similar to the proof of (3.14) and (3.15), we could obtain the following equation:

$$\Delta\psi(k,\lambda) = H_k(\lambda) - \lambda\sum_{s=0}^{k}\frac{\omega(s)}{p(k)}\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right)+\frac{p(-1)}{p(k)}. \tag{3.16}$$

Here, $H_k(\lambda)$ satisfies: for $k<1$, $H_k(\lambda)=0$ and for $k\geqslant 1$, $H_k(\lambda)$ is a k degree polynomial of λ and its lowest degree is 2.

Then, by (3.14), (3.15) and (3.16), it can be seen that

$$\begin{aligned} F(\lambda) =& k_3\varphi(N-1,\lambda)+k_1\Delta\psi(N-1,\lambda)-k_2\psi(N-1,\lambda)-2\cos\alpha \\ =& K_{N-1}(\lambda)-\lambda k_3\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}-\lambda k_1\sum_{s=0}^{N-1}\frac{\omega(s)}{p(N-1)}\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right) \\ &-\lambda k_2\sum_{s=0}^{N-2}\left(\frac{\omega(s)}{p(s)}+\frac{\omega(s)}{p(s+1)}+\cdots+\frac{\omega(s)}{p(N-2)}\right)\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right). \end{aligned} \tag{3.17}$$

Here, $K_{N-1}(\lambda)$ is a $N-1$ degree polynomial of λ and its lowest degree is 2. Now, (3.17) implies that

$$\begin{aligned} F'(0) =& -k_3\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}-k_1\sum_{s=0}^{N-1}\frac{\omega(s)}{p(N-1)}\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right) \\ &-k_2\sum_{s=0}^{N-2}\left(\frac{\omega(s)}{p(s)}+\frac{\omega(s)}{p(s+1)}+\cdots+\frac{\omega(s)}{p(N-2)}\right)\left(1+\frac{p(-1)}{p(0)}+\cdots+\frac{p(-1)}{p(s-1)}\right) \\ =& -\left(k_3\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}+k_1\sum_{s=0}^{N-1}\omega(s)\sum_{t=-1}^{s-1}\frac{1}{p(t)}+k_2\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}\sum_{t=-1}^{s-1}\frac{p(-1)}{p(t)}\right). \end{aligned}$$

Thus, if (3.11) holds, then $F'(0)<0$; if (3.12) holds, then $F'(0)>0$ and if (3.13) holds, then $F'(0)=0$. $\square$

REMARK 3.7. Let $k_2=0$ in (3.11)-(3.13), then the left sides of these three inequalities or equations reduce to

$$M := k_3\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}+k_1\sum_{s=0}^{N-1}\omega(s)\sum_{t=-1}^{s-1}\frac{1}{p(t)}. \tag{3.18}$$

Furthermore, if $k_3=k_1=1$, then by the assumption $p(-1)=p(N-1)$, we have

$$M=\sum_{s=0}^{N-2}\omega(s)\sum_{t=s}^{N-2}\frac{1}{p(t)}+\sum_{s=0}^{N-1}\omega(s)\sum_{t=-1}^{s-1}\frac{1}{p(t)}=\sum_{s=0}^{N-1}\omega(s)\sum_{t=0}^{N-1}\frac{1}{p(t)}. \tag{3.19}$$

Combining this with the fact that $p(t)>0$ on $[-1,N-1]_{\mathbb{Z}}$, we easily get three assertions under the assumptions $k_3=k_1=1$ and $k_2=0$:

(i) if $\sum_{s=0}^{N-2}\omega(s)>0$, then $F'(0)<0$;

(ii) if $\sum_{s=0}^{N-2}\omega(s)<0$, then $F'(0)>0$;

(iii) if $\sum_{s=0}^{N-2}\omega(s)=0$, then $F'(0)=0$.

These above three conclusions also have been obtained by Gao and Ma [37] (Lemma 2.1) for $\alpha=0$. Therefore, Lemma 3.6 is more general than Lemma 2.1 of Gao and Ma [37].

4. Main results

In this section, we demonstrate and prove our main results. From Lemma 2.2, we know that $\omega(N-1)$ is greater or less than 0 will influence the number of the positive and negative eigenvalues of the problem (2.1), (2.2), which in turn will influence the number of the positive and negative eigenvalues of the problem (1.1), (1.2). Therefore, we will demonstrate our main results into two cases: $\omega(N-1)>0$ and $\omega(N-1)<0$. Moreover, for $\nu\in\{+,-\}$, let us use $\lambda(K)_{k,\nu}$ to denote the zeros of $F(\lambda)$ with $\alpha=0$, $\lambda(-K)_{k,\nu}$ to denote the zeros of $F(\lambda)$ with $\alpha=\pi$ and $\lambda(e^{i\alpha}K)_{k,\nu}$ to denote the zeros of $F(\lambda)$ with $-\pi<\alpha<\pi$ and $\alpha\neq 0,\pi$.

THEOREM 4.1. *Suppose (H1)-(H4) hold,* $\omega(N-1)>0$ *and*

(H6) $k_3+k_1-2\geqslant k_2\sum_{s=0}^{N-1}\frac{p(-1)}{p(s)}$.

Then, for $\alpha=0$, $F(\lambda)$ *has at least two zeros* $\lambda(K)_{1,-}$ *and* $\lambda(K)_{1,+}$ *in* $(\mu_{1,-},\mu_{1,+})$, *and have exactly* $n-2$ *positive zeros in* $(\mu_{1,+},\infty)$ *and have exactly* $N-n-1$ *negative zeros in* $(-\infty,\mu_{1,-})$. *Then these* N *eigenvalues can be ordered as follows.*

(a) If N *and* n *are both even numbers, then*

$$\begin{aligned}&\mu_{N-n,-}\leqslant\lambda(K)_{N-n,-}<\mu_{N-n-1,-}<\lambda(K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\leqslant\lambda(K)_{N-n-2,-}\\&<\mu_{N-n-3,-}<\lambda(K)_{N-n-3,-}\cdots\leqslant\lambda(K)_{4,-}<\mu_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}\\&<\mu_{1,-}<\lambda(K)_{1,-}\leqslant 0\leqslant\lambda(K)_{1,+}<\mu_{1,+}<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\mu_{3,+}\\&<\lambda(K)_{4,+}\leqslant\mu_{4,+}\leqslant\cdots\leqslant\mu_{n-4,+}\leqslant\lambda(K)_{n-3,+}<\mu_{n-3,+}<\lambda(K)_{n-2,+}\leqslant\mu_{n-2,+}\\&\leqslant\lambda(K)_{n-1,+}<\mu_{n-1,+}<\lambda(K)_{n,+}.\end{aligned}\tag{4.1}$$

(b) If N *is an even number and* n *is an odd number, then*

$$\begin{aligned}&\mu_{N-n,-}<\lambda(K)_{N-n,-}\leqslant\mu_{N-n-1,-}\leqslant\lambda(K)_{N-n-1,-}<\mu_{N-n-2,-}<\lambda(K)_{N-n-2,-}\\&\leqslant\mu_{N-n-3,-}\leqslant\lambda(K)_{N-n-3,-}\cdots\leqslant\lambda(K)_{4,-}<\mu_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}\\&<\mu_{1,-}<\lambda(K)_{1,-}\leqslant 0\leqslant\lambda(K)_{1,+}<\mu_{1,+}<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\mu_{3,+}\\&<\lambda(K)_{4,+}\leqslant\mu_{4,+}\leqslant\cdots<\mu_{n-4,+}<\lambda(K)_{n-3,+}\leqslant\mu_{n-3,+}\leqslant\lambda(K)_{n-2,+}<\mu_{n-2,+}\\&<\lambda(K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(K)_{n,+}.\end{aligned}\tag{4.2}$$

(c) If N is an odd number and n is an even number, then

$$\begin{aligned}&\mu_{N-n,-}<\lambda(K)_{N-n,-}\leqslant\mu_{N-n-1,-}\leqslant\lambda(K)_{N-n-1,-}<\mu_{N-n-2,-}<\lambda(K)_{N-n-2,-}\\&\leqslant\mu_{N-n-3,-}\leqslant\lambda(K)_{N-n-3,-}\leqslant\cdots\leqslant\lambda(K)_{4,-}<\mu_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}\\&<\mu_{1,-}<\lambda(K)_{1,-}\leqslant 0\leqslant\lambda(K)_{1,+}<\mu_{1,+}<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\mu_{3,+}\\&<\lambda(K)_{4,+}\leqslant\mu_{4,+}\leqslant\cdots\leqslant\mu_{n-4,+}\leqslant\lambda(K)_{n-3,+}<\mu_{n-3,+}<\lambda(K)_{n-2,+}\leqslant\mu_{n-2,+}\\&\leqslant\lambda(K)_{n-1,+}<\mu_{n-1,+}<\lambda(K)_{n,+}.\end{aligned} \tag{4.3}$$

(d) If N and n are both odd numbers, then

$$\begin{aligned}&\mu_{N-n,-}\leqslant\lambda(K)_{N-n,-}<\mu_{N-n-1,-}\leqslant\lambda(K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\leqslant\lambda(K)_{N-n-2,-}\\&<\mu_{N-n-3,-}<\lambda(K)_{N-n-3,-}\leqslant\cdots\leqslant\lambda(K)_{4,-}<\mu_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}\\&<\mu_{1,-}<\lambda(K)_{1,-}\leqslant 0\leqslant\lambda(K)_{1,+}<\mu_{1,+}<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\mu_{3,+}\\&<\lambda(K)_{4,+}\leqslant\mu_{4,+}\leqslant\cdots\leqslant\mu_{n-4,+}\leqslant\lambda(K)_{n-3,+}<\mu_{n-3,+}<\lambda(K)_{n-2,+}\leqslant\mu_{n-2,+}\\&\leqslant\lambda(K)_{n-1,+}<\mu_{n-1,+}<\lambda(K)_{n,+}.\end{aligned} \tag{4.4}$$

Proof. By Lemma 3.3, $F(0)\geqslant 0$. Combining this with Lemma 3.1, we get that $F(\lambda)$ has at least two zeros $\lambda(K)_{1,-}$ and $\lambda(K)_{1,+}$ in $(\mu_{1,-},\mu_{1,+})$, which satisfy

$$\mu_{1,+}>\lambda(K)_{1,+}\geqslant 0\geqslant\lambda(K)_{1,-}>\mu_{1,-}.$$

Now, our proof will be divided into two cases.

Case 1. N is an even number.

Case 1.1. n is an even number. Then $n-1$ is an odd number and $N-n$ is an even number. By Lemma 3.1, we have

(i) $F(\lambda)$ has at least two zeros $\lambda(K)_{2k,+}$ and $\lambda(K)_{2k+1,+}$ in $(\mu_{2k-1,+},\mu_{2k+1,+})$, ($k=1,2,\cdots,\frac{n-2}{2}$), which satisfy $\lambda(K)_{2k,+}\leqslant\mu_{2k,+}\leqslant\lambda(K)_{2k+1,+}$.

(ii) $F(\lambda)$ has at least two zeros $\lambda(K)_{2j+1,-}$ and $\lambda(K)_{2j,-}$ in $(\mu_{2j+1,-},\mu_{2j-1,-})$, ($j=1,2,\cdots,\frac{N-n-2}{2}$) which satisfy $\lambda(K)_{2j+1,-}\leqslant\mu_{2j,-}\leqslant\lambda(K)_{2j,-}$.

Now, we will prove that there exist at least one zero $\lambda(K)_{n,+}$ of $F(\lambda)$ which satisfies $\lambda(K)_{n,+}>\mu_{n-1,+}$ and at least one zero $\lambda(K)_{N-n,-}$ of $F(\lambda)$ which satisfies $\mu_{N-n,-}\leqslant\lambda(K)_{N-n,-}<\mu_{N-n-1,-}$. From (3.1)-(3.4), we get that

$$\psi(N,\lambda)=(-1)^n\frac{\Pi_{t=0}^{N-1}\omega(t)}{\Pi_{t=0}^{N-1}p(t)}\lambda^N+A(\lambda),$$

where $A(\lambda)$ is a polynomial of λ with degree $N-1$. Similarly, we could also get that $\varphi(N-1,\lambda)$ and $\psi(N-1,\lambda)$ are both polynomials of λ with degree $N-1$. Therefore,

$$\begin{aligned}F(\lambda)=&k_3\varphi(N-1,\lambda)+k_1\psi(N,\lambda)+(k_1-k_2)\psi(N-1,\lambda)-2\cos\alpha\\=&(-1)^nk_1A(N)\lambda^N+B(\lambda)-2\cos\alpha,\end{aligned}$$

where $B(\lambda)$ is a polynomial of λ whose degree is $N-1$. Since N and n are both even number and $k_1 > 0$ (according to (H3)), $F(\lambda) \to +\infty$ as $\lambda \to \pm\infty$. Meanwhile, since $n-1$ is an odd number, according to Lemma 3.1, $F(\mu_{n-1,+}) < 0$. Therefore, $F(\lambda)$ has at least one real zero $\lambda(K)_{n,+}$ with $\lambda(K)_{n,+} > \mu_{n-1,+}$. Since $N-n$ is an even number, we get that $F(\mu_{N-n,-}) \geqslant 0$ and $F(\mu_{N-n-1,-}) > 0$. Combining this with the fact that $\lim_{\lambda\to-\infty} F(\lambda) = +\infty$, we obtain that there exists another real zero $\lambda(K)_{N-n,-}$ of $F(\lambda)$ with $\mu_{N-n,-} \leqslant \lambda(K)_{N-n,-} < \mu_{N-n-1,-}$.

Furthermore, as a N degree polynomial of λ, $F(\lambda)$ has at most N real zeros. Therefore, $F(\lambda)$ has exactly N real zeros. According to the above discussion, these N eigenvalues satisfy (4.1).

Case 1.2. n is an odd number. Then $n-1$ is an even number and $N-n$ is an odd number. By Lemma 3.1, we have

(i) $F(\lambda)$ has at least two zeros $\lambda(K)_{2k,+}$ and $\lambda(K)_{2k+1,+}$ in $(\mu_{2k-1,+}, \mu_{2k+1,+})$, $(k = 1,2,\cdots,\frac{n-3}{2})$, which satisfy $\lambda(K)_{2k,+} \leqslant \mu_{2k,+} \leqslant \lambda(K)_{2k+1,+}$.

(ii) $F(\lambda)$ has at least two zeros $\lambda(K)_{2j+1,-}$ and $\lambda(K)_{2j,-}$ in $(\mu_{2j+1,-}, \mu_{2j-1,-})$, $(j = 1,2,\cdots,\frac{N-n-1}{2})$ which satisfy $\lambda(K)_{2j+1,-} \leqslant \mu_{2j,-} \leqslant \lambda(K)_{2j,-}$.

Moreover, by the definition of $F(\lambda)$, we can see that $F(\lambda) \to +\infty$ as $\lambda \to +\infty$. Combining this with the fact that $F(\mu_{n-2,+}) < 0$, we obtain there exists another zero $\lambda(K)_{n,+}$ of $F(\lambda)$ with $\lambda(K)_{n,+} > \mu_{n-1,+}$.

Furthermore, as a polynomial of λ with degree N, $F(\lambda)$ has at most N real zeros. Therefore, $F(\lambda)$ has exactly N real zeros which satisfy (4.2).

Case 2. N is an odd number.

Case 2.1. n is an even number. Then $n-1$ is an odd number and $N-n$ is an odd number. Similar to the discussion of *Case 1*, (4.3) holds.

Case 2.2. n is an odd number. Then $n-1$ is an even number and $T-n$ an even number. Furthermore, similar to the which discussion of *Case 1*, (4.4) holds. □

As a direct consequence, we could get the following corollary.

COROLLARY 4.2. *Suppose that (H1)-(H4), (H6) hold and* $\omega(N-1) > 0$. *If* $\lambda(K)_{k,\nu}$ *is a double zero of* $F(\lambda)$ *with* $\alpha = 0$, *then either*

(i) $\lambda(K)_{k,\nu} = 0$;

or

(ii) $\lambda(K)_{k,\nu} = \mu_{i,\nu}$ *for some even* $i \in \{1,\cdots\max\{N-n, n-1\}\}$.

REMARK 4.3. In Theorem 4.1, if the strict inequality holds in (H6), then by Lemma 3.3, the two principal eigenvalues $\lambda(K)_{1,+}$ and $\lambda(K)_{1,-}$ satisfy: $\lambda(K)_{1,+} > 0 > \lambda(K)_{1,-}$.

On the other hand, if the equation holds in (H6), i.e., (H5) holds, then the relations between $\lambda(K)_{1,+}$, $\lambda(K)_{1,-}$ and 0 could be determined by the conditions on k_i $(i = 1,2,3)$ and α in Lemma 3.6. Actually, if (3.11) holds, then $\lambda(K)_{1,-} < 0 = \lambda(K)_{1,+}$. If (3.12) holds, then $\lambda(K)_{1,-} = 0 < \lambda(K)_{1,+}$. If (3.13) holds, then $\lambda(K)_{1,-} = 0 = \lambda(K)_{1,+}$.

So, in Theorem 4.1 and the following main results, we only give the non-strict inequality for $\lambda(K)_{1,+}$, $\lambda(K)_{1,-}$ and 0.

Since all the proofs are similar to the proof of Theorem 4.1 with obvious change, we only list other two main results, i.e., the result for $F(\lambda)=0$ with $\alpha=\pi$ and the result for $F(\lambda)=0$ with $\alpha\in(-\pi,0)\cup(0,\pi)$.

THEOREM 4.4. *Suppose that (H1)-(H4) hold,* $\omega(N-1)>0$ *and*

(H7) $k_3+k_1+2\geqslant k_2\sum_{s=0}^{N-1}\frac{p(-1)}{p(s)}$.

Then, for $\alpha=\pi$, $F(\lambda)$ *has at least two zeros* $\lambda(-K)_{1,-}$ *and* $\lambda(-K)_{1,+}$ *in* $(\mu_{1,-},\mu_{1,+})$, *exactly* $n-2$ *positive zeros in* $(\mu_{1,+},\infty)$ *and exactly* $N-n-1$ *negative zeros in* $(-\infty,\mu_{1,-})$. *Moreover, these* N *eigenvalues can be ordered as follows.*

(a) If N *and* n *are both even numbers, then*

$$\begin{aligned}&\mu_{N-n,-}<\lambda(-K)_{N-n,-}\leqslant\mu_{N-n-1,-}\leqslant\lambda(-K)_{N-n-1,-}<\mu_{N-n-2,-}\\&<\lambda(-K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\lambda(-K)_{N-n-3,-}<\cdots\leqslant\mu_{3,-}\leqslant\lambda(-K)_{3,-}<\mu_{2,-}\\&<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\leqslant\lambda(-K)_{1,-}\leqslant 0\leqslant\lambda(-K)_{1,+}\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\mu_{2,+}\\&<\lambda(-K)_{3,+}\leqslant\cdots<\mu_{n-4,+}<\lambda(-K)_{n-3,+}\leqslant\mu_{n-3,+}\leqslant\lambda(-K)_{n-2,+}<\mu_{n-2,+}\\&<\lambda(-K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(-K)_{n,+}.\end{aligned}$$

(b) If N *is an even number and* n *is an odd number, then*

$$\begin{aligned}&\mu_{N-n,-}\leqslant\lambda(-K)_{N-n,-}<\mu_{N-n-1,-}<\lambda(-K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\\&\leqslant\lambda(-K)_{N-n-2,-}<\mu_{N-n-3,-}<\lambda(-K)_{N-n-3,-}\leqslant\cdots\leqslant\mu_{3,-}\leqslant\lambda(-K)_{3,-}<\mu_{2,-}\\&<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\leqslant\lambda(-K)_{1,-}\leqslant 0\leqslant\lambda(-K)_{1,+}\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\mu_{2,+}\\&<\lambda(-K)_{3,+}\leqslant\cdots\leqslant\mu_{n-4,+}\leqslant\lambda(-K)_{n-3,+}<\mu_{n-3,+}<\lambda(-K)_{n-2,+}\leqslant\mu_{n-2,+}\\&\leqslant\lambda(-K)_{n-1,+}<\mu_{n-1,+}<\lambda(-K)_{n,+}.\end{aligned}$$

(c) If N *is an odd number and* n *is an even number, then*

$$\begin{aligned}&\mu_{N-n,-}\leqslant\lambda(-K)_{N-n,-}<\mu_{N-n-1,-}<\lambda(-K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\\&\leqslant\lambda(-K)_{N-n-2,-}<\mu_{N-n-3,-}<\lambda(-K)_{N-n-3,-}\leqslant\cdots\leqslant\mu_{3,-}\leqslant\lambda(-K)_{3,-}<\mu_{2,-}\\&<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\leqslant\lambda(-K)_{1,-}\leqslant 0\leqslant\lambda(-K)_{1,+}\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\mu_{2,+}\\&<\lambda(-K)_{3,+}\leqslant\cdots<\mu_{n-4,+}<\lambda(-K)_{n-3,+}\leqslant\mu_{n-3,+}\leqslant\lambda(-K)_{n-2,+}<\mu_{n-2,+}\\&<\lambda(-K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(-K)_{n,+}.\end{aligned}$$

(d) If N *and* n *are both odd numbers, then*

$$\begin{aligned}&\mu_{N-n,-}<\lambda(-K)_{N-n,-}\leqslant\mu_{N-n-1,-}\leqslant\lambda(-K)_{N-n-1,-}<\mu_{N-n-2,-}\\&<\lambda(-K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\lambda(-K)_{N-n-3,-}<\cdots\leqslant\mu_{3,-}\leqslant\lambda(-K)_{3,-}<\mu_{2,-}\\&<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\leqslant\lambda(-K)_{1,-}\leqslant 0\leqslant\lambda(-K)_{1,+}\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\mu_{2,+}\\&<\lambda(-K)_{3,+}\leqslant\mu_{3,+}\leqslant\cdots\leqslant\mu_{n-4,+}\leqslant\lambda(-K)_{n-3,+}<\mu_{n-3,+}<\lambda(-K)_{n-2,+}\\&\leqslant\mu_{n-2,+}\leqslant\lambda(-K)_{n-1,+}<\mu_{n-1,+}<\lambda(-K)_{n,+}.\end{aligned}$$

COROLLARY 4.5. *Suppose that (H1)-(H4), (H6) hold and* $\omega(N-1)>0$. *If* $\lambda(-K)_{k,\nu}$ *is a double zero of* $F(\lambda)$ *with* $\alpha=\pi$, *then* $\lambda(-K)_{k,\nu}=\mu_{i,\nu}$ *for some odd* $i\in\{1,\cdots,\max\{N-n,n-1\}\}$.

THEOREM 4.6. *Suppose that (H1)-(H4) hold,* $\omega(N-1)>0$ *and*

(H8) $\quad k_3+k_1-2\cos\alpha\geqslant k_2\sum_{s=0}^{N-1}\frac{p(-1)}{p(s)}, \qquad \alpha\in(-\pi,0)\cup(0,\pi).$

Then, for $\alpha\in(-\pi,0)\cup(0,\pi)$, $F(\lambda)$ *has at least two zeros* $\lambda(e^{i\alpha}K)_{1,-}$ *and* $\lambda(e^{i\alpha}K)_{1,+}$ *in* $(\mu_{1,-},\mu_{1,+})$, *and have exactly* $n-2$ *positive zeros in* $(\mu_{1,+},\infty)$ *and have exactly* $N-n-1$ *negative zeros in* $(-\infty,\mu_{1,-})$. *Then these* N *eigenvalues can be ordered as follows*

$$\begin{aligned}&\mu_{N-n,-}<\lambda(e^{i\alpha}K)_{N-n,-}<\mu_{N-n-1,-}<\lambda(e^{i\alpha}K)_{N-n-1,-}<\mu_{N-n-2,-}<\cdots<\mu_{3,-}\\&<\lambda(e^{i\alpha}K)_{3,-}<\mu_{2,-}<\lambda(e^{i\alpha}K)_{2,-}<\mu_{1,-}<\lambda(e^{i\alpha}K)_{1,-}\leqslant 0\leqslant\lambda(e^{i\alpha}K)_{1,+}<\mu_{1,+}\\&<\lambda(e^{i\alpha}K)_{2,+}<\mu_{2,+}<\lambda(e^{i\alpha}K)_{3,+}<\mu_{3,+}<\cdots<\lambda(e^{i\alpha}K)_{n-2,+}<\mu_{n-2,+}\\&<\lambda(e^{i\alpha}K)_{n-1,+}<\mu_{n-1,+}<\lambda(e^{i\alpha}K)_{n,+}.\end{aligned}$$

Based on Theorem 4.1, Theorem 4.4 and Theorem 4.6, we could obtain the interlacing properties for the eigenvalues of (1.1), (1.2) as α changes as follows. Until now, we know that the existence of the eigenvalues $\lambda(K)_{1,\nu}$, $\lambda(-K)_{1,\nu}$ and $\lambda(e^{i\alpha}K)_{1,\nu}$ mainly depend on $F(0)\geqslant 0$. From (H6)-(H8), it is easy to see that if (H6) holds, then (H7) and (H8) hold. Furthermore, $F(0)\geqslant 0$ for each $\alpha\in(-\pi,\pi)$. Therefore, in the following discussion, we always suppose that (H6) holds for discussing the interlacing properties of the eigenvalues.

THEOREM 4.7. *Suppose that (H1)-(H4), (H6) hold and* $\omega(N-1)>0$. *If* N *and* n *are both even numbers, then the eigenvalues* $\lambda(-K)_{k,\nu}$, $\lambda(e^{i\alpha}K)_{k,\nu}$ *and* $\lambda(-K)_{k,\nu}$ *can be ordered as follows*

$$\begin{aligned}&\mu_{N-n,-}\leqslant\lambda(K)_{N-n,-}<\lambda(e^{i\alpha}K)_{N-n,-}<\lambda(-K)_{N-n,-}\leqslant\mu_{N-n-1,-}\leqslant\lambda(-K)_{N-n-1,-}\\&<\lambda(e^{i\alpha}K)_{N-n-1,-}<\lambda(K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\leqslant\lambda(K)_{N-n-2,-}<\lambda(e^{i\alpha}K)_{N-n-2,-}\\&<\lambda(-K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\cdots<\lambda(e^{i\alpha}K)_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}\\&<\lambda(e^{i\alpha}K)_{2,-}<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\leqslant\lambda(-K)_{1,-}<\lambda(e^{i\alpha}K)_{1,-}<\lambda(K)_{1,-}\leqslant 0\leqslant\lambda(K)_{1,+}\\&<\lambda(e^{i\alpha}K)_{1,+}<\lambda(-K)_{1,+}\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\lambda(e^{i\alpha}K)_{2,+}<\lambda(K)_{2,+}\leqslant\mu_{2,+}\\&\leqslant\lambda(K)_{3,+}<\lambda(e^{i\alpha}K)_{3,+}<\lambda(-K)_{3,+}\leqslant\mu_{3,+}\leqslant\cdots\leqslant\lambda(-K)_{n-2,+}<\lambda(e^{i\alpha}K)_{n-2,+}\\&<\lambda(K)_{n-2,+}\leqslant\mu_{n-2,+}\leqslant\lambda(K)_{n-1,+}<\lambda(e^{i\alpha}K)_{n-1,+}<\lambda(-K)_{n-1,+}\leqslant\mu_{n-1,+}\\&\leqslant\lambda(-K)_{n,+}<\lambda(e^{i\alpha}K)_{n,+}<\lambda(K)_{n,+}.\end{aligned}$$

Proof. We only need to prove that for $\nu\in\{+,-\}$,

$$|\lambda(K)_{2k-1,\nu}|<|\lambda(e^{i\alpha}K)_{2k-1,\nu}|<|\lambda(-K)_{2k-1,\nu}|, \tag{4.5}$$

and

$$|\lambda(-K)_{2k,\nu}|<|\lambda(e^{i\alpha}K)_{2k,\nu}|<|\lambda(K)_{2k,\nu}|. \tag{4.6}$$

At first, let us consider the value of $F(\lambda)$ at $\lambda=\lambda(K)_{k,\nu}$, $\lambda=\lambda(e^{i\alpha}K)_{k,\nu}$ and $\lambda=\lambda(-K)_{k,\nu}$ for different α. For the sake of convenience, let $F_{\alpha=0}(\lambda)$ be the value of $F(\lambda)$ at λ for $\alpha=0$, $F_{\alpha=\pi}(\lambda)$ be the value of $F(\lambda)$ at λ for $\alpha=\pi$ and $F_{\alpha\in(-\pi,0)\cup(0,\pi)}(\lambda)$ be the value of $F(\lambda)$ at λ for $\alpha\in(-\pi,0)\cup(0,\pi)$.

(a) Let $\lambda=\lambda(K)_{k,\nu}$. If $\alpha=0$, then $F_{\alpha=0}(\lambda(K)_{k,\nu})=0$, which implies that

$$k_3\varphi(N-1,\lambda(K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(K)_{k,\nu})-k_2\psi(N-1,\lambda(K)_{k,\nu})-2=0. \quad (4.7)$$

By (4.7), if $\alpha=\pi$, then

$$\begin{aligned}&F_{\alpha=\pi}(\lambda(K)_{k,\nu})\\=&k_3\varphi(N-1,\lambda(K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(K)_{k,\nu})-k_2\psi(N-1,\lambda(K)_{k,\nu})+2 \quad (4.8)\\=&4>0.\end{aligned}$$

If $-\pi<\alpha<\pi$ and $\alpha\neq 0$, then

$$\begin{aligned}&F_{\alpha\in(-\pi,0)\cup(0,\pi)}(\lambda(K)_{k,\nu})\\=&k_3\varphi(N-1,\lambda(K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(K)_{k,\nu})-k_2\psi(N-1,\lambda(K)_{k,\nu})-2\cos\alpha\\=&2-2\cos\alpha>0.\end{aligned} \quad (4.9)$$

(b) Let $\lambda=\lambda(e^{i\alpha}K)_{k,\nu}$. If $\alpha\neq 0,\pi$, then $F_{\alpha\in(-\pi,0)\cup(0,\pi)}(\lambda(e^{i\alpha}K)_{k,\nu})=0$, which implies that

$$k_3\varphi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})-k_2\psi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})=2\cos\alpha. \quad (4.10)$$

By (4.10), if $\alpha=0$, then

$$\begin{aligned}&F_{\alpha=0}(\lambda(e^{i\alpha}K)_{k,\nu})\\=&k_3\varphi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})-k_2\psi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})-2\\=&2\cos\alpha_0-2<0, \qquad \text{for some } \alpha_0\neq 0,\pi.\end{aligned} \quad (4.11)$$

If $\alpha=\pi$, then

$$\begin{aligned}&F_{\alpha=\pi}(\lambda(e^{i\alpha}K)_{k,\nu})\\=&k_3\varphi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})-k_2\psi(N-1,\lambda(e^{i\alpha}K)_{k,\nu})+2\\=&2\cos\alpha_0+2>0, \qquad \text{for some } \alpha_0\neq 0,\pi.\end{aligned} \quad (4.12)$$

(c) Let $\lambda=\lambda(-K)_{k,\nu}$. If $\alpha=\pi$, then $F_{\alpha=\pi}(\lambda(-K)_{k,\nu})=0$, which implies that

$$k_3\varphi(N-1,\lambda(-K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(-K)_{k,\nu})-k_2\psi(N-1,\lambda(-K)_{k,\nu})=-2. \quad (4.13)$$

By (4.13), if $\alpha=0$, then

$$\begin{aligned}&F_{\alpha=0}(\lambda(-K)_{k,\nu})\\=&k_3\varphi(N-1,\lambda(-K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(-K)_{k,\nu})-k_2\psi(N-1,\lambda(-K)_{k,\nu})-2\\=&-2-2=-4<0.\end{aligned} \quad (4.14)$$

If $-\pi<\alpha<\pi$ and $\alpha\neq 0$, then

$$\begin{aligned}&F_{\alpha\in(-\pi,0)\cup(0,\pi)}(\lambda(-K)_{k,\nu})\\ =&k_3\varphi(N-1,\lambda(-K)_{k,\nu})+k_1\Delta\psi(N-1,\lambda(-K)_{k,\nu})-k_2\psi(N-1,\lambda(-K)_{k,\nu})-2\cos\alpha\\ =&-2-2\cos\alpha<0.\end{aligned} \tag{4.15}$$

Second, we prove (4.5) and (4.6) in the case that $\nu=+$. Meanwhile, we only discuss the relations of $\lambda(K)_{k,+}$ and $\lambda(e^{i\alpha}K)_{k,+}$. Other cases could be treated similarly.

Firstly, let us prove that $\lambda(K)_{1,+}<\lambda(e^{i\alpha}K)_{1,+}$. Suppose on the contrary that $\lambda(K)_{1,+}\geqslant\lambda(e^{i\alpha}K)_{1,+}$. If $\lambda(K)_{1,+}=\lambda(e^{i\alpha}K)_{1,+}$, then

$$F_{\alpha=0}(\lambda(e^{i\alpha}K)_{1,+})=F_{\alpha=0}(\lambda(K)_{1,+})=0. \tag{4.16}$$

However, by (4.11), $F_{\alpha=0}(\lambda(e^{i\alpha}K)_{1,+})<0$. A contradiction. Therefore,

$$\lambda(K)_{1,+}>\lambda(e^{i\alpha}K)_{1,+}. \tag{4.17}$$

Obviously, $\lambda(K)_{1,+}>0$. Furthermore, by (H6), we get that the strict inequality in (H8) holds. Then by Lemma 3.3, $F_{\alpha\in(-\pi,0)\cup(0,\pi)}(0)>0$. Therefore, Remark 4.3 and Theorem 4.6 imply that $\lambda(e^{i\alpha}K)_{1,+}>0$, subsequently, $\lambda(K)_{1,+}>\lambda(e^{i\alpha}K)_{1,+}>0$.

Now, by the fact that $\lambda(K)_{1,+}$ is the first nonnegative zero of $F_{\alpha=0}(\lambda)$, we have $F_{\alpha=0}(\lambda(e^{i\alpha}K)_{1,+})>0$, which contradicts (4.11). Thus, $\lambda(K)_{1,+}<\lambda(e^{i\alpha}K)_{1,+}$.

Secondly, we claim that $\lambda(K)_{2k-1,+}<\lambda(e^{i\alpha}K)_{2k-1,+}$ for $k\in\{2,\cdots,\frac{n-2}{2}\}$. From Theorem 4.1 and Theorem 4.6, we get that

$$\mu_{2k-2,+}<\lambda(e^{i\alpha}K)_{2k-1,+}<\mu_{2k-1,+},\quad \lambda(K)_{2k-2,+}\leqslant\mu_{2k-2,+}\leqslant\lambda(K)_{2k-1,+}<\mu_{2k-1,+}.$$

If $\mu_{2k-2,+}=\lambda(K)_{2k-1,+}$, then it is obvious that $\lambda(K)_{2k-1,+}<\lambda(e^{i\alpha}K)_{2k-1,+}$. Now, suppose that $\mu_{2k-2,+}<\lambda(K)_{2k-1,+}$. By Lemma 3.1, $F_{\alpha=0}(\mu_{2k-2,+})\geqslant 0$, this combines with $F_{\alpha=0}(0)\geqslant 0$ implies that for each $\lambda\in(\lambda(K)_{2k-2,+},\lambda(K)_{2k-1,+})$, $F_{\alpha=0}(\lambda)\geqslant 0$. However, by (4.11), when $\alpha=0$, $F_{\alpha=0}(\lambda(e^{i\alpha}K)_{2k-1,+})<0$. Combining this with $\lambda(e^{i\alpha}K)_{2k-1,+}>\mu_{2k-2,+}$, it is not difficult to see that $\lambda(K)_{2k-1,+}<\lambda(e^{i\alpha}K)_{2k-1,+}$ for $k\in\{2,\cdots,\frac{n-2}{2}\}$.

Thirdly, we prove that $\lambda(e^{i\alpha}K)_{2k,+}<\lambda(K)_{2k,+}$ for $k\in\{1,2,\cdots,\frac{n-2}{2}\}$. From Theorem 4.1 and Theorem 4.6,

$$\mu_{2k-1,+}<\lambda(e^{i\alpha}K)_{2k,+}<\mu_{2k,+},\qquad \mu_{2k-1,+}<\lambda(K)_{2k,+}\leqslant\mu_{2k,+}\leqslant\lambda(K)_{2k+1,+}.$$

If $\mu_{2k,+}=\lambda(K)_{2k,+}$, then it is obvious that $\lambda(e^{i\alpha}K)_{2k,+}<\lambda(K)_{2k,+}$. Now, suppose that $\lambda(K)_{2k,+}<\mu_{2k,+}$. By Lemma 3.1, $F_{\alpha=0}(\mu_{2k,+})\geqslant 0$, this combines with $F_{\alpha=0}(0)\geqslant 0$ implies that for each $\lambda\in(\lambda(K)_{2k,+},\lambda(K)_{2k+1,+})$, $F_{\alpha=0}(\lambda)\geqslant 0$. Meanwhile, by (4.11), when $\alpha=0$, $F_{\alpha=0}(\lambda(e^{i\alpha}K)_{2k-1,+})<0$. Combining this with $\lambda(e^{i\alpha}K)_{2k,+}<\mu_{2k,+}$, we could obtain that $\lambda(e^{i\alpha}K)_{2k,+}<\lambda(K)_{2k,+}$ for $k\in\{1,2,\cdots,\frac{n-2}{2}\}$.

Finally, we claim that $\lambda(e^{i\alpha}K)_{n,+}<\lambda(K)_{n,+}$. Suppose on the contrary that $\lambda(K)_{n,+}\geqslant\lambda(e^{i\alpha}K)_{n,+}$. Since $\lambda(K)_{n,+}$ is the last zero of $F_{\alpha=0}(\lambda)$ and $F_{\alpha=0}(\lambda)\to+\infty$ as $\lambda\to+\infty$, we know that $F_{\alpha=0}(\lambda(e^{i\alpha}K)_{n,+})>0$. However, this contradicts (4.11). Therefore, $\lambda(e^{i\alpha}K)_{n,+}<\lambda(K)_{n,+}$. $\square$

THEOREM 4.8. *Suppose that (H1)-(H4), (H6) hold and* $\omega(N-1)>0$. *Then the eigenvalues* $\lambda(-K)_{k,\nu}$, $\lambda(e^{i\alpha}K)_{k,\nu}$ *and* $\lambda(-K)_{k,\nu}$ *can be ordered as follows.*

(a) If N *is an even number and* n *is an odd number, then*

$$\begin{aligned}&\mu_{N-n,-}\leqslant\lambda(-K)_{N-n,-}<\lambda(e^{i\alpha}K)_{N-n,-}<\lambda(K)_{N-n,-}\leqslant\mu_{N-n-1,-}\\ &\leqslant\lambda(K)_{N-n-1,-}<\lambda(e^{i\alpha}K)_{N-n-1,-}<\lambda(-K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\\ &\leqslant\lambda(-K)_{N-n-2,-}<\lambda(e^{i\alpha}K)_{N-n-2,-}<\lambda(K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\cdots\\ &<\lambda(e^{i\alpha}K)_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}<\lambda(e^{i\alpha}K)_{2,-}<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\\ &\leqslant\lambda(-K)_{1,-}<\lambda(e^{i\alpha}K)_{1,-}<\lambda(K)_{1,-}\leqslant 0\leqslant\lambda(K)_{1,+}<\lambda(e^{i\alpha}K)_{1,+}<\lambda(-K)_{1,+}\\ &\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\lambda(e^{i\alpha}K)_{2,+}<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\lambda(e^{i\alpha}K)_{3,+}\\ &<\lambda(-K)_{3,+}\leqslant\mu_{3,+}\leqslant\cdots\leqslant\lambda(K)_{n-2,+}<\lambda(e^{i\alpha}K)_{n-2,+}<\lambda(-K)_{n-2,+}\leqslant\mu_{n-2,+}\\ &\leqslant\lambda(-K)_{n-1,+}<\lambda(e^{i\alpha}K)_{n-1,+}<\lambda(K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(K)_{n,+}<\lambda(e^{i\alpha}K)_{n,+}\\ &<\lambda(-K)_{n,+}.\end{aligned}$$

(b) If N *and* n *are both odd numbers, then*

$$\begin{aligned}&\mu_{N-n,-}\leqslant\lambda(K)_{N-n,-}<\lambda(e^{i\alpha}K)_{N-n,-}<\lambda(-K)_{N-n,-}\leqslant\mu_{N-n-1,-}\\ &\leqslant\lambda(-K)_{N-n-1,-}<\lambda(e^{i\alpha}K)_{N-n-1,-}<\lambda(K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\\ &\leqslant\lambda(K)_{N-n-2,-}<\lambda(e^{i\alpha}K)_{N-n-2,-}<\lambda(-K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\cdots\\ &<\lambda(e^{i\alpha}K)_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}<\lambda(e^{i\alpha}K)_{2,-}<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\\ &\leqslant\lambda(-K)_{1,-}<\lambda(e^{i\alpha}K)_{1,-}<\lambda(K)_{1,-}\leqslant 0\leqslant\lambda(K)_{1,+}<\lambda(e^{i\alpha}K)_{1,+}<\lambda(-K)_{1,+}\\ &\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\lambda(e^{i\alpha}K)_{2,+}<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\lambda(e^{i\alpha}K)_{3,+}\\ &<\lambda(-K)_{3,+}\leqslant\mu_{3,+}\leqslant\cdots\leqslant\lambda(K)_{n-2,+}<\lambda(e^{i\alpha}K)_{n-2,+}<\lambda(-K)_{n-2,+}\leqslant\mu_{n-2,+}\\ &\leqslant\lambda(-K)_{n-1,+}<\lambda(e^{i\alpha}K)_{n-1,+}<\lambda(K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(K)_{n,+}<\lambda(e^{i\alpha}K)_{n,+}\\ &<\lambda(-K)_{n,+}.\end{aligned}$$

(c) If N *is an odd number and* n *is an even number, then*

$$\begin{aligned}&\mu_{N-n,-}\leqslant\lambda(-K)_{N-n,-}<\lambda(e^{i\alpha}K)_{N-n,-}<\lambda(K)_{N-n,-}\leqslant\mu_{N-n-1,-}\\ &\leqslant\lambda(K)_{N-n-1,-}<\lambda(e^{i\alpha}K)_{N-n-1,-}<\lambda(-K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\\ &\leqslant\lambda(-K)_{N-n-2,-}<\lambda(e^{i\alpha}K)_{N-n-2,-}<\lambda(-K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\cdots\\ &<\lambda(e^{i\alpha}K)_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}<\lambda(e^{i\alpha}K)_{2,-}<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\\ &\leqslant\lambda(-K)_{1,-}<\lambda(e^{i\alpha}K)_{1,-}<\lambda(K)_{1,-}\leqslant 0\leqslant\lambda(K)_{1,+}<\lambda(e^{i\alpha}K)_{1,+}<\lambda(-K)_{1,+}\\ &\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\lambda(e^{i\alpha}K)_{2,+}<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\lambda(e^{i\alpha}K)_{3,+}\\ &<\lambda(-K)_{3,+}\leqslant\mu_{3,+}\leqslant\cdots\leqslant\lambda(-K)_{n-2,+}<\lambda(e^{i\alpha}K)_{n-2,+}<\lambda(K)_{n-2,+}\leqslant\mu_{n-2,+}\\ &\leqslant\lambda(K)_{n-1,+}<\lambda(e^{i\alpha}K)_{n-1,+}<\lambda(-K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(-K)_{n,+}<\lambda(e^{i\alpha}K)_{n,+}\\ &<\lambda(K)_{n,+}.\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 4.7 with obvious change, so we omit it here. □

Finally, we just list the interlacing properties of the eigenvalues of (1.1), (1.2) for the case $\omega(N-1)<0$, since the discussion is similar to the discussion for $\omega(N-1)>0$. The main difference between these two kinds of cases is the different number of the positive and negative eigenvalues under these two cases.

THEOREM 4.9. *Suppose that (H1)-(H4), (H6) hold and* $\omega(N-1)<0$. *Then eigenvalues* $\lambda(-K)_{k,\nu}$, $\lambda(e^{i\alpha}K)_{k,\nu}$ *and* $\lambda(-K)_{k,\nu}$ *can be ordered as follows.*

(a) If N *and* n *are both even numbers, then*

$$\begin{aligned}
&\lambda(K)_{N-n,-}<\lambda(e^{i\alpha}K)_{N-n,-}<\lambda(-K)_{N-n,-}\leqslant\mu_{N-n-1,-}\leqslant\lambda(-K)_{N-n-1,-}\\
&<\lambda(e^{i\alpha}K)_{N-n-1,-}<\lambda(K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\leqslant\lambda(K)_{N-n-2,-}<\lambda(e^{i\alpha}K)_{N-n-2,-}\\
&<\lambda(-K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\cdots<\lambda(e^{i\alpha}K)_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}\\
&<\lambda(e^{i\alpha}K)_{2,-}<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\leqslant\lambda(-K)_{1,-}<\lambda(e^{i\alpha}K)_{1,-}<\lambda(K)_{1,-}\leqslant 0\\
&\leqslant\lambda(K)_{1,+}<\lambda(e^{i\alpha}K)_{1,+}<\lambda(-K)_{1,+}\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\lambda(e^{i\alpha}K)_{2,+}\\
&<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\cdots\leqslant\mu_{n-2,+}\leqslant\lambda(K)_{n-1,+}<\lambda(e^{i\alpha}K)_{n-1,+}\\
&<\lambda(-K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(-K)_{n,+}<\lambda(e^{i\alpha}K)_{n,+}<\lambda(K)_{n,+}\leqslant\mu_{n,+}.
\end{aligned}$$

(b) If N *is an even number and* n *is an odd number, then*

$$\begin{aligned}
&\lambda(-K)_{N-n,-}<\lambda(e^{i\alpha}K)_{N-n,-}<\lambda(K)_{N-n,-}\leqslant\mu_{N-n-1,-}\leqslant\lambda(K)_{N-n-1,-}\\
&<\lambda(e^{i\alpha}K)_{N-n-1,-}<\lambda(-K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\leqslant\lambda(-K)_{N-n-2,-}<\lambda(e^{i\alpha}K)_{N-n-2,-}\\
&<\lambda(K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\cdots<\lambda(e^{i\alpha}K)_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}\\
&<\lambda(e^{i\alpha}K)_{2,-}<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\leqslant\lambda(-K)_{1,-}<\lambda(e^{i\alpha}K)_{1,-}<\lambda(K)_{1,-}\leqslant 0\\
&\leqslant\lambda(K)_{1,+}<\lambda(e^{i\alpha}K)_{1,+}<\lambda(-K)_{1,+}\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\lambda(e^{i\alpha}K)_{2,+}\\
&<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\cdots\leqslant\mu_{n-2,+}\leqslant\lambda(-K)_{n-1,+}<\lambda(e^{i\alpha}K)_{n-1,+}\\
&<\lambda(K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(K)_{n,+}<\lambda(e^{i\alpha}K)_{n,+}<\lambda(-K)_{n,+}\leqslant\mu_{n,+}.
\end{aligned}$$

(c) If N *is an odd number and* n *is an even number, then*

$$\begin{aligned}
&\lambda(-K)_{N-n,-}<\lambda(e^{i\alpha}K)_{N-n,-}<\lambda(K)_{N-n,-}\leqslant\mu_{N-n-1,-}\leqslant\lambda(K)_{N-n-1,-}\\
&<\lambda(e^{i\alpha}K)_{N-n-1,-}<\lambda(-K)_{N-n-1,-}\leqslant\mu_{N-n-2,-}\leqslant\lambda(-K)_{N-n-2,-}<\lambda(e^{i\alpha}K)_{N-n-2,-}\\
&<\lambda(K)_{N-n-2,-}\leqslant\mu_{N-n-3,-}\leqslant\cdots<\lambda(e^{i\alpha}K)_{3,-}<\lambda(K)_{3,-}\leqslant\mu_{2,-}\leqslant\lambda(K)_{2,-}\\
&<\lambda(e^{i\alpha}K)_{2,-}<\lambda(-K)_{2,-}\leqslant\mu_{1,-}\leqslant\lambda(-K)_{1,-}<\lambda(e^{i\alpha}K)_{1,-}<\lambda(K)_{1,-}\leqslant 0\\
&\leqslant\lambda(K)_{1,+}<\lambda(e^{i\alpha}K)_{1,+}<\lambda(-K)_{1,+}\leqslant\mu_{1,+}\leqslant\lambda(-K)_{2,+}<\lambda(e^{i\alpha}K)_{2,+}\\
&<\lambda(K)_{2,+}\leqslant\mu_{2,+}\leqslant\lambda(K)_{3,+}<\cdots\leqslant\mu_{n-2,+}\leqslant\lambda(K)_{n-1,+}<\lambda(e^{i\alpha}K)_{n-1,+}\\
&<\lambda(-K)_{n-1,+}\leqslant\mu_{n-1,+}\leqslant\lambda(-K)_{n,+}<\lambda(e^{i\alpha}K)_{n,+}<\lambda(K)_{n,+}\leqslant\mu_{n,+}.
\end{aligned}$$

(d) If N and n are both odd numbers, then

$$
\begin{aligned}
&\lambda(K)_{N-n,-} < \lambda(e^{i\alpha}K)_{N-n,-} < \lambda(-K)_{N-n,-} \leqslant \mu_{N-n-1,-} \leqslant \lambda(-K)_{N-n-1,-} \\
&<\lambda(e^{i\alpha}K)_{N-n-1,-} < \lambda(K)_{N-n-1,-} \leqslant \mu_{N-n-2,-} \leqslant \lambda(K)_{N-n-2,-} < \lambda(e^{i\alpha}K)_{N-n-2,-} \\
&<\lambda(-K)_{N-n-2,-} \leqslant \mu_{N-n-3,-} \leqslant \cdots < \lambda(e^{i\alpha}K)_{3,-} < \lambda(K)_{3,-} \leqslant \mu_{2,-} \leqslant \lambda(K)_{2,-} \\
&<\lambda(e^{i\alpha}K)_{2,-} < \lambda(-K)_{2,-} \leqslant \mu_{1,-} \leqslant \lambda(-K)_{1,-} < \lambda(e^{i\alpha}K)_{1,-} < \lambda(K)_{1,-} \leqslant 0 \\
&\leqslant \lambda(K)_{1,+} < \lambda(e^{i\alpha}K)_{1,+} < \lambda(-K)_{1,+} \leqslant \mu_{1,+} \leqslant \lambda(-K)_{2,+} < \lambda(e^{i\alpha}K)_{2,+} \\
&<\lambda(K)_{2,+} \leqslant \mu_{2,+} \leqslant \lambda(K)_{3,+} < \cdots \leqslant \mu_{n-2,+} \leqslant \lambda(-K)_{n-1,+} < \lambda(e^{i\alpha}K)_{n-1,+} \\
&<\lambda(K)_{n-1,+} \leqslant \mu_{n-1,+} \leqslant \lambda(K)_{n,+} < \lambda(e^{i\alpha}K)_{n,+} < \lambda(-K)_{n,+} \leqslant \mu_{n,+}.
\end{aligned}
$$

REMARK 4.10. Until now, we only discuss the case that $F(0) \geqslant 0$. If $F(0) < 0$, we could also obtain the real eigenvalues of (1.1), (1.2). However, compared to the case that $F(0) \geqslant 0$, we could only obtain that (1.1), (1.2) has at least $N-2$ real eigenvalues, similar to the case $F(0) \geqslant 0$, we also could obtain the interlacing properties for these $N-2$ real eigenvalues. For example, suppose that (H1)-(H4) hold and

(H9) $k_3 + k_1 - 2\cos\alpha < k_2 \sum_{s=0}^{N-1} \frac{p(-1)}{p(s)}$.

Then, by (H9), $F(0) < 0$. Furthermore, since $F(\mu_{1,+}) < 0$ and $F(\mu_{1,-}) < 0$, we could not obtain two real zeros in $(\mu_{1,-}, \mu_{1,+})$ of $F(\lambda)$ as in the proof of Theorem 4.1, Theorem 4.4, Theorem 4.6-Theorem 4.9, but other eigenvalues also exist and satisfy the same interlacing properties in this case. This implies that we may lose the principal eigenvalues for (1.1), (1.2).

REMARK 4.11. In this remark, we shall point out the relations of our problems and the discrete left-definite problems. Define the Hilbert space $X := \{y | y : [-1, N]_{\mathbb{Z}} \to \mathbb{C},\text{ and } y \text{ satisfy } (1.2)\}$ with the inner product

$$\langle u, v\rangle = \sum_{t=0}^{N-1} u(t)\bar{v}(t), \qquad u, v \in X.$$

Then X is Banach space with the induced norm $\|u\| = \sqrt{\sum_{t=0}^{N-1} |u(t)|^2}$. Define $L : X \to X$ by

$$Ly := -\nabla(p(n)\Delta y(n)) + q(n)y(n).$$

Then for $y \in X$ and $y \neq \theta$,

$$
\begin{aligned}
\langle Ly, \bar{y}\rangle &= -\sum_{t=0}^{N-1} \nabla(p(t)\Delta y(t))\bar{y}(t) + \sum_{t=0}^{N-1} q(t)y(t)\bar{y}(t) \\
&= \sum_{t=0}^{N-1} p(t-1)\bar{y}(t)\Delta y(t-1) - \sum_{t=0}^{N-1} p(t)\bar{y}(t)\Delta y(t) + \sum_{t=0}^{N-1} q(t)y(t)\bar{y}(t) \\
&= \sum_{t=-1}^{N-2} p(t)\bar{y}(t+1)\Delta y(t) - \sum_{t=-1}^{N-2} p(t)\bar{y}(t)\Delta y(t) + \sum_{t=0}^{N-1} q(t)y(t)\bar{y}(t) \\
&\quad + p(-1)\bar{y}(-1)\Delta y(-1) - p(N-1)\bar{y}(N-1)\Delta y(N-1).
\end{aligned}
$$

Combining the boundary condition (1.2) with the assumption $p(-1) = p(N-1)$, we have

$$\begin{aligned}\langle Ly, y\rangle &= \sum_{t=-1}^{N-2} p(t)|\Delta y(t)|^2 + \sum_{t=0}^{N-1} q(t)|y(t)|^2 + p(-1)[\bar{y}(-1)\Delta y(-1) - \bar{y}(N-1)\Delta y(N-1)] \\ &= \sum_{t=-1}^{N-2} p(t)|\Delta y(t)|^2 + \sum_{t=0}^{N-1} q(t)|y(t)|^2 + p(-1)[(1-k_1k_3)\bar{y}(-1)\Delta y(-1) - k_1k_2|y_{-1}|^2].\end{aligned}$$

Now, if $q(t) \not\equiv 0$ and $q(t) \geqslant 0$ on $[0, N-1]_{\mathbb{Z}}$, we could get a sharp condition for $\langle Ly, \bar{y}\rangle > 0$. That is, $k_1k_3 = 1$ and $k_2 = 0$. This is only a special case of our Lemma 3.4. Therefore, the problem we discuss here is more general than the left-definite case.

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