Operators and Matrices
Volume 14, Number 2 (2020), 523–533 doi:10.7153/oam-2020-14-37

ON POWER DRAZIN NORMAL AND DRAZIN QUASI–NORMAL HILBERT SPACE OPERATORS

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(*Communicated by I. M. Spitkovsky*)

Abstract. A Drazin invertible Hilbert space operator $T \in B(\mathscr{H})$, with Drazin inverse T_d, is (n,m)-power D-normal, $T \in [(n,m)DN]$, if $[T_d^n, T^{*m}] = T_d^n T^{*m} - T^{*m} T_d^n = 0$; T is (n,m)-power D-quasinormal, $T \in [(n,m)DQN]$, if $[T_d^n, T^{*m}T] = 0$. Operators $T \in [(n,m)DN]$ have a representation $T = T_1 \oplus T_0$, where T_1 is similar to an invertible normal operator and T_0 is nilpotent. Using this representation, we have a keener look at the structure of $[(n,m)DN]$ and $[(n,m)DQN]$ operators. It is seen that $T \in [(n,m)DN]$ if and only if $T \in [(n,m)DQN]$, and if $[T,X] = 0$ for some operators $X \in B(\mathscr{H})$ and $T \in [(1,1)DN]$, then $[T_d^*, X] = 0$. Given simply polar operators $S,T \in [(1,1)DN]$ and an operator $A = \begin{pmatrix} T & C \\ 0 & S \end{pmatrix} \in B(\mathscr{H} \oplus \mathscr{H})$, $A \in [(1,1)DN]$ if and only if C has a representation $C = 0 \oplus C_{22}$.

1. Introduction

Let $B(\mathscr{H})$ denote the algebra of operators, i.e. bounded linear transformations, on a complex infinite dimensional Hilbert space $\mathscr{H}$ into itself. For $S,T \in B(\mathscr{H})$, let $[S,T] = ST - TS$ denote the commutator of S,T. An operator $A \in B(\mathscr{H})$ is normal if $[A^*,A] = 0$. The spectral mapping theorem guarantees the existence of normal nth roots of a normal operator $A \in B(\mathscr{H})$; however, normal A may have other non-normal nth roots. If $T \in B(\mathscr{H})$ is an nth root of a normal operator $A \in B(\mathscr{H})$, then an application of the Fuglede theorem [9, 10] to $[T^n, T] = 0$ implies $[T^n, T^*] = 0$. Conversely, $[T^n, T^*] = 0$ implies T^n is normal. Recall, [3], that $T \in B(\mathscr{H})$ is Drazin invertible if there exists an operator $T_d \in B(\mathscr{H})$ such that

$$[T_d, T] = 0,\ T_d^2 T = T_d,\ T^{p+1} T_d = T^p$$

for some integer $p \geqslant 1$. The operator T_d is then the Drazin inverse of T and p is the Drazin index of T. A generalization of $[T^n, T^*] = 0$ is obtained upon replacing T by T_d: T is Drazin normal , $T \in [DN]$, if $[T_d, T^*] = 0$ [2] and T is (n,m)-Drazin normal, for some integer $m \geqslant 1$, $T \in [(n,m)DN]$, if $[T_d^n, T^{*m}] = 0$ [13].

It is clear that if we let the positive integer k denote the *least common multiple* of n and m, $k = \mathrm{LCM}(n,m)$, then $T \in [(n,m)DN]$ implies T_d^k is normal. As an nth root of

Mathematics subject classification (2010): 47A15, 47B15, 47B20.
Keywords and phrases: Drazin invertible operator, structure of $[(n,m)DN]$ operators, n-th root of normal operator, commutativity theorem.

a normal operator, T_d, has a well defined structure [6, 8, 12]. Add to this the fact that as a Drazin invertible operator, T has a direct sum decomposition of type $T = T_1 \oplus T_0$, T_1 invertible and T_0 nilpotent (of some order), and T_d has a decomposition $T_d = T_1^{-1} \oplus 0$, it follows that T_1 is similar to a normal operator [14]. Using this characterisation, we study the structure of $[(n,m)DN]$ operators in this note to prove that the (so called) class $[(n,m)DQN]$ of (n,m) D-quasinormal operators T, $[T_d^n, T^{*m}T] = 0$, studied by [2, 13] coincides with the class of $[(n,m)DN]$ operators. It is seen that $T \in [(n,m)DN] \wedge [(n+1,m)DN]$ (resp., $T \in [(n,m)DN] \wedge [(n,m+1)DN]$) if and only if $T \in [(k,m)DN]$ (resp., $T \in [(n,k)DN]$) for all integers $k \geqslant 1$; an m-partially isometric $[(n,m)DN]$ contraction is the direct sum of a unitary with a nilpotent; $[T,X] = 0$ implies $[T_d^*, X] = 0$ for $T \in [DN]$ and $X \in B(\mathscr{H})$. More generally, if $A, B \in B(\mathscr{H})$ are such that $TA = BT$ for an operator $T \in [DN]$, and if either of the hypotheses $AT = TB$ and $T_d(A-B) = (B-A)T_d$ is satisfied, then $T_d^*A = BT_d^*$ and $AT_d^* = T_d^*B$. Given operators $S, T \in [(n,m)DN]$, we prove a sufficient conditiion for the upper triangular operator $A = \begin{pmatrix} T & C \\ 0 & S \end{pmatrix}$ to be an $[(n,m)DN]$ operator; it is seen that this condition is necessary too in the case in which $n = m = 1$, and both S and T have a simple pole at 0.

2. Results

Throughout the following, S, T shall denote operators in $B(\mathscr{H})$, n and m shall denote positive integers, and I shall denote the identity map. The spectrum of T will be denoted by $\sigma(T)$ and $\text{iso}\sigma(T)$ shall denote the isolated points of the spectrum of T. Many of the properties of $[(n,m)DN]$ operators lie on the surface. For example, $T \in [(n,m)DN]$ implies $T^k \in [(n,m)DN]$ for all integers $k \geqslant 1$, since

$$(T^k)_d = T_d^k, \ [T_d^n, T^{*m}] = 0 \Longrightarrow [T_d^{kn}, T^{*mk}] = 0.$$

If $S, T \in [(n,m)DN]$ and $[S,T] = 0 = [S^*, T]$, then $(TS)_d = T_dS_d = S_dT_d = (ST)_d$,

$$[T_d^n, T^{*m}] = 0 = [S_d^n, S^{*m}] \Longrightarrow [(TS)_d^n, (TS)^{*m}] = 0,$$

and this (result) in turn implies (for tensor product $T \otimes S$ of T and S) that

$$[(T \otimes S)_d^n, (T \otimes S)^{*m}] = [(T_d^n \otimes I)(I \otimes S_d^n), (T^{*m} \otimes I)(I \otimes S^{*m})] = 0.$$

For an understanding of some of the not so apparent structural properties of operators $T \in [(n,m)DN] \vee [(n,m)DQN]$, we start by recalling that T is Drazin invertible if and only if T has finite ascent and finite descent [3, 15]. Equivalently, T is Drazin invertible if and only if $0 \in \text{iso}\sigma(T)$ and there exists an integer $p \geqslant 1$, called the Drazin index of T, such that

$$\mathscr{H} = T^p(\mathscr{H}) \oplus T^{-p}(0) = \mathscr{H}_1 \oplus \mathscr{H}_0, \ T = T|_{T^p(\mathscr{H})} \oplus T|_{T^{-p}(0)} = T_1 \oplus T_0.$$

Here, T_1 is (evidently) invertible and T_0 is p-nilpotent. (In the case in which $0 \notin \sigma(T)$, we allow ourselves a misuse of language and let T^{-1} denote the Drazin inverse of T). Denoting as before the Drazin inverse of T by T_d, T_d has a direct sum representation

$$T_d = T_1^{-1} \oplus 0 \in B(\mathscr{H}_1 \oplus \mathscr{H}_0)$$

[3, Theorem 2.2.3]. Evidently,

$$\begin{aligned} T \in [(n,m)DN] &\iff [T_d^n, T^{*m}] = 0 \\ &\iff [T_1^{-n} \oplus 0, T_1^{*m} \oplus T_0^{*m}] = 0 \\ &\iff [T_1^{-n}, T_1^{*m}] \oplus 0 = 0 \\ &\iff [T_1^n, T_1^{*m}] = 0. \end{aligned}$$

Hence:

PROPOSITION 2.1. *$T \in [(n,m)DN]$ if and only if $T_1 \in [(n,m)DN]$.*

The following theorem provides further information on the structure of $[(n,m)DN]$ operators T.

THEOREM 2.2. *For every $T \in [(n,m)DN]$, there exists a direct sum decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_0$ of $\mathscr{H}$ and a decomposition $T = T|_{H_1} \oplus T|_{\mathscr{H}_0} = T_1 \oplus T_0$ of T such that T is similar to the direct sum of a normal operator in $B(\mathscr{H}_1)$ with a nilpotent operator (of the order of the Drazin index of T) and T_d is similar to a normal operator.*

Proof. Assuming p to be the Drazin index of T, define the (closed) subspaces $\mathscr{H}_1$ and $\mathscr{H}_0$ and the operators T_1 and T_0 as above. Then

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_0, \ T = T_1 \oplus T_0 \in B(\mathscr{H}_1 \oplus \mathscr{H}_0)$$

(with T_1 invertible and T_0 p-nilpotent). Let $s = \mathrm{LCM}(n,m)$. Then

$$[T_d^n, T^{*m}] = 0 \Longrightarrow [T_d^n, T_d^{*m}] = 0 \Longrightarrow [T_d^s, T_d^{*s}] = 0,$$

i.e., T_d^s is normal. Since

$$T_d^s \text{ is normal} \iff T_1^{-s} \text{ is normal} \iff T_1^s \text{ is normal},$$

it follows from [14] that there exists an invertible normal operator $N_1 \in B(\mathscr{H}_1)$ and an invertible operator $S_1 \in B(\mathscr{H}_1)$ such that $T_1 = S_1^{-1} N_1 S_1$. Letting $S = S_1 \oplus I|_{\mathscr{H}_0}$ and $N = N_1^{-1} \oplus 0$, we have $T_d = S^{-1} N S$.

Theorem 2.2 leads to the simplification of the proofs of a number of results from [2, 13]. Postponing this exercise for the time being, we start here with the following proposition which (contrary to the claim in [13, 2]) proves that the classes $[(n,m)DN]$ and $[(n,m)DQN]$ of Hilbert space operators coincide.

PROPOSITION 2.3. *$T \in [(n,m)DQN] \iff T \in [(n,m)DN]$.*

Proof. Following the notation above,

$$\begin{aligned} T \in [(n,m)DQN] &\iff [T_d^n, T^{*m}T] = 0 \\ &\iff [T_1^{-n} \oplus 0, (T_1^{*m} \oplus T_0^{*m})(T_1 \oplus T_0)] = 0 \\ &\iff [T_1^{-n}, T_1^{*m}T_1] = 0 \\ &\iff [T_1^{-n}, T_1^{*m}] = 0 \\ &\iff [T_1^{-n} \oplus 0, T_1^{*m} \oplus T_0^{*m}] = 0 \\ &\iff [T_d^n, T^{*m}] = 0 \\ &\iff T \in [(n,m)DN]. \end{aligned}$$

This completes the proof.

REMARK 2.4. Defining the invertible operator S as in the proof of Theorem 2.2, it is seen that the operators $T \in [(n,m)DQN] \vee [(n,m)DN]$ are similar to the direct sum of a normal operator with a nilpotent operator. Hence, for operators $T \in [(n,m)DQN] \vee [(n,m)DN]$, both T and T^* satisfy Bishop– Eschmeier– Putinar properties $(\beta)_\varepsilon$ and (β). (The interested reader will find all pertinent information related to these properties, and results on operators satisfying these properties, in references [7, 11, 4].) In particular, such operators T are decomposable (hence have the single-valued extension property). Furthermore, because of similarity to the direct sum of a normal and a nilpotent operator, points $\lambda \in \mathrm{iso}\sigma(T)$ for such T are poles of the resolvent of the operator: simple poles if $\lambda \neq 0$ and a pole of order p at 0. In consequence, operators T satisfy most, generalized and classical, Browder and Weyl type theorems. (See [1] for information on Browder and Weyl type theorems.)

By definition, $T \in [(n,m)DN] \wedge [(n+1,m)DN]$ if and only if

$$\begin{aligned} &T_d(T_d^nT^{*m}) = (T^{*m}T_d^n)T_d = (T_d^nT^{*m})T_d \\ \iff &T_1^{-(n+1)}T_1^{*m} = (T_1^{*m}T_1^{-n})T_1^{-1} = (T_1^{-n}T_1^{*m})T_1^{-1} \\ \iff &[T_1^{-1}, T_1^{*m}] = 0 \iff T_1 \in [(1,m)DN] \\ \iff &T \in [(1,m)DN]; \end{aligned}$$

again, $T \in [(n,m)DN] \wedge [(n,m+1)DN]$ if and only if

$$\begin{aligned} &(T^{*m}T_d^n)T^* = (T_d^nT^{*m})T^* = T^{*(m+1)}T_d^n \\ \iff &T_1^{*m}T_1^{-n}T_1^* = T_1^{-n}T_1^{*(m+1)} = T_1^{*(m+1)}T_1^{-n} \\ \iff &[T_1^{-n}, T_1^*] = 0 \iff T_1 \in [(n,1)DN] \\ \iff &T \in [(n,1)DN]. \end{aligned}$$

Hence:

PROPOSITION 2.5. *$T \in [(n,m)DN] \wedge [(n+1,m)DN]$ if and only if $T \in [(k,m)DN]$ and $T \in [(n,m)DN] \wedge [(n,m+1)DN]$ if and only if $T \in [(n,k)DN]$ for all integers $k \geqslant 1$.*

Proposition 2.5 generalizes [13, Propositions 2.5 –2.9]. We remark here that the hypotheses T is injective in [13, Proposition 2.6] and T^* is injective in [13, Proposition 2.9] are redundant.

An operator $A \in B(\mathscr{H})$ is an m-partial isometry for some integer $m \geqslant 1$ if $A^m A^{*m} A^m = A^m$. An invertible m-partial isometry is unitary. Hence, for operators $T \in [(n,m)DN]$ for which T is an m-partial isometry, T_1^m is a unitary, $\sigma(T) \subseteq \partial\mathbb{D} \cup \{0\}$ and $T = T_1 \oplus T_0$, where $\partial\mathbb{D}$ denote the boundary of the unit disc in $\mathbb{C}$, T_1 is similar to a unitary operator [14] and T_0 is nilpotent. Furthermore, since $T_1^{*m} = T_1^{-m}$,

$$\begin{aligned} T \in [(n,m)DN] &\Longleftrightarrow [T_1^{-n}, T_1^{*m}] = 0 \\ &\Longleftrightarrow T_1^{-n-m} T_1^{*m} = T_1^{*2m} T_1^{-n} = T_1^{*m} T_1^{-n-m} \\ &\Longleftrightarrow T \in [(m+n,m)DN]. \end{aligned}$$

It is evident that an m-partially isometric operator $T \in [(n,m)DN]$ for $m = 1$ is the direct sum of a unitary operator with a nilpotent: a similar conclusion holds for a general m in the case in which T is a contraction.

Recall that every contraction $A \in B(\mathscr{H})$ has a direct sum decomposition $A = A_u \oplus A_c$ into its unitary and cnu (=completely non-unitary) parts. A is a cnu $C_{.0}$ contraction if $||A^{*n}x|| \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in \mathscr{H}$ [10, Page 110]. The operator A is k-paranormal for some integer $k \geqslant 2$ if $||Ax||^k \leqslant ||A^k x|| ||x||^{k-1}$ for all $x \in \mathscr{H}$. It is known, see [5, Page 319], that k-paranormal contractions have $C_{.0}$ cnu parts.

PROPOSITION 2.6. *If $T \in [(n,m)DN]$ is an m-partially isometric operator, then $T \in [(1,1)DN]$ (equivalently, $T \in [DN]$) and T has a representation $T = U \oplus T_0$, where $U \in B(\mathscr{H}_1)$ is a unitary and $T_0 \in B(\mathscr{H}_0)$ is a nilpotent.*

Proof. If $T \in [(n,m)DN]$ is m-partially isometric, then (see above) T_1^m is unitary. This, since T is a contraction implies T_1 is a contraction, implies

$$||T_1 x||^m \leqslant ||x||^m = ||T_1^m x|| ||x||^{m-1}$$

for all $x \in \mathscr{H}_1$. Consequently, T_1 is m-paranormal. Since T_1 has a non-trivial $C_{.0}$ cnu part forces T_1^m to have a non-trivial $C_{.0}$ cnu part, we must have that T_1 is unitary. Hence $T = U \oplus T_0$ for some unitary U and nilpotent $T_0 \in B(\mathscr{H}_0)$. Finally,

$$T_1^{*2} = T_1^{-1} T_1^* = T_1^* T_1^{-1} \Longleftrightarrow [T_1^{-1}, T_1^*] = 0 \Longleftrightarrow [T_d, T^*] = 0,$$

i.e., $T \in [(1,1)DN]$.

Commutativity properties. For operators $T \in [DN]$ (equivalently, $T \in [(1,1)DN]$), T_d is normal, hence if $[T_d, A] = 0$ for an operator $A \in B(\mathscr{H})$, then $[T_d^*, A] = 0$ (by the Fuglede theorem [9, 10]). Again, if $T \in [DN]$ is injective, then it is necessarily invertible and $T_d = T^{-1}$. Hence, T is normal and if $[T, A] = 0$ for some operator $A \in B(\mathscr{H})$, then $[T^*, A] = 0 = [T_d^*, A]$. The operator $T \in [DN]$ is in general not normal, and $TA = AT$ does not always imply $T^*A = AT^*$; however, $[T, A] = 0$ and $T \in [DN]$ implies $[T_d^*, A] = 0$, as the following argument shows. The operator $T \in [DN]$ has a direct sum representation $T = T_1 \oplus T_0$, T_1 invertible normal and T_0 nilpotent, and the Drazin inverse T_d has a direct sum representation $T_d = T_1^{-1} \oplus 0$. Letting A have the corresponding matrix representation $A = [A_{ij}]_{i,j=1}^2$, it is seen that $[T,A] = 0$ forces $A_{12} = A_{21} = 0$, and then

$$[T,A] = 0 \Longrightarrow [T_d,A] = 0 \Longleftrightarrow [T_1,A_{11}] = 0 \Longleftrightarrow [T_1^*,A_{11}] = 0 \Longleftrightarrow [T_d^*,A] = 0.$$

This conclusion does not extend to $T \in [DN]$ such that $TA = BT$ for some operators $A,B \in B(\mathscr{H})$.

EXAMPLE 2.7. Define operators $T,A,B \in B(\mathbb{C}^4)$ by

$$T = M \oplus N,\ A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix},\ B = \begin{pmatrix} B_1 & B_3 \\ 0 & 0 \end{pmatrix},$$

where $M,N,A_i (1 \leqslant i \leqslant 3), B_i (i = 1,3)$ are the $B(\mathbb{C}^2)$ operators

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ B_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},\ B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$T_d = M_d \oplus 0, [T_d,T^*] = 0 (\Longleftrightarrow T \in [DN]), \text{ and } TA = BT,$$

but

$$T^*A \neq BT^* \text{ and } T_d^*A \neq BT_d^*.$$

Additional hypotheses are required for $T \in [DN]$ and $TA = BT$ to imply $T_d^*A = BT_d^*$. The following theorem considers a couple of such hypotheses.

THEOREM 2.8. *Given operators $A,B,T \in B(\mathscr{H})$ such that $AT = TB$, if $T \in [DN]$ and either of the hypotheses $BT = TA$ and $(A-B)T_d = T_d(B-A)$ is satisfied, then $T_d^*A = BT_d^*$ and $T_d^*B = AT_d^*$.*

Proof. If $T \in [DN]$, then $T_dT^* = T^*T_d$, $T \in B(\mathscr{H}_1 \oplus \mathscr{H}_0)$ has a decomposition $T = T_1 \oplus T_0$, T_1 is invertible normal, T_0 is nilpotent, and $T_d = T_1^{-1} \oplus 0 \in B(\mathscr{H}_1 \oplus \mathscr{H}_0)$. Let $A,B \in B(\mathscr{H}_1 \oplus \mathscr{H}_0)$ have the matrix representations

$$A = [A_{ij}]_{i,j=1}^2 \text{ and } B = [B_{ij}]_{i,j=1}^2.$$

Then $AT = TB$ implies

$$A_{11}T_1 = T_1B_{11},\ A_{12}T_2 = T_1B_{12},\ A_{21}T_1 = T_2B_{21},\ A_{22}T_2 = T_2B_{22}.$$

Since $T_2^p = 0$ for some integer $p \geqslant 1$ and T_1 is invertible

$$A_{12}T_2 = T_1B_{12} \Longrightarrow T_1^pB_{12} = 0 \Longleftrightarrow B_{12} = 0 \text{ and}$$
$$A_{21}T_1 = T_2B_{21} \Longrightarrow A_{21}T_1^p = 0 \Longleftrightarrow A_{21} = 0.$$

(a) Assume to start with that $BT = TA$. Then

$$B_{21}T_1 = 0 \Longleftrightarrow B_{21} = 0 \Longrightarrow B = B_{11} \oplus B_{22},$$
$$T_1A_{12} = 0 \Longleftrightarrow A_{12} = 0 \Longrightarrow A = A_{11} \oplus A_{22},$$

$B_{11}T_1 = T_1A_{11}$ (and $B_{22}T_2 = T_2A_{22}$). Hence, since T_1 is normal,

$$(A_{11}+B_{11})T_1 = T_1(A_{11}+B_{11}) \Longleftrightarrow (A_{11}+B_{11})T_1^* = T_1^*(A_{11}+B_{11}) \text{ and}$$
$$(A_{11}-B_{11})T_1 = -T_1(A_{11}-B_{11}) \Longleftrightarrow (A_{11}-B_{11})T_1^* = -T_1^*(A_{11}-B_{11}).$$

Consequently,

$$A_{11}T_1^* = T_1^*B_{11} \Longleftrightarrow AT_d^* = T_d^*B \text{ and}$$
$$B_{11}T_1^* = T_1^*A_{11} \Longleftrightarrow BT_d^* = T_d^*A.$$

(b) If instead $(A-B)T_d = T_d(B-A)$, then

$$-B_{21}T_1^{-1} = 0 \Longleftrightarrow B_{21} = 0, \; -T_1^{-1}A_{12} = 0 \Longleftrightarrow A_{12} = 0$$

and

$$(A_{11}-B_{11})T_1^{-1} = -T_1^{-1}(A_{11}-B_{11}) \Longleftrightarrow (A_{11}-B_{11})T_1 = -T_1(A_{11}-B_{11})$$
$$\Longleftrightarrow (A_{11}-B_{11})T_1^* = -T_1^*(A_{11}-B_{11}).$$

Since we already have ($A_{11}T_1 = T_1B_{11} \Longleftrightarrow$) $A_{11}T_1^* = T_1^*B_{11}$, once again we have $AT_d^* = T_d^*B$ and $BT_d^* = T_d^*A$.

Theorem 2.8 is an improved version of [13, Theorem 4.4]: it tells us that hypothesis (4.1) and any one of the hypotheses (4.2) and (4.3) of [13, Theorem 4.4] guarantees the validity of the theorem.

If $S,T \in [(n,m)DN]$ and $\text{LCM}(n,m) = k$, then S_d^k and T_d^k are normal, hence $T_dA = AS_d$ for an operator $A \in B(\mathscr{H})$ implies $T_d^kA - AS_d^k = 0 = T_d^{*k}A - AS_d^{*k}$. This, however, does not guarantee $T_d^*A - AS_d^* = 0$ (contrary to the claim made in [13, Theorem 4.3]).

EXAMPLE 2.9. For operators $S,T \in B(\mathbb{C}^2)$, let

$$S = T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$T_d = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \; T_d^2T^{*3} = T^{*3}T_d^2$$

(so that $S = T \in [(2,3)DN]$). Since $T_d^{*3} = -I$, $T_d^{*3}A = AT_d^{*3}$ for all $A \in B(\mathbb{C}^2)$. If, however, we let $A = T$, then

$$[T_d, A] = 0 \;\text{ and } T_d^*A \neq AT_d^*.$$

Observe that $T_d = T^{-1}$, hence $TA = AT$ and $T^*A \neq AT^*$.

The following theorem considers operators $S,T \in [(n,m)DN]$ such that S,T are intertwined by a quasiaffinity (i.e., an injective operator with a dense range) to prove that S,T are similar to the perturbation of a normal operator by nilpotent operators.

THEOREM 2.10. *If $S,T \in B(\mathscr{H})$ are such that X is a quasiaffinity, S and T are $[(n,m)DN]$ operators and $SX = XT$, then there exist a normal operator N, nilpotent operators S_0 and T_0, and invertible operators $A,B \in B(\mathscr{H})$ such that $S = A^{-1}(N \oplus S_0)A$ and $T = B^{-1}(N \oplus T_0)B$.*

Proof. There exist positive integers p,q such that

$$S = S_1 \oplus S_0 \in B(S^q(\mathscr{H}) \oplus S^{-q}(0)),\ T = T_1 \oplus T_0 \in B(T^p(\mathscr{H}) \oplus T^{-p}(0)),$$

where S_0 is q nilpotent, T_0 is p nilpotent, $S_1 = A_1^{-1}N_1A_1$ and $T_1 = A_2^{-1}N_2A_2$ for some normal operators $N_1 \in B(S^q(\mathscr{H}))$ and $N_2 \in B(T^p(\mathscr{H}))$, and invertible operators $A_1 \in B(S^q(\mathscr{H}))$ and $A_2 \in B(T^p(\mathscr{H}))$. Define the invertible operators $A,B_1 \in B(\mathscr{H})$ by

$$A = A_1 \oplus I|_{S^{-q}(0)},\ B_1 = A_2 \oplus I|_{T^{-p}(0)}.$$

Then

$$A^{-1}(N_1 \oplus S_0)A_1X = XB_1^{-1}(N_2 \oplus T_0)B_1 \Longleftrightarrow (N_1 \oplus S_0)Y = Y(N_2 \oplus T_0),$$

where we have set $AXB_1^{-1} = Y$. Evidently, $Y : T^p(\mathscr{H}) \oplus T^{-p}(0) \longrightarrow S^q(\mathscr{H}) \oplus S^{-q}(0)$ is a quasiaffinity. Let Y have the matrix representation $Y = [Y_{ij}]_{i,j=1}^2$. Then, since N_1,N_2 are invertible and S_0,T_0 are nilpotent, a straightforward argument shows that

$$Y_{12} = Y_{21} = 0,\ Y = Y_{11} \oplus Y_{22},\ Y_{11} \text{ and } Y_{22} \text{ are quasiaffinities.}$$

Furthermore,

$$S_0Y_{22} = Y_{22}T_0$$

(so that indeed $p = q$) and

$$N_1Y_{11} = Y_{11}N_2 \Longleftrightarrow N_1^*Y_{11} = Y_{11}N_2^*.$$

But then N_1 and N_2 are unitarily equivalent normal operators, i.e., there exists a unitary U and a normal operator N such that $N_1 = N$ and $N_2 = U^*NU$. Now define the operator B by $B = UA_2 \oplus I|_{T^{-p}(0)}$. Then $S = A^{-1}(N \oplus S_0)A$ and $T = B^{-1}(N \oplus T_0)B$.

If $S,T \in [(n,m)DN]$, then $S \oplus T \in [(n,m)DN]$. This fails for upper triangular operator matrices (with a non-trivial entry in the $(1,2)$-place).

EXAMPLE 2.11. Consider operators $T,C \in B(\mathbb{C}^2)$ and $A \in B(\mathbb{C}^4)$ defined by $T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ (as in Example 2.9) and

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\ A = \begin{pmatrix} T & C \\ 0 & T \end{pmatrix}.$$

Then $T \in [(2,3)DN]$ and $A_d = \begin{pmatrix} T_d & X \\ 0 & T_d \end{pmatrix}$, where $X = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. A simple calculation shows that $A \notin [(2,3)DN]$.

If $S,T \in B(\mathscr{H})$ are $[(n,m)DN]$ operators such that S has Drazin index q and T has Drazin index p, then $S = S_1 \oplus S_0 \in B(S^q(\mathscr{H}) \oplus S^{-q}(0))$ and $T = T_1 \oplus T_0 \in B(T^p(\mathscr{H}) \oplus T^{-p}(0))$. Let $C : S^q(\mathscr{H}) \oplus S^{-q}(0) \longrightarrow T^p(\mathscr{H}) \oplus T^{-p}(0)$ have the matrix representation $C = [C_{ij}]_{i,j=1}^2$. Then the operator $A = \begin{pmatrix} T & C \\ 0 & S \end{pmatrix}$, is Drazin invertible with Drazin inverse $A_d = \begin{pmatrix} T_d & X \\ 0 & S_d \end{pmatrix}$, where X is the operator

$$\begin{aligned} X &= \left[\sum_{j=0}^{q-1} T_d^{j+2} C S^j\right](I - SS_d) + (I - TT_d)\left[\sum_{j=0}^{p-1} T^j C S_d^{j+2}\right] - T_d C S_d \\ &= \begin{pmatrix} -T_1^{-1} C_{11} S_1^{-1} & \sum_{j=0}^{q-1} T_1^{-j-2} C_{12} B_2^j \\ \sum_{j=0}^{p-1} T_2^j C_{21} S_1^{-j-2} & 0 \end{pmatrix} \end{aligned}$$

[3, 2.3.12 Theorem, Page 29]. The following theorem considers the case $n = m = p = q = 1$ to give a necessary and sufficient condition for $A \in [DN]$.

THEOREM 2.12. *Given operators $S,T \in B(\mathscr{H})$ such that $S,T \in [DN]$, S and T have Drazin index 1 and $C : S(\mathscr{H}) \oplus S^{-1}(0) \longrightarrow T(\mathscr{H}) \oplus T^{-1}(0)$ has the matrix representation $C = [C_{ij}]_{i,j=1}^2$, a necessary and sufficient condition for the operator $A \in B(\mathscr{H} \oplus \mathscr{H})$ to be a $[DN]$ operator is that $C = 0 \oplus C_{22}$.*

Proof. If S,T have Drazin index 1, then $S = S_1 \oplus 0$, $T = T_1 \oplus 0$, S_1 and T_1 are normal invertible and the operator X (above) has the form

$$X = \begin{pmatrix} -T_1^{-1} C_{11} S_1^{-1} & T_1^{-2} C_{12} \\ C_{21} S_1^{-2} & 0 \end{pmatrix}.$$

Given $S,T \in [DN]$, $A \in [DN]$ if and only if

$$\begin{aligned} [A_d, A^*] = 0 &\Longleftrightarrow \begin{pmatrix} T_d T^* + XC^* & XS^* \\ S_d C^* & S_d S^* \end{pmatrix} = \begin{pmatrix} T^* T_d & T^* X \\ C^* T_d & C^* X + S^* S_d \end{pmatrix} \\ &\Longleftrightarrow S_d C^* = C^* T_d,\ XS^* = T^* X,\ XC^* = 0 = C^* X. \end{aligned}$$

The equality

$$\begin{aligned} S_d C^* = C^* T_d &\Longleftrightarrow S_1^{-1} C_{11}^* = C_{11} T_1^{-1},\ S_1^{-1} C_{21}^* = 0 = C_{12}^* T_1^{-1} \\ &\Longleftrightarrow S_1^{-1} C_{11}^* = C_{11}^* T_1^{-1},\ C_{12} = C_{21} = 0; \end{aligned}$$

$$\begin{aligned} XS^* = T^* X &\Longleftrightarrow T_1^{-1} C_{11} S_1^{-1} S_1^* = T_1^* T_1^{-1} C_{11} S_1^{-1},\ C_{21} S_1^{-2} S_1^* = 0 = T_1^* T_1^{-2} C_{12} \\ &\Longleftrightarrow C_{12} = C_{21} = 0,\ T_1^{-1} C_{11} S_1^* S_1^{-1} = T_1^{-1} T_1^* C_{11} S_1^{-1} \\ &\Longleftrightarrow C_{12} = C_{21} = 0,\ C_{11} S_1^* = T_{11}^* C_{11}. \end{aligned}$$

Considering finally the equalities $XC^*=0=C^*X$, if $C_{12}=C_{21}=0$ and $C_{11}^*T_1=S_1C_{11}^*$, then

$$\begin{aligned} XC^*=0=C^*X &\iff T^{-1}C_{11}S_1^{-1}C_{11}^*=0=C_{11}^*T_1^{-1}C_{11}S_1^{-1} \\ &\iff C_{11}S_1^{-1}C_{11}^*=0=C_{11}^*T_1^{-1}C_{11} \\ &\iff T_1^{-1}C_{11}C_{11}^*=0=C_{11}^*C_{11}S_1^{-1} \\ &\iff C_{11}=0. \end{aligned}$$

It being straightforward to verify that $XC^*=0=C^*X$ and $C_{11}=0$ implies $C_{12}=C_{21}=0$ and $C_{11}^*T_1=S_1C_{11}^*$, it follows that a necessary and sufficient condition for $A\in[DN]$ is that $C=0\oplus C_{22}\in B(S(\mathscr{H})\oplus S^{-1}(0),T(\mathscr{H})\oplus T^{-1}(0))$.

The proof above, in particular our consideration of the equation $XS^*=T^*X$, exploited the fact that S_1 and T_1 are normal. Since this no longer holds for $S,T\in[(n,m)DN]$ for general $n,m>1$, the necessity of the condition $C=0\oplus C_{22}$ is not clear (for the general case). The following theorem, however, shows that this condition is sufficient. Let S have Drazin index q, T have Drazin index p, and let $C\in B(S^q(\mathscr{H})\oplus S^{-q}(0),T^p(\mathscr{H})\oplus T^{-p}(0))$ have the direct sum decomposition $C=0\oplus C_{22}$.

THEOREM 2.13. *If $S,T\in[(n,m)DN]$, then $A\in[(n,m)DN]$.*

Proof. The hypothesis $C=0\oplus C_{22}$ forces $X=0$, and then $A_d^n=T_d^n\oplus S_d^n$. Define the operator L by

$$L=0\oplus\sum_{j=0}^{m-1}S_2^{*j}C_{22}^*T_2^{*(m-1-j)}.$$

Then

$$\begin{aligned} A_d^nA^{*m} &= \begin{pmatrix} T_d^nT^{*m} & 0 \\ S_d^nL & S_d^nS^{*m} \end{pmatrix} = T_d^nT^{*m}\oplus S_d^nS^{*m}=T^{*m}T_d^n\oplus S^{*m}S_d^n \\ &= \begin{pmatrix} T^{*m}T_d^n & 0 \\ LT_d^n & S^{*m}S_d^n \end{pmatrix} = A^{*m}A_d^n, \end{aligned}$$

i.e., $A\in[(n,m)DN]$.

Theorem 2.13 is a generalized version of [13, Theorem 2.7] (which contrary to the claim made by the authors does not prove the necessity of the stated conditions).

Acknowledgement. The second named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2019R1F1A1057574).

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(Received October 30, 2019)

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