Operators and Matrices
Volume 14, Number 3 (2020), 601–634 doi:10.7153/oam-2020-14-40

SCHRÖDINGER OPERATORS WITH POTENTIAL WAVEGUIDES ON PERIODIC GRAPHS

OLAF POST AND NATALIA SABUROVA *

Abstract. We consider discrete Schrödinger operators with periodic potentials on periodic graphs perturbed by guided positive potentials, which are periodic in some directions and finitely supported in other ones. The spectrum of the unperturbed operator is a union of a finite number of non-degenerate bands and eigenvalues of infinite multiplicity. It is known that the spectrum of the perturbed operator consists of the spectrum of the unperturbed one and the additional guided spectrum, which is also a union of a finite number of bands. We estimate the positions of the guided bands *in gaps* of the unperturbed operator in terms of eigenvalues of Schrödinger operators on some finite graphs. We also determine sufficient conditions for the guided potentials under which the guided bands do not appear in gaps of the unperturbed problem.

1. Introduction

Discrete Laplace and Schrödinger operators on periodic graphs have attracted a lot of attention due to their applications to the study of electronic properties of real crystalline structures, see, e.g., [11], [12], [34] and the survey [7]. However, the arrangement of atoms or molecules in most crystalline materials is not perfect. The regular patterns are interrupted by crystalline defects. The defects may have different dimensions: point, linear, volume defects. These defects are the most important features for engineering material and are manipulated to control its behavior. In particular they allow one to obtain conductivity of the material for those frequencies (energies) at which it was not in purely periodic structure. Such effects have a lot of applications, for example waveguides in photonic crystal structures (see e.g. in [10], [24], [25] and references therein).

It is well known that the spectrum of the Laplacian on periodic discrete graphs has band structure with a finite number of flat bands (eigenvalues of infinite multiplicity) [16], [20], [26], [36]. The spectrum of the Schrödinger operators with finitely supported potentials on periodic graphs consists of the spectrum of the Laplacian and a finite number of eigenvalues of finite multiplicity. The discrete spectrum of Schrödinger operators with finitely supported potentials on the d-dimensional lattice $\mathbb{Z}^d$, the simplest periodic graph, was studied in [15], [21]. The absence of eigenvalues embedded in the essential spectrum of the operators was proved in [3] for some kind of graphs including

Mathematics subject classification (2010): 47A10, 47A58, 47A75.
Keywords and phrases: Discrete Schrödinger operator, periodic graph, guided waves, spectral gaps.
This paper was written during Natalia Saburova's stay at Trier University, Germany. She is grateful to the university for the hospitality. Her study was supported by the grant No. 1.12792.2018/12.2 of the Ministry of Science and Higher Education of the Russian Federation and the German Academic Exchange Service (DAAD) and by the RFBR grant No. 19-01-00094.
* Corresponding author.

the square, triangular, diamond and Kagome lattices. The inverse scattering problem for discrete Schrödinger operators with finitely supported potentials was considered in [23] on the lattice $\mathbb{Z}^d$ and in [2] on the hexagonal lattice, where a reconstruction procedure for the potential from the scattering matrix for all energies was derived.

Discrete Schrödinger operators with periodic potentials on periodic graphs perturbed by *guided* non-positive potentials, which are periodic in some directions and finitely supported in other ones were considered in [28]. For example, on the lattice $\mathbb{Z}^2$ the support of a (non-trivial) guided potential is a strip. It was shown that the spectrum of the perturbed Schrödinger operator consists of the spectrum of the unperturbed one and the so-called *guided* spectrum. The additional guided spectrum is a union of a finite number of bands and the corresponding wave-functions are located along the support of the guided potentials and decrease in perpendicular directions. The authors studied the guided spectrum *below* the spectrum of the unperturbed operator. They estimated the position of guided bands and their lengths in terms of some geometric parameters of graphs. They determined asymptotics of the guided spectrum for large guided potentials and showed that the possible number of guided bands, their length and positions can be rather arbitrary for some specific potentials. But in [28] there are no results about the guided spectrum which may appear *in gaps* of the unperturbed problem. It seems that, in general, this is a much more difficult problem, see also the discussion in [14] about similar problems for continuous models.

We should also mention a series of papers [31] – [33], where the author considered discrete periodic operators with different kinds of defects (localized, parallel, perpendicular) and derived an algorithm of finding the spectrum based on algebraic operations on the finite matrix-valued functions and integration. Note that the structure with an infinite line defect embedded in the square lattice was considered in [35]. In this case the dispersion relations for defect modes generated by a line defect can be computed in explicit form.

In this paper we consider discrete Schrödinger operators with periodic potentials on periodic graphs perturbed by guided positive potentials. We estimate the positions of the guided bands lying *in gaps* of the unperturbed operator in terms of eigenvalues of Schrödinger operators on some finite graphs. We also determine sufficient conditions for the guided potentials under which the guided bands do not appear in gaps of the unperturbed problem. It should be mentioned that the obtained results also hold true in the case of finitely supported potentials and even in this simplest case the results are new to the best of our knowledge.

1.1. Schrödinger operators with periodic potentials

Let $G=(\mathscr{V},\mathscr{E})$ be a connected infinite graph, possibly having loops and multiple edges, and embedded[1] into the space $\mathbb{R}^{\widetilde{d}}$. Here $\mathscr{V}$ is the set of its vertices and $\mathscr{E}$ is the set of its unoriented edges. From the set $\mathscr{E}$ we construct the set $\mathscr{A}$ of oriented edges

[1]Note that the embedding of G into $\mathbb{R}^{\widetilde{d}}$ plays no role in the analysis of Δ, H_0, H etc. as only the incidence structure of the graph matters. Nevertheless, the embedding simplifies some notation, and allows e.g. a simple definition of edge indices, see Section 3.

by considering each edge in $\mathscr{E}$ to have two orientations. An edge starting at a vertex u and ending at a vertex v from $\mathscr{V}$ will be denoted by the ordered pair $(u,v)\in\mathscr{A}$.[2] We define the degree $\varkappa_v$ of the vertex $v\in\mathscr{V}$ as the number of all edges from $\mathscr{A}$ starting at v.

Let $\widetilde{\Gamma}$ be a lattice of rank $\widetilde{d}$ in $\mathbb{R}^{\widetilde{d}}$ with basis $a_1,\ldots,a_{\widetilde{d}}$, i.e.,

$$\widetilde{\Gamma}=\left\{a: a=\sum_{s=1}^{\widetilde{d}} n_s a_s,\ n_s\in\mathbb{Z},\ s\in\mathbb{N}_{\widetilde{d}}\right\},\qquad \mathbb{N}_{\widetilde{d}}=\{1,\ldots,\widetilde{d}\},$$

and let

$$\Omega=\left\{x\in\mathbb{R}^{\widetilde{d}}: x=\sum_{s=1}^{\widetilde{d}} t_s a_s,\ 0\leqslant t_s<1,\ s\in\mathbb{N}_{\widetilde{d}}\right\} \tag{1.1}$$

be the *fundamental cell* of the lattice $\widetilde{\Gamma}$. We define an equivalence relation on $\mathbb{R}^{\widetilde{d}}$ as follows:

$$x\equiv y \pmod{\widetilde{\Gamma}}\quad\Leftrightarrow\quad x-y\in\widetilde{\Gamma}\qquad \forall x,y\in\mathbb{R}^{\widetilde{d}}.$$

We consider *locally finite* $\widetilde{\Gamma}$*-periodic graphs* G, i.e., graphs satisfying the following conditions:

1) $G=G+a$ for any $a\in\widetilde{\Gamma}$;

2) the *quotient graph* $G_*=(\mathscr{V}_*,\mathscr{E}_*)=G/\widetilde{\Gamma}$ is finite.

The basis vectors $a_1,\ldots,a_{\widetilde{d}}$ of the lattice $\widetilde{\Gamma}$ are called the *periods* of G. Below the coordinates of all vectors of $\mathbb{R}^{\widetilde{d}}$ will be given with respect to the basis $a_1,\ldots,a_{\widetilde{d}}$. The quotient graph G_* is a graph on the $\widetilde{d}$-dimensional torus $\mathbb{R}^{\widetilde{d}}/\widetilde{\Gamma}$.

Edges of the periodic graph G connecting the vertices from a fundamental cell Ω with the vertices outside Ω will be called *bridges*. Bridges always exist and provide the connectivity of the periodic graph. Edges of the quotient graph corresponding to bridges of the periodic one will be also called bridges.

Let $\ell^2(\mathscr{V})$ be the Hilbert space of all functions $f\colon\mathscr{V}\to\mathbb{C}$ equipped with the norm

$$\|f\|^2_{\ell^2(\mathscr{V})}=\sum_{v\in\mathscr{V}}|f(v)|^2<\infty.$$

We consider a discrete Schrödinger operator H_0 with a periodic potential W on $f\in\ell^2(\mathscr{V})$ as *an unperturbed operator* defined by

$$H_0=\Delta+W, \tag{1.2}$$

where Δ is the discrete combinatorial Laplace operator given by

$$\big(\Delta f\big)(v)=\sum_{(v,u)\in\mathscr{A}}\big(f(v)-f(u)\big),\qquad \forall f\in\ell^2(\mathscr{V}),\qquad \forall v\in\mathscr{V}, \tag{1.3}$$

[2]This is a slight abuse of notation as the two vertices u and v do not determine *multiple* edges. Nevertheless, this abuse allows us to avoid introducing initial and terminal functions associating to an edge its initial and terminal vertex.

and the sum in (1.3) is taken over all oriented edges starting at the vertex $v \in \mathscr{V}$. The potential W is real-valued and $\widetilde{\Gamma}$-periodic, i.e.,

$$(Wf)(v) = W(v)f(v), \qquad W(v+a) = W(v), \qquad \forall (v,a) \in \mathscr{V} \times \widetilde{\Gamma}.$$

It is known that H_0 is self-adjoint and has the following decomposition into a constant fiber direct integral for some unitary operator $\widetilde{U}\colon \ell^2(\mathscr{V}) \to \widetilde{\mathscr{H}}$ [26]:

$$\widetilde{\mathscr{H}} = \int_{\mathbb{T}^{\widetilde{d}}}^{\oplus} \ell^2(\mathscr{V}_*) \frac{d\vartheta}{(2\pi)^{\widetilde{d}}}, \qquad \widetilde{U} H_0 \widetilde{U}^{-1} = \int_{\mathbb{T}^{\widetilde{d}}}^{\oplus} \widetilde{H}_0(\vartheta) \frac{d\vartheta}{(2\pi)^{\widetilde{d}}}, \qquad \widetilde{H}_0(\vartheta) = \widetilde{\Delta}(\vartheta) + W, \tag{1.4}$$

where $\mathbb{T}^{\widetilde{d}} = \mathbb{R}^{\widetilde{d}}/(2\pi\mathbb{Z})^{\widetilde{d}}$, the fiber Schrödinger operator $\widetilde{H}_0(\vartheta)$ acts on the fiber space $\ell^2(\mathscr{V}_*)$, the fiber Laplacian $\widetilde{\Delta}(\vartheta)$ is given by

$$\big(\widetilde{\Delta}(\vartheta) f\big)(v) = \sum_{\mathbf{e}=(v,u) \in \mathscr{A}_*} \Big(f(v) - e^{i\langle \widetilde{\tau}(\mathbf{e}), \vartheta \rangle} f(u) \Big), \qquad f \in \ell^2(\mathscr{V}_*). \tag{1.5}$$

Here $\widetilde{\tau}(\mathbf{e}) \in \mathbb{Z}^{\widetilde{d}}$ is the index of the edge $\mathbf{e} \in \mathscr{A}_*$ of the quotient graph $G_* = (\mathscr{V}_*, \mathscr{E}_*)$ defined by (3.5) as $n = 0$; $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{\widetilde{d}}$. Note that $\widetilde{\Delta}(0)$ is the Laplacian on the quotient graph G_*.

For a self-adjoint operator A, $\sigma(A)$, $\sigma_{\mathrm{ess}}(A)$, $\sigma_{\mathrm{ac}}(A)$, $\sigma_{\mathrm{p}}(A)$, and $\sigma_{\mathrm{fb}}(A)$ denote its spectrum, essential spectrum, absolutely continuous spectrum, point spectrum (eigenvalues of finite multiplicity), and the set of all its flat bands (eigenvalues of infinite multiplicity), respectively.

Let $\#M$ denote the number of elements in a set M. Each fiber operator $\widetilde{H}_0(\vartheta)$, $\vartheta \in \mathbb{T}^{\widetilde{d}}$, has p eigenvalues $\lambda_k(\vartheta)$, $k \in \mathbb{N}_p := \{1, 2, \ldots, p\}$, $p = \#\mathscr{V}_*$, which are labeled (counting multiplicities) by

$$\lambda_1(\vartheta) \leqslant \ldots \leqslant \lambda_p(\vartheta).$$

Each $\lambda_k(\cdot)$, $k \in \mathbb{N}_p$, is a real and piecewise analytic function on the torus $\mathbb{T}^{\widetilde{d}}$ and creates the *spectral band* $\sigma_k \equiv \sigma_k(H_0)$ given by

$$\sigma_k \equiv \sigma_k(H_0) = [\lambda_k^-, \lambda_k^+] = \lambda_k(\mathbb{T}^{\widetilde{d}}). \tag{1.6}$$

Note that if $\lambda_k(\cdot) = \Lambda_k = \mathrm{const}$ on some subset of $\mathbb{T}^{\widetilde{d}}$ of positive Lebesgue measure, then Λ_k is an eigenvalue of H_0 on G of infinite multiplicity. We call $\{\Lambda_k\}$ a *flat band*. Thus, the spectrum of the Schrödinger operator H_0 on the periodic graph G has the form

$$\sigma(H_0) = \bigcup_{\vartheta \in \mathbb{T}^{\widetilde{d}}} \sigma\big(\widetilde{H}_0(\vartheta)\big) = \bigcup_{k=1}^{p} \sigma_k(H_0) = \sigma_{\mathrm{ac}}(H_0) \cup \sigma_{\mathrm{fb}}(H_0). \tag{1.7}$$

A non-empty interval $I \subset \mathbb{R}$ such that $I \cap \sigma(H_0) = \varnothing$ will be called a *spectral gap* (or just a *gap*) of H_0. Note that we do not assume that the gap I is maximal, i.e., if $I \subset \widetilde{I}$, then we do not make a statement about the existence of the spectrum inside $\widetilde{I}$.

1.2. Schrödinger operators with guided potentials

Let d be an integer satisfying $0 \leqslant d < \widetilde{d}$. We introduce two sublattices Γ and Γ_0 of the lattice $\widetilde{\Gamma}$:

- Γ is a lattice of rank d with basis $a_1, \ldots, a_d$;
- Γ_0 is a lattice of rank $d_0 = \widetilde{d} - d$ with basis $a_{d+1}, \ldots, a_{\widetilde{d}}$.

We consider a family of *guided Schrödinger operators* H_t, $t > 0$, on the periodic graph $G = (\mathscr{V}, \mathscr{E})$ given by

$$H_t = H_0 + tQ, \qquad \big(Qf\big)(v) = Q(v)f(v), \qquad f \in \ell^2(\mathscr{V}),$$

where H_0 is the unperturbed Schrödinger operator given by (1.2), and $Q \geqslant 0$ is a *guided potential* if it fulfills the following properties:

1) the support of Q satisfies:

$$\operatorname{supp} Q \subset \mathbb{R}^d \times [0,1)^{d_0}, \qquad d_0 = \widetilde{d} - d;$$

2) Q is Γ-periodic, i.e.,

$$Q(v+a) = Q(v), \qquad \forall\, (v,a) \in \mathscr{V} \times \Gamma.$$

REMARK 1.1. i) Recall that the coordinates of all vectors of $\mathbb{R}^{\widetilde{d}}$ are given with respect to the basis $a_1, \ldots, a_{\widetilde{d}}$ of the lattice $\widetilde{\Gamma}$ (the periods of the graph G). In other words, the guided potential Q is periodic in the directions $a_1, \ldots, a_d$ and finitely supported in the directions $a_{d+1}, \ldots, a_{\widetilde{d}}$.

ii) Note that one might need to choose a sublattice of the original lattice $\widetilde{\Gamma}$ in order to make sure that Q is supported in $\mathbb{R}^d \times [0,1)^{d_0}$ and periodic with respect to Γ.

iii) If $d = 0$, then the potential Q has finite support containing in the fundamental cell $\Omega = [0,1)^{\widetilde{d}}$.

Let $d > 0$. We define the infinite *fundamental graph* $C = G/\Gamma$ of the $\widetilde{\Gamma}$-periodic graph G, which is a graph on the cylinder $\mathbb{R}^{\widetilde{d}}/\Gamma$. We also call the fundamental graph C *a discrete cylinder* or just *a cylinder*. The cylinder $C = (\mathscr{V}_{\rm c}, \mathscr{E}_{\rm c})$ has the vertex set $\mathscr{V}_{\rm c} = \mathscr{V}/\Gamma$, the set $\mathscr{E}_{\rm c} = \mathscr{E}/\Gamma$ of unoriented edges and the set $\mathscr{A}_{\rm c} = \mathscr{A}/\Gamma$ of oriented edges. Note that C is a Γ_0-periodic graph with periods $a_{d+1}, \ldots, a_{\widetilde{d}}$.

We identify the vertices of the cylinder C with the vertices of the periodic graph G from the strip $S = [0,1)^d \times \mathbb{R}^{d_0}$. We denote this infinite vertex set by the same symbol $\mathscr{V}_{\rm c}$:

$$\mathscr{V}_{\rm c} = \mathscr{V} \cap S, \qquad S = [0,1)^d \times \mathbb{R}^{d_0}, \qquad d_0 = \widetilde{d} - d.$$

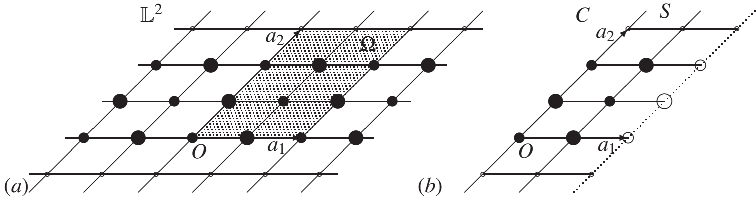

Figure 1: a) The square lattice $\mathbb{L}^2$ ($\widetilde{d}=2$); the fundamental cell Ω is shaded; the big black vertices are support of the guided potential Q ($d=1$); b) the discrete cylinder $C=\mathbb{L}^2/\Gamma$ (the vertices on the left and right side of the strip S are identified).

EXAMPLE 1.2. For the square lattice $\mathbb{L}^2$ with periods a_1, a_2 (see Fig. 1a), the discrete cylinder $C=\mathbb{L}^2/\Gamma=(\mathscr{V}_c,\mathscr{E}_c)$ is shown in Fig. 1b, Γ is the lattice generated by the vector a_1. The support of the guided potential Q is shown by big black vertices of the graph; Q is periodic in the direction a_1 and has finite support in the direction a_2.

For $d>0$ we define the torus $\mathbb{T}^d=\mathbb{R}^d/(2\pi\mathbb{Z})^d$ and describe the basic spectral properties of the guided Schrödinger operators [28].

PROPOSITION 1.3. *i) The guided Schrödinger operator* $H_t=H_0+tQ$, $t>0$, *has the following decomposition into a constant fiber direct integral for some unitary operator* $U\colon \ell^2(\mathscr{V})\to\mathscr{H}$*:*

$$\mathscr{H}=\int_{\mathbb{T}^d}^{\oplus}\ell^2(\mathscr{V}_c)\frac{d\vartheta}{(2\pi)^d},\qquad UH_tU^{-1}=\int_{\mathbb{T}^d}^{\oplus}H_t(\vartheta)\frac{d\vartheta}{(2\pi)^d},\qquad H_t(\vartheta)=H_0(\vartheta)+tQ, \tag{1.8}$$

where the fiber Schrödinger operator $H_t(\vartheta)$ *acts on the fiber space* $\ell^2(\mathscr{V}_c)$,

$$H_0(\vartheta)=\Delta(\vartheta)+W,\qquad \vartheta\in\mathbb{T}^d,$$

is the fiber operator for H_0*, the fiber Laplacian* $\Delta(\vartheta)$ *is given by*

$$\big(\Delta(\vartheta)f\big)(v)=\sum_{\mathbf{e}=(v,u)\in\mathscr{A}_c}\Big(f(v)-e^{i\langle\tau(\mathbf{e}),\vartheta\rangle}f(u)\Big),\qquad f\in\ell^2(\mathscr{V}_c), \tag{1.9}$$

and the potential Q *on* $\ell^2(\mathscr{V}_c)$ *has finite support. Here* $\tau(\mathbf{e})\in\mathbb{Z}^d$ *is the vector consisting of the first* d *components of the index* $\widetilde{\tau}(\mathbf{e})\in\mathbb{Z}^{\widetilde{d}}$ *of the edge* $\mathbf{e}\in\mathscr{A}_c$*, defined by (3.2), (3.4);* $\langle\cdot,\cdot\rangle$ *is the inner product in* $\mathbb{R}^d$.

ii) For each $(\vartheta,t)\in\mathbb{T}^d\times\mathbb{R}_{>0}$ *the spectrum of the fiber operator* $H_t(\vartheta)$ *has the form*

$$\sigma\big(H_t(\vartheta)\big)=\sigma_{ac}\big(H_t(\vartheta)\big)\cup\sigma_{fb}\big(H_t(\vartheta)\big)\cup\sigma_p\big(H_t(\vartheta)\big),$$
$$\sigma_{ac}\big(H_t(\vartheta)\big)=\sigma_{ac}\big(H_0(\vartheta)\big),\qquad \sigma_{fb}\big(H_t(\vartheta)\big)=\sigma_{fb}\big(H_0(\vartheta)\big),$$

$\sigma_p\big(H_t(\vartheta)\big)$ *is the set of all eigenvalues* $\lambda_1(\vartheta,t)\leqslant\lambda_2(\vartheta,t)\leqslant\ldots$ *of* $H_t(\vartheta)$ *with finite multiplicity.*

REMARK 1.4. i) Since $\sigma(H_0) = \bigcup_{\vartheta\in\mathbb{T}^d} \sigma\big(H_0(\vartheta)\big)$ and

$$\sigma\big(H_0(\vartheta)\big) = \sigma_{\rm ess}\big(H_0(\vartheta)\big) = \sigma_{\rm ess}\big(H_t(\vartheta)\big),$$

spectral gaps of the unperturbed Schrödinger operator H_0 remain spectral gaps in the essential spectrum of $H_t(\vartheta)$ for all $(\vartheta,t)\in\mathbb{T}^d\times\mathbb{R}_{>0}$.

ii) The eigenvalues $\lambda_m(\vartheta,t)$, $m=1,2,\ldots$, of the perturbed fiber operator $H_t(\vartheta) = H_0(\vartheta)+tQ$ may lie above the spectrum of the unperturbed operator $H_0(\vartheta)$, in the spectrum of $H_0(\vartheta)$ and in gaps of $H_0(\vartheta)$. Since the perturbation Q in (1.8) has finite support of size $\rho = \#\operatorname{supp} Q\restriction_{\mathscr{V}_c}$, each gap of $H_0(\vartheta)$ contains at most ρ eigenvalues (counting multiplicities) of $H_t(\vartheta)$ (see, e.g., [4]).

iii) The fiber Laplacian $\Delta(\vartheta)$, $\vartheta\in\mathbb{T}^d$, can be considered as a magnetic Laplacian with a periodic magnetic potential $\alpha(\mathbf{e}) = \langle\tau(\mathbf{e}),\vartheta\rangle$, $\mathbf{e}\in\mathscr{A}_c$, on the cylinder C (see e.g. [9, 17, 18, 29]).

Proposition 1.3 and standard arguments (see Theorem XIII.85 in [38]) describe the spectrum of the guided Schrödinger operator H_t if $d>0$. Since $H_t(\vartheta)$ is self-adjoint and analytic in $\vartheta\in\mathbb{T}^d$, each $\lambda_m(\cdot,t)$ is a real and piecewise analytic function on the torus $\mathbb{T}^d$ and creates the *guided band* $\mathfrak{s}_m(H_t)$ given by

$$\mathfrak{s}_m(H_t) = [\lambda_m^-(t),\lambda_m^+(t)] = \lambda_m(\mathbb{T}^d,t), \qquad m=1,2,\ldots.$$

Thus, the spectrum of the guided Schrödinger operator H_t on the graph G has the form

$$\sigma(H_t) = \bigcup_{\vartheta\in\mathbb{T}^d} \sigma\big(H_t(\vartheta)\big) = \sigma(H_0)\cup\mathfrak{s}(H_t),$$

where $\sigma(H_0)$ is the spectrum of the unperturbed Schrödinger operator H_0 given by (1.7), and

$$\mathfrak{s}(H_t) = \bigcup_{\vartheta\in\mathbb{T}^d} \sigma_{\rm p}\big(H_t(\vartheta)\big) = \bigcup_m \mathfrak{s}_m(H_t) = \mathfrak{s}_{\rm ac}(H_t)\cup\mathfrak{s}_{\rm fb}(H_t). \tag{1.10}$$

Here $\mathfrak{s}_{\rm ac}(H_t)$ and $\mathfrak{s}_{\rm fb}(H_t)$ are the absolutely continuous part and the flat band part of the guided spectrum $\mathfrak{s}(H_t)$, respectively. The guided spectrum $\mathfrak{s}(H_t)$ may partly lie above the spectrum of the unperturbed operator H_0, in the spectrum of H_0 and in gaps of H_0. We will study the guided spectrum $\mathfrak{s}(H_t)$ lying *in gaps* of H_0.

The spectrum of $H_t(\vartheta)$ inside the gaps of H_0 consists of a finite number of isolated eigenvalues $\lambda_{m_1}(\vartheta,t),\lambda_{m_2}(\vartheta,t),\ldots$ of finite multiplicity which are piecewise analytic functions in the parameter t. These functions $\lambda_{m_j}(\vartheta,\cdot)$, $j=1,2,\ldots$, are called *eigenvalue branches* of the operator family $H_t(\vartheta)$, $t\geqslant 0$. Similarly we define *guided band branches* $\mathfrak{s}_{m_j}(t)$ as guided bands $\mathfrak{s}_{m_j}(H_t)$ considering as functions of t:

$$\mathfrak{s}_{m_j}(t) \equiv \mathfrak{s}_{m_j}(H_t) = \lambda_{m_j}(\mathbb{T}^d,t), \qquad j=1,2,\ldots. \tag{1.11}$$

REMARK 1.5. If $d=0$, then for each $t>0$ the spectrum of the Schrödinger operator $H_t = H_0+tQ$ with finitely supported potentials Q on the graph G has the form

$$\sigma(H_t) = \sigma(H_0)\cup\sigma_{\rm p}(H_t),$$

where $\sigma(H_0)$ is the spectrum of the unperturbed Schrödinger operator H_0 given by (1.7), and $\sigma_p(H_t)$ is the set of all eigenvalues $\lambda_1(t) \leqslant \lambda_2(t) \leqslant \ldots$ of H_t with finite multiplicity.

1.3. Eigenvalue bracketing for periodic Schrödinger operators

Since our goal is to study the guided spectrum of the perturbed Schrödinger operator H_t lying *in* gaps of the unperturbed operator H_0, we formulate some sufficient conditions for H_0 to have spectral gaps (see also [9] and references therein). In order to do this we estimate the position of the bands $\sigma_k(H_0)$, $k \in \mathbb{N}_p$, of H_0 defined by (1.6) in terms of eigenvalues of two Schrödinger operators on finite graphs:

- the Schrödinger operator H_0^- on some subgraph G_*^- of the quotient graph G_*;
- the Dirichlet Schrödinger operator H_0^+ on G_*.

In order to define these operators we represent the quotient graph $G_* = (\mathscr{V}_*, \mathscr{E}_*)$ as a union of two graphs with the same vertex set $\mathscr{V}_*$:

$$G_* = G_{\mathfrak{b}} \cup G_*^-, \qquad G_{\mathfrak{b}} = (\mathscr{V}_*, \mathscr{B}_*), \qquad G_*^- = (\mathscr{V}_*, \mathscr{E}_* \setminus \mathscr{B}_*), \tag{1.12}$$

where $\mathscr{B}_*$ is the set of all bridges of the quotient graph G_*. In other words, the graph G_*^- is obtained from G_* by deleting all bridges and preserving the vertex set $\mathscr{V}_*$. Let $\mathscr{V}_{\mathfrak{b}} \subset \mathscr{V}_*$ be a set of the quotient graph vertices such that each bridge from $\mathscr{B}_*$ is incident to at least one vertex of the set $\mathscr{V}_{\mathfrak{b}}$. For each $t \geqslant 0$ we consider two operators:

- $H_t^- = \Delta^- + W + tQ$ is the Schrödinger operator on the graph G_*^-, where Δ^- is the Laplacian defined by (1.3) on the graph G_*^-;
- H_t^+ is the Schrödinger operator $H_t = \Delta + W + tQ$ on the quotient graph G_* with Dirichlet boundary conditions $f\restriction_{\mathscr{V}_{\mathfrak{b}}} = 0$, where $f\restriction_{\mathscr{V}_{\mathfrak{b}}}$ is the restriction of $f \in \ell^2(\mathscr{V}_*)$ onto $\mathscr{V}_{\mathfrak{b}}$.

We label the eigenvalues of the operators H_t^- and H_t^+ in non-decreasing order (counting multiplicities), respectively,

$$\mu_1^-(t) \leqslant \ldots \leqslant \mu_p^-(t), \qquad p = \#\mathscr{V}_*,$$

and

$$\mu_1^+(t) \leqslant \ldots \leqslant \mu_{p-r}^+(t), \qquad r = \#\mathscr{V}_{\mathfrak{b}}, \qquad r \geqslant 1.$$

Since $Q \geqslant 0$, $\mu_j^\pm(t)$ are non-decreasing functions of $t \geqslant 0$.

Combining the results from [27] and [30], we obtain the following statement:

PROPOSITION 1.6. *i) Each band $\sigma_k(H_0)$, $k \in \mathbb{N}_p$, of the unperturbed Schrödinger operator H_0 defined by (1.6) satisfies*

$$\begin{aligned} &\sigma_k(H_0) \subset \big[\mu_k^-, \mu_k^+\big], && k = 1, \ldots, p-r, \\ &\sigma_k(H_0) \subset \big[\mu_k^-, \mu_k^- + 2\mathfrak{b}_+\big], && k = p-r+1, \ldots, p, \end{aligned} \tag{1.13}$$

where

$$\mu_k^\pm = \mu_k^\pm(0), \qquad \mathfrak{b}_+ = \max_{v \in \mathscr{V}_*} \mathfrak{b}_v,$$

and $\mathfrak{b}_v$ *is the number of bridges of* G_* *incident to the vertex* $v \in \mathscr{V}_*$.

ii) If $\mu_k^+ < \mu_{k+1}^-$ *for some* $k = 1, \dots, p-r$*, then the interval* $(\mu_k^+, \mu_{k+1}^-) \neq \varnothing$ *is a gap of* H_0.

REMARK 1.7. i) The vertex set $\mathscr{V}_\mathfrak{b}$ is not uniquely defined. In order to get a better localization of the spectrum in (1.13) we have to choose $\mathscr{V}_\mathfrak{b}$ as small as possible.

ii) We exclude the trivial case when $\mathscr{V}_\mathfrak{b} = \mathscr{V}_*$. In this case the spectrum of the operator H_t^+ is empty for any $t \geqslant 0$.

iii) In [9] the authors also obtained a similar localization of the bands $\sigma_k(\Delta)$, $k \in \mathbb{N}_p$, for the Laplacian Δ on periodic graphs, also for other graph weights.

From now on we assume that $\mu_k^+ < \mu_{k+1}^-$ for some $k \in \mathbb{N}_{p-r}$, i.e., the interval

$$I_k = (\mu_k^+, \mu_{k+1}^-) \neq \varnothing \quad \text{for some} \quad k \in \mathbb{N}_{p-r} \tag{1.14}$$

is a spectral gap of the unperturbed Schrödinger operator H_0.

The paper is organized as follows. In Section 2 we formulate our main results:

- a localization of the guided band branches (1.11) of the family of the guided Schrödinger operators $H_t = H_0 + tQ$, $t > 0$, in the gap I_k of the unperturbed operator H_0 in terms of the eigenvalue branches $\mu_j^\pm(t)$ of the Schrödinger operators $H_t^\pm$ (Theorem 2.1);
- sufficient conditions for the guided potentials Q under which the guided bands of H_t do not appear in the gap I_k (Theorem 2.3).

  The proof of these results is based on the direct integral decomposition (1.8)–(1.9) for the guided Schrödinger operator H_t, where the fiber operators $H_t(\vartheta)$ act on the infinite fundamental graph C.

In Section 3 we approximate the infinite graph C by a sequence of finite graphs C_n, $n = 0, 1, 2, \dots$, and, using the Birman-Schwinger principle, prove convergence of spectra of Schrödinger operators on C_n to the spectrum of $H_t(\vartheta)$ in gaps of $H_0(\vartheta)$. These results are used in the proofs of Theorems 2.1 and 2.3.

Section 4 is devoted to the proofs of the main results (Theorems 2.1 and 2.3), where we essentially use the monotonicity and continuity of the eigenvalues of the fiber operators $H_t(\vartheta)$ with respect to the parameter t.

Section 5 is devoted to examples. First we apply the obtained results in the simplest case of the one-dimensional Schrödinger operator with a periodic potential perturbed by a finitely supported potential. Next we consider the guided Schrödinger operator H_t on the hexagonal lattice and the square lattice with four vertices in the fundamental domain and describe the guided spectrum of H_t.

In the appendix we recall some well-known properties of self-adjoint operators, needed to prove our results.

It should be noted that in the proof we also use some ideas from a series of papers of Alama, Deift and Hempel [1], [8], [13], where the existence of eigenvalues for Schrödinger operators $H_t = H_0 + tQ$, $t \in \mathbb{R}$, on $\mathbb{R}^d$ in gaps of H_0 was investigated.

2. Main results

In this section we formulate our main results. First we consider the perturbed Schrödinger operators $H_t = H_0 + tQ$, $t > 0$, where the guided potential Q has *maximal* support, i.e., $\mathscr{V}_* \subset \operatorname{supp} Q$. In this case all eigenvalue branches $\mu_j^-(t)$, $j \in \mathbb{N}_p$, and $\mu_j^+(t)$, $j \in \mathbb{N}_{p-r}$, of the operator families H_t^- and H_t^+, $t \geqslant 0$, defined in Subsection 1.3, are strictly increasing functions of t and

$$\mu_j^{\pm}(\mathbb{R}_{\geqslant 0}) = [\mu_j^{\pm}, +\infty), \quad \text{where} \quad \mu_j^{\pm} = \mu_j^{\pm}(0)$$

(see Lemma 4.1). Denote by $t_j^{\pm}\colon [\mu_j^{\pm}, +\infty) \to \mathbb{R}_{\geqslant 0}$ the inverse functions of $\mu_j^{\pm}(t)$.

In the next theorem we show that the interval I_k defined by (1.14) contains k guided band branches of H_t, $t > 0$, and obtain a localization of these branches in terms of the eigenvalue branches $\mu_j^{\pm}(t)$ of $H_t^{\pm}$. We also determine values of t for which the guided Schrödinger operator H_t has no guided spectrum in I_k.

THEOREM 2.1. *Let the gap condition (1.14) be fulfilled, $\lambda \in I_k$, and assume that the guided potential Q has maximal support, i.e., $\mathscr{V}_* \subset \operatorname{supp} Q$. Then the following statements hold true.*

i) There exist k guided band branches $\mathfrak{s}_1^{(k)}(t), \dots, \mathfrak{s}_k^{(k)}(t)$ of the operator family $H_t = H_0 + tQ$, $t > 0$, crossing the level λ. Moreover, in the gap I_k each of these branches satisfies

$$\mathfrak{s}_j^{(k)}(t) \subset [\mu_j^-(t), \mu_j^+(t)], \qquad j = 1, \dots, k. \tag{2.1}$$

ii) For each $t \in \mathbb{R}_{>0} \setminus T$, where

$$T = \bigcup_{j=1}^{k} [t_j^+(\lambda), t_j^-(\lambda)],$$

the level λ does not belong to the guided spectrum $\mathfrak{s}(H_t)$ of H_t. Here $t_j^{\pm}(\mu)$ are the inverse functions of $\mu_j^{\pm}(t)$. In particular, if $t > t_1^-(\mu_{k+1}^-)$, then there are no guided bands of H_t in I_k.

REMARK 2.2. i) The guided band branches of the family of the perturbed Schrödinger operators $H_t = H_0 + tQ$, $t \geqslant 0$, with a maximally supported potential Q (i.e., $\mathscr{V}_* \subset \operatorname{supp} Q$) in gaps of H_0 are shown schematically in Fig. 2. The horizontal axis represents the parameter t and the vertical axis represents the spectrum of H_t. For each $\lambda \in I_k$ there exist k guided band branches $\mathfrak{s}_j^{(k)}(t)$, $j = 1, \dots, k$, crossing the level λ. In the interval I_k each of these branches $\mathfrak{s}_j^{(k)}(t)$ lies between the eigenvalue branches

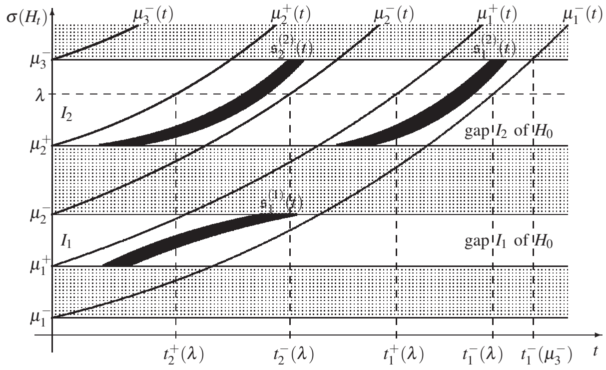

Figure 2: A schematic diagram of the dependence of the guided bands of H_t on t (the guided band branches) in gaps of H_0.

$\mu_j^-(t)$ and $\mu_j^+(t)$ of the Schrödinger operators H_t^- and H_t^+. But it is valid only in I_k. The behavior of the guided band branches outside the interval I_k is difficult to control. In particular, it is good if the gap I_k is maximal.

ii) The spectral localization (2.1) is not very precise as only for some values of t, it gives a smaller set than the gap (μ_k^+,μ_{k+1}^-), see Fig. 2.

iii) The guided spectrum of $H_t = H_0 + tQ$ in gaps of the unperturbed operator H_0 has quite different properties compared to the guided spectrum above the spectrum of H_0. It is known [28] that for t large enough there are always $\#\operatorname{supp} Q\restriction_{\mathscr{V}_*}$ guided bands of H_t above the spectrum of H_0. On the other hand, by Theorem 2.1.*ii*), if $\mathscr{V}_* \subset \operatorname{supp} Q$, then for t large enough there is no guided spectrum of H_t in gaps of H_0.

Now we consider the perturbed Schrödinger operators $H_t = H_0 + tQ$, $t > 0$, with a non-maximally supported guided potential Q. Recall that $\mathscr{V}_\mathfrak{b} \subset \mathscr{V}_*$ is a vertex set of the quotient graph G_* such that each bridge of G_* is incident to at least one vertex of the set $\mathscr{V}_\mathfrak{b}$. We determine sufficient conditions for the guided potentials Q under which the guided bands of the guided Schrödinger operators $H_t = H_0 + tQ$, $t > 0$, do not appear in the gap I_k of the unperturbed operator H_0.

THEOREM 2.3. *Let the gap condition (1.14) be fulfilled and let* $\operatorname{supp} Q\restriction_{\mathscr{V}_*} \subset \mathscr{V}_\mathfrak{b}$. *Then for each $t > 0$ the guided Schrödinger operator $H_t = H_0 + tQ$ has no guided spectrum in the gap I_k of H_0, i.e., I_k is also a gap of H_t for all $t > 0$.*

REMARK 2.4. i) The spectrum of the operator H_t, of course, does not depend on the choice of the set $\mathscr{V}_\mathfrak{b}$. But the interval I_k, in general, depends on this choice. In particular, one could vary $\mathscr{V}_\mathfrak{b}$ in order to get a better spectral localization.

ii) The results similar to Theorems 2.1 and 2.3 hold true for the case $d = 0$, i.e., for the spectrum of the Schrödinger operator $H_t = H_0 + tQ$ with finitely supported potentials Q on the periodic graph G. We only need to replace the words “the guided band branches of H_t” and “the guided spectrum of H_t” by the words “the eigenvalue branches of H_t" and "the point spectrum of H_t", respectively. Note that even for the case of finitely supported potentials Q the results of Theorems 2.1 and 2.3 are new (to the best of our knowledge).

iii) We consider periodic graphs embedded into $\mathbb{R}^{\widetilde{d}}$. But the obtained results stay valid for abstract periodic graphs, see also Footnote 1.

3. Operators on approximating graphs

3.1. Approximating graphs and edge indices

We recall that the cylinder $C = G/\Gamma$ is a Γ_0-periodic graph with periods $a_{d+1},\dots,a_{\widetilde{d}}$. We will approximate an infinite cylinder $C = (\mathscr{V}_c, \mathscr{E}_c)$ by a sequence of the finite graphs $C_n = (\mathscr{V}_n, \mathscr{E}_n)$ defined by

$$C_n = C/(2^n\Gamma_0), \qquad n \in \mathbb{N}_0, \qquad \mathbb{N}_0 = \{0,1,2,\dots\}.$$

REMARK 3.1. i) The quotient graph $G_* = G/\widetilde{\Gamma}$ of the $\widetilde{\Gamma}$-periodic graph G coincides with the graph $C_0 = C/\Gamma_0$.

ii) The sequence of the approximating graphs C_n, $n \in \mathbb{N}_0$, is also called a *tower of covering graphs*.

We define an *edge index*, which was introduced in [26]. The indices are important to study the spectrum of the Laplacians and Schrödinger operators on periodic graphs, since fiber operators are expressed in terms of edges indices (see (1.5), (1.9)).

For any vertex $v \in \mathscr{V}$ of the $\widetilde{\Gamma}$-periodic graph G the following unique representation holds true:

$$v = v_0 + [v], \qquad v_0 \in \Omega, \qquad [v] \in \widetilde{\Gamma}, \tag{3.1}$$

where Ω is the fundamental cell of the lattice $\widetilde{\Gamma}$ defined by (1.1). In other words, each vertex $v \in \mathscr{V}$ can be obtained from a vertex $v_0 \in \Omega$ by the shift by a vector $[v] \in \widetilde{\Gamma}$. For any oriented edge $\mathbf{e} = (u,v) \in \mathscr{A}$ of the periodic graph G we define the *edge index* $\widetilde{\tau}(\mathbf{e})$ as the integer vector given by

$$\widetilde{\tau}(\mathbf{e}) = [v]_{\mathbb{A}} - [u]_{\mathbb{A}} \in \mathbb{Z}^{\widetilde{d}}, \tag{3.2}$$

where $[v]_{\mathbb{A}}$ is the coordinate vector of $[v]$ with respect to the basis $\mathbb{A} = \{a_1,\dots,a_{\widetilde{d}}\}$ of the lattice $\widetilde{\Gamma}$, and $[v] \in \widetilde{\Gamma}$ is defined by (3.1). Note that edges connecting vertices inside the fundamental cell Ω have zero indices.

On the set $\mathscr{A}$ of all oriented edges of the periodic graph G we define the surjection

$$\mathfrak{f}\colon \mathscr{A} \to \mathscr{A}_c = \mathscr{A}/\Gamma, \tag{3.3}$$

which maps each $\mathbf{e} \in \mathscr{A}$ to its equivalence class $\mathbf{e}_c = \mathfrak{f}(\mathbf{e})$ which is an oriented edge of the cylinder $C = G/\Gamma$. For an oriented edge $\mathbf{e}_c \in \mathscr{A}_c$ we define the edge index $\widetilde{\tau}(\mathbf{e}_c)$ by

$$\widetilde{\tau}(\mathbf{e}_c) = \widetilde{\tau}(\mathbf{e}) \quad \text{for some } \mathbf{e} \in \mathscr{A} \quad \text{such that } \mathbf{e}_c = \mathfrak{f}(\mathbf{e}). \tag{3.4}$$

In other words, edge indices of the cylinder C are induced by edge indices of the periodic graph G.

Denote by $\mathscr{A}_n$ the set of all oriented edges of the graph C_n, $n \in \mathbb{N}_0$. Similarly to (3.3), on the set $\mathscr{A}_c$ of all oriented edges of the cylinder C we define the surjections

$$\mathfrak{f}_n : \mathscr{A}_c \to \mathscr{A}_n = \mathscr{A}_c/(2^n\Gamma_0), \qquad n \in \mathbb{N}_0,$$

which map each $\mathbf{e} \in \mathscr{A}_c$ to its equivalence class $\mathbf{e}_n = \mathfrak{f}_n(\mathbf{e})$ which is an oriented edge of the approximating graph $C_n = C/(2^n\Gamma_0)$. For the edge $\mathbf{e}_n \in \mathscr{A}_n$ we define the edge index $\widetilde{\tau}(\mathbf{e}_n)$ by

$$\widetilde{\tau}(\mathbf{e}_n) = \widetilde{\tau}(\mathbf{e}) \quad \text{for some } \mathbf{e} \in \mathscr{A}_c \quad \text{such that } \mathbf{e}_n = \mathfrak{f}_n(\mathbf{e}), \tag{3.5}$$

i.e., edge indices of the approximating graphs are induced by edge indices of the cylinder C.

Edge indices, generally speaking, depend on the embedding of G into $\mathbb{R}^{\widetilde{d}}$ and on the choice of the basis $\mathbb{A}$ of the lattice $\widetilde{\Gamma}$. Recall that the fundamental cell Ω is determined by the basis $\mathbb{A}$, see (1.1). But once the embedding of G into $\mathbb{R}^{\widetilde{d}}$ and the basis of $\widetilde{\Gamma}$ are fixed, edge indices of the cylinder C and the approximating graphs C_n are uniquely determined by (3.4) and (3.5), respectively, since

$$\widetilde{\tau}(\mathbf{e}+a) = \widetilde{\tau}(\mathbf{e}), \qquad \forall (\mathbf{e},a) \in \mathscr{A} \times \widetilde{\Gamma}.$$

3.2. Schrödinger operators on approximating graphs

For each $(\vartheta, n) \in \mathbb{T}^d \times \mathbb{N}_0$ we define the operator $\Delta_n(\vartheta)$ on the approximating graph $C_n = (\mathscr{V}_n, \mathscr{E}_n)$:

$$\big(\Delta_n(\vartheta)f\big)(v) = \sum_{\mathbf{e}=(v,u)\in\mathscr{A}_n} \Big(f(v) - e^{i\langle\tau(\mathbf{e}),\vartheta\rangle} f(u)\Big), \qquad f \in \ell^2(\mathscr{V}_n), \tag{3.6}$$

where $\tau(\mathbf{e}) \in \mathbb{Z}^d$ is the vector consisting of the first d components of the index $\widetilde{\tau}(\mathbf{e}) \in \mathbb{Z}^{\widetilde{d}}$ of the edge $\mathbf{e} \in \mathscr{A}_n$, defined by (3.5).

For each $(\vartheta, t) \in \mathbb{T}^d \times \mathbb{R}_{\geqslant 0}$ we consider the sequence of operators

$$H_{t,n}(\vartheta) = \Delta_n(\vartheta) + W + tQ, \qquad n \in \mathbb{N}_0, \tag{3.7}$$

on the approximating graphs C_n. The operator $H_{t,n}(\vartheta)$ has p_n eigenvalues $\lambda_j\big(H_{t,n}(\vartheta)\big)$, $j \in \mathbb{N}_{p_n}$, $p_n = \#\mathscr{V}_n$, which are labeled (counting multiplicities) by

$$\lambda_1\big(H_{t,n}(\vartheta)\big) \leqslant \ldots \leqslant \lambda_{p_n}\big(H_{t,n}(\vartheta)\big),$$
$$p_n = \#\mathscr{V}_n = 2^{nd_0}p, \qquad d_0 = \widetilde{d} - d, \qquad p = \#\mathscr{V}_*.$$

Since $Q \geqslant 0$, for each $(\vartheta, n, j) \in \mathbb{T}^d \times \mathbb{N}_0 \times \mathbb{N}_{p_n}$ the eigenvalue $\lambda_j\big(H_{t,n}(\vartheta)\big)$ is a continuous piecewise analytic and non-decreasing function of $t \geqslant 0$.

We formulate some properties of the operators $H_{t,n}(\vartheta)$ needed to prove the main results.

PROPOSITION 3.2. *Let* $(\vartheta,n)\in\mathbb{T}^d\times\mathbb{N}_0$. *Then:*

i) The operator $\Delta_n(\vartheta)$ *defined by (3.6) is a magnetic Laplacian on the finite graph* $C_n=(\mathscr{V}_n,\mathscr{E}_n)$ *with magnetic vector potential* $\alpha(\mathbf{e})=\langle\tau(\mathbf{e}),\vartheta\rangle$, $\mathbf{e}\in\mathscr{A}_n$.

ii) The quadratic form of $\Delta_n(\vartheta)$ *is given by*

$$\langle\Delta_n(\vartheta)f,f\rangle_{\ell^2(\mathscr{V}_n)}=\frac{1}{2}\sum_{\mathbf{e}=(v,u)\in\mathscr{A}_n}\Big|f(v)-e^{i\langle\tau(\mathbf{e}),\vartheta\rangle}f(u)\Big|^2,\qquad f\in\ell^2(\mathscr{V}_n).\tag{3.8}$$

iii) The operator $H_{0,n}(\vartheta)=\Delta_n(\vartheta)+W$ *satisfies*

$$\|H_{0,n}(\vartheta)\|\leqslant 2\varkappa_+ +\max_{v\in\mathscr{V}_*}|W(v)|,\qquad \textit{where}\qquad \varkappa_+=\max_{v\in\mathscr{V}_*}\varkappa_v<\infty.\tag{3.9}$$

Proof. *i*) The discrete magnetic Laplacian Δ_α on a graph $G=(V,E)$ has the form

$$\big(\Delta_\alpha f\big)(v)=\sum_{\mathbf{e}=(v,u)\in A}\Big(f(v)-e^{i\alpha(\mathbf{e})}f(u)\Big),\qquad f\in\ell^2(V),\tag{3.10}$$

where A is the set of all oriented edges of G, and the *magnetic vector potential* $\alpha: A\to\mathbb{R}$ satisfies the condition $\alpha(\underline{\mathbf{e}})=-\alpha(\mathbf{e})$ for all $\mathbf{e}\in A$. Here $\underline{\mathbf{e}}=(u,v)\in A$ is the inverse edge of $\mathbf{e}=(v,u)\in A$.

Comparing (3.6) with (3.10), we obtain the required statement.

ii) The quadratic form $\langle\Delta_\alpha f,f\rangle_{\ell^2(V)}$ associated with the magnetic Laplacian (3.10) is given by (see [19], [28]):

$$\langle\Delta_\alpha f,f\rangle_{\ell^2(V)}=\frac{1}{2}\sum_{\mathbf{e}=(v,u)\in A}\Big|f(v)-e^{i\alpha(\mathbf{e})}f(u)\Big|^2.$$

Applying this to the operator $\Delta_n(\vartheta)$, we obtain (3.8).

iii) It is known (see, e.g., [19]) that the magnetic Laplacian (3.10) on a graph G satisfies $\|\Delta_\alpha\|\leqslant 2\varkappa_+$, where $\varkappa_+$ is the maximum vertex degree of G. Then we obtain

$$\|H_{0,n}(\vartheta)\|=\|\Delta_n(\vartheta)+W\|\leqslant\|\Delta_n(\vartheta)\|+\|W\|\leqslant 2\varkappa_+ +\max_{v\in\mathscr{V}_*}|W(v)|.\qquad\square$$

PROPOSITION 3.3. *Let* $(\vartheta,n,t)\in\mathbb{T}^d\times\mathbb{N}_0\times\mathbb{R}_{\geqslant 0}$. *Then the eigenvalues* $\lambda_j\big(H_{t,n}(\vartheta)\big)$ *of the operator* $H_{t,n}(\vartheta)$ *on the approximating graph* C_n *defined by (3.7) satisfy*

$$\lambda_j(H_{t,n}^-)\leqslant\lambda_j\big(H_{t,n}(\vartheta)\big)\leqslant\lambda_j(H_{t,n}^+),\qquad\forall\, j=1,\ldots,2^{nd_0}(p-r),\tag{3.11}$$

where

$$H_{t,n}^{\pm}=H_t^{\pm}\oplus\underbrace{H_0^{\pm}\oplus\ldots\oplus H_0^{\pm}}_{2^{nd_0}-1},\tag{3.12}$$

and the Schrödinger operators $H_t^{\pm}$ *are defined in Subsection 1.3.*

Proof. We equip each unoriented edge $\mathbf{e} \in \mathscr{E}_n$ of the graph $C_n = (\mathscr{V}_n, \mathscr{E}_n)$ with some orientation and represent C_n as a union of two graphs with the same vertex set $\mathscr{V}_n$:

$$C_n = C_{n,\mathfrak{b}} \cup C_n^-, \qquad C_{n,\mathfrak{b}} = (\mathscr{V}_n, \mathscr{B}_n), \qquad C_n^- = (\mathscr{V}_n, \mathscr{E}_n \setminus \mathscr{B}_n),$$

where $\mathscr{B}_n$ is the set of all edges $\mathbf{e}$ of the graph C_n with nonzero indices $\widetilde{\tau}(\mathbf{e})$ defined by (3.5). Due to the construction, the graph C_n^- consists of 2^{nd_0} connected components each of which is isomorphic to the graph G_*^- defined in (1.12). Let $\mathscr{V}_{n,\mathfrak{b}} \subset \mathscr{V}_n$ be the set of all vertices of C_n which are Γ_0-equivalent to vertices of the set $\mathscr{V}_\mathfrak{b}$. Recall that $\mathscr{V}_\mathfrak{b} \subset \mathscr{V}_*$ is the vertex set of the quotient graph G_* such that each bridge (an edge with nonzero index) of G_* is incident to at least one vertex of the set $\mathscr{V}_\mathfrak{b}$.

Similarly to $H_t^{\pm}$, we define two operators:

- $H_{t,n}^- = \Delta_n^- + W + tQ$ is the Schrödinger operator on the graph C_n^-, where Δ_n^- is the Laplacian defined by (1.3) on the graph C_n^-;
- $H_{t,n}^+$ is the operator $H_{t,n}(\vartheta) = \Delta_n(\vartheta) + W + tQ$ on the graph C_n with Dirichlet boundary conditions $f\restriction_{\mathscr{V}_{n,\mathfrak{b}}} = 0$, where $f\restriction_{\mathscr{V}_{n,\mathfrak{b}}}$ is the restriction of $f \in \ell^2(\mathscr{V}_n)$ to $\mathscr{V}_{n,\mathfrak{b}}$.

Due to the definitions, these operators $H_{t,n}^{\pm}$ satisfy (3.12).

Since the quadratic form of $H_{t,n}(\vartheta)$ is just an extension of the quadratic form associated with $H_{t,n}^+$ to a larger domain, we have $H_{t,n}(\vartheta) \leqslant H_{t,n}^+$. This proves the upper estimate in (3.11).

Now we prove the lower estimate in (3.11). Using (3.7) and (3.8), we obtain

$$\begin{aligned}
\langle H_{t,n}(\vartheta) f, f\rangle_{\ell^2(\mathscr{V}_n)} &= \langle (\Delta_n(\vartheta) + W + tQ) f, f\rangle_{\ell^2(\mathscr{V}_n)} \\
&= \frac{1}{2} \sum_{\mathbf{e}=(v,u)\in\mathscr{A}_n} \left| f(v) - e^{i\langle \tau(\mathbf{e}),\,\vartheta\rangle} f(u)\right|^2 + \langle (W+tQ) f, f\rangle_{\ell^2(\mathscr{V}_n)} \\
&= \frac{1}{2} \sum_{(v,u)\in\mathscr{A}_n\setminus\mathscr{B}_n} |f(v) - f(u)|^2 + \langle (W+tQ) f, f\rangle_{\ell^2(\mathscr{V}_n)} \\
&\quad + \frac{1}{2} \sum_{\mathbf{e}=(v,u)\in\mathscr{B}_n} \left| f(v) - e^{i\langle \tau(\mathbf{e}),\,\vartheta\rangle} f(u)\right|^2 \\
&\geqslant \frac{1}{2} \sum_{(v,u)\in\mathscr{A}_n\setminus\mathscr{B}_n} |f(v) - f(u)|^2 + \langle (W+tQ) f, f\rangle_{\ell^2(\mathscr{V}_n)} \\
&= \langle H_{t,n}^- f, f\rangle_{\ell^2(\mathscr{V}_n)}, \qquad \forall f \in \ell^2(\mathscr{V}_n).
\end{aligned}$$

Thus, in the sense of quadratic forms, $H_{t,n}^- \leqslant H_{t,n}(\vartheta)$, and, we obtain the lower estimate in (3.11). □

REMARK 3.4. i) Let $(n,t) \in \mathbb{N}_0 \times \mathbb{R}_{\geqslant 0}$. The spectrum of the operator $H_{t,n}^-$ defined in (3.12) consists of the eigenvalues μ_m^-, $m \in \mathbb{N}_p$, of H_0^- each of which is repeated $2^{nd_0} - 1$ times and the eigenvalues $\mu_m^-(t)$, $m \in \mathbb{N}_p$, of H_t^-. Then the eigenvalues

$\lambda_1(H_{t,n}^-) \leqslant \ldots \leqslant \lambda_{2^{nd_0}p}(H_{t,n}^-)$ of $H_{t,n}^-$ labeled in non-decreasing order satisfy

$$\begin{aligned}
&\lambda_{2^{nd_0}(m-1)+1}(H_{t,n}^-) = \ldots = \lambda_{2^{nd_0}m-m}(H_{t,n}^-) = \mu_m^-, && m \in \mathbb{N}_p,\\
&\lambda_{2^{nd_0}m-(m-l)}(H_{t,n}^-) = \min\big\{\max\{\mu_l^-(t),\mu_m^-\},\mu_{m+1}^-\big\}, && m \in \mathbb{N}_{p-1}, \quad l \in \mathbb{N}_m,\\
&\lambda_{2^{nd_0}p-(p-l)}(H_{t,n}^-) = \max\{\mu_l^-(t),\mu_p^-\}, && l \in \mathbb{N}_p.
\end{aligned} \tag{3.13}$$

Similarly, the spectrum of the operator $H_{t,n}^+$ defined in (3.12) consists of the eigenvalues μ_m^+, $m \in \mathbb{N}_{p-r}$, of H_0^+ each of which is also repeated $2^{nd_0}-1$ times and the eigenvalues $\mu_m^+(t)$, $m \in \mathbb{N}_{p-r}$, of H_t^+. Then the eigenvalues $\lambda_1(H_{t,n}^+) \leqslant \ldots \leqslant \lambda_{2^{nd_0}(p-r)}(H_{t,n}^+)$ of $H_{t,n}^+$ satisfy

$$\begin{aligned}
&\lambda_{2^{nd_0}(m-1)+1}(H_{t,n}^+) = \ldots = \lambda_{2^{nd_0}m-m}(H_{t,n}^+) = \mu_m^+, && m \in \mathbb{N}_{p-r},\\
&\lambda_{2^{nd_0}m-(m-l)}(H_{t,n}^+) = \min\big\{\max\{\mu_l^+(t),\mu_m^+\},\mu_{m+1}^+\big\}, && m \in \mathbb{N}_{p-r-1}, \quad l \in \mathbb{N}_m,\\
&\lambda_{2^{nd_0}(p-r)-(p-r-l)}(H_{t,n}^+) = \max\{\mu_l^+(t),\mu_{p-r}^+\}, && l \in \mathbb{N}_{p-r}.
\end{aligned} \tag{3.14}$$

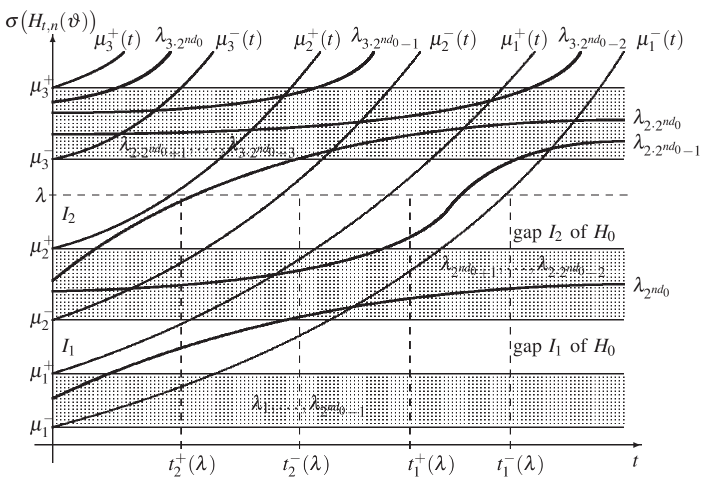

Figure 3: A schematic diagram of the dependence of the eigenvalues $\lambda_j \equiv \lambda_j\big(H_{t,n}(\vartheta)\big)$, $j = 1,\ldots,2^{nd_0}(p-r)$, of $H_{t,n}(\vartheta)$ on t (the eigenvalue branches).

ii) From (3.11) it follows that for all $(\vartheta,n) \in \mathbb{T}^d \times \mathbb{N}_0$ the graph of each eigenvalue branch $\lambda_j(t) \equiv \lambda_j\big(H_{t,n}(\vartheta)\big)$, $j = 1,\ldots,2^{nd_0}(p-r)$, of the operator family $H_{t,n}(\vartheta)$, $t \geqslant 0$, lies between the graphs of the corresponding eigenvalue branches $\lambda_j(H_{t,n}^\pm)$ of the operator families $H_{t,n}^\pm$, $t \geqslant 0$. The behavior of the functions $\lambda_j(t)$ is shown schematically in Fig. 3 for the case when $p-r=3$ and $I_k = (\mu_k^+,\mu_{k+1}^-) \neq \varnothing$, $k=1,2$. Due to the first identities in (3.13) and (3.14), for each $m = 1,2,3$ the graphs of the eigenvalue branches

$$\lambda_{2^{nd_0}(m-1)+1}(t),\ldots,\lambda_{2^{nd_0}m-m}(t)$$

lie in the half-strip $\mathbb{R}_{\geqslant 0}\times[\mu_m^-,\mu_m^+]$. The graphs of the remaining eigenvalue branches are shown in the figure. For example, by the second identities in (3.13) and (3.14) as $m=2$ and $l=1$, the eigenvalue branch $\lambda_{2\cdot 2^{nd_0}-1}(t)$ satisfies

$$\min\big\{\max\big\{\mu_1^-(t),\mu_2^-\big\},\mu_3^-\big\}\leqslant\lambda_{2\cdot 2^{nd_0}-1}(t)\leqslant\min\big\{\max\big\{\mu_1^+(t),\mu_2^+\big\},\mu_3^+\big\}.$$

3.3. Convergence of Birman-Schwinger kernels

Let the gap condition (1.14) be fulfilled, and $\lambda\in I_k$. For each $(\vartheta,n)\in\mathbb{T}^d\times\mathbb{N}_0$ we define the Birman-Schwinger kernels $K(\vartheta)$ and $K_n(\vartheta)$ associated with $H_0(\vartheta)$ and $H_{0,n}(\vartheta)$, respectively:

$$K(\vartheta):=Q^{1/2}\big(H_0(\vartheta)-\lambda\big)^{-1}Q^{1/2},\qquad K_n(\vartheta):=Q^{1/2}\big(H_{0,n}(\vartheta)-\lambda\big)^{-1}Q^{1/2},\tag{3.15}$$

where we think of $\big(H_{0,n}(\vartheta)-\lambda\big)^{-1}$ as being extended by 0 on $\ell^2(\mathscr{V}_c\setminus\mathscr{V}_n)$.

In this subsection we obtain a convergence result for the Birman-Schwinger kernels (3.15).

PROPOSITION 3.5. *Let the gap condition (1.14) be fulfilled, $\lambda\in I_k$ and $\vartheta\in\mathbb{T}^d$. Then the following statements hold true.*

i) The sequence of the Birman-Schwinger kernels $K_n(\vartheta)$, $n\in\mathbb{N}_0$, associated with $H_{0,n}(\vartheta)$ converges in norm to the Birman-Schwinger kernel $K(\vartheta)$ for $H_0(\vartheta)$ as $n\to\infty$.

ii) The spectra of $K(\vartheta)$ and $K_n(\vartheta)$, $n\in\mathbb{N}_0$, satisfy

$$\sigma\big(K(\vartheta)\big)=\lim_{n\to\infty}\sigma\big(K_n(\vartheta)\big).\tag{3.16}$$

REMARK 3.6. The identity (3.16) is understood in the following sense: for any $t\in\sigma\big(K(\vartheta)\big)$ there exists a sequence $t_n\in\sigma\big(K_n(\vartheta)\big)$, $n\in\mathbb{N}_0$, such that $\lim\limits_{n\to\infty}t_n=t$ and, conversely, if $t_n\in\sigma\big(K_n(\vartheta)\big)$, $n\in\mathbb{N}_0$, and $\lim\limits_{n\to\infty}t_n=t$ for some t, then $t\in\sigma\big(K(\vartheta)\big)$.

In order to prove this proposition we need the following lemma.

LEMMA 3.7. *Let $(\vartheta,t)\in\mathbb{T}^d\times\mathbb{R}_{\geqslant 0}$. Then the sequence of the operators $H_{t,n}(\vartheta)$, $n\in\mathbb{N}_0$, on the approximating graphs $C_n=(\mathscr{V}_n,\mathscr{E}_n)$ (when $\ell^2(\mathscr{V}_n)$ is naturally embedded in $\ell^2(\mathscr{V}_c)$) converges strongly to the operator $H_t(\vartheta)$ on the cylinder $C=(\mathscr{V}_c,\mathscr{E}_c)$ as $n\to\infty$.*

Proof. First we prove this statement for $t=0$. Denote by $\ell^2_{\rm fin}(\mathscr{V}_c)$ the set of all finitely supported functions $f\in\ell^2(\mathscr{V}_c)$. For each $f\in\ell^2_{\rm fin}(\mathscr{V}_c)$ and sufficiently large $n\in\mathbb{N}_0$, $\mathscr{V}_n$ contains the support of f and, consequently, $H_0(\vartheta)f=H_{0,n}(\vartheta)f$. Then we have

$$\big\|H_{0,n}(\vartheta)f-H_0(\vartheta)f\big\|\to 0\quad\text{as}\quad n\to\infty,\qquad\forall f\in\ell^2_{\rm fin}(\mathscr{V}_c).$$

The set $\ell^2_{\rm fin}(\mathscr{V}_{\rm c})$ is dense in $\ell^2(\mathscr{V}_{\rm c})$ and, due to (3.9), $H_{0,n}(\vartheta)$ are uniformly bounded. Then, by Proposition A.1.*i*), we deduce that

$$\big\|H_{0,n}(\vartheta)f - H_0(\vartheta)f\big\| \to 0 \quad \text{as} \quad n\to\infty, \qquad \forall f\in\ell^2(\mathscr{V}_{\rm c}).$$

Now let $t>0$. Then, using the identity $H_{t,n}(\vartheta)=H_{0,n}(\vartheta)+tQ$, we have

$$\big\|H_{t,n}(\vartheta)f - H_t(\vartheta)f\big\| = \big\|H_{0,n}(\vartheta)f - H_0(\vartheta)f\big\| \to 0 \text{ as } n\to\infty, \quad \forall f\in\ell^2(\mathscr{V}_{\rm c}). \quad \square$$

Proof of Proposition 3.5. *i*) Lemma 3.7 and the boundedness of the operators $H_0(\vartheta)$ and $H_{0,n}(\vartheta)$, $n\in\mathbb{N}_0$, give that $H_{0,n}(\vartheta)\to H_0(\vartheta)$ in strong resolvent sense as $n\to\infty$. Then for any $f\in\ell^2(\mathscr{V}_{\rm c})$ we have

$$\big\|\big(H_{0,n}(\vartheta)-\lambda\big)^{-1}f - \big(H_0(\vartheta)-\lambda\big)^{-1}f\big\| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.17}$$

Using (3.15), we obtain

$$\begin{aligned}\big\|K_n(\vartheta)-K(\vartheta)\big\| &= \big\|Q^{1/2}\big(H_{0,n}(\vartheta)-\lambda\big)^{-1}Q^{1/2} - Q^{1/2}\big(H_0(\vartheta)-\lambda\big)^{-1}Q^{1/2}\big\| \\ &\leqslant \big\|Q^{1/2}\big\|\cdot\big\|\big(H_{0,n}(\vartheta)-\lambda\big)^{-1}Q^{1/2} - \big(H_0(\vartheta)-\lambda\big)^{-1}Q^{1/2}\big\|.\end{aligned}$$

Since $Q^{1/2}$ is a compact operator, applying Proposition A.1.*ii*) to (3.17), we deduce that $\big\|K_n(\vartheta)-K(\vartheta)\big\|\to 0$ as $n\to\infty$.

ii) This item follows directly from the previous one and Proposition A.1.*iii*). $\square$

4. Proof of the main results

4.1. Localization of the guided bands

We consider the case when the guided potential Q has maximal support, i.e., it satisfies the condition $\mathscr{V}_*\subset\operatorname{supp}Q$, and prove Theorem 2.1 about a localization of the guided bands of $H_t=H_0+tQ$. We need the following lemma.

LEMMA 4.1. *Let $\mathscr{V}_*\subset\operatorname{supp}Q$. Then all eigenvalue branches $\mu_j^\pm(t)$ of the operator families $H_t^\pm$, $t\geqslant 0$, are strictly increasing functions of t and*

$$\mu_j^\pm(\mathbb{R}_{\geqslant 0}) = \big[\mu_j^\pm,+\infty\big), \qquad \textit{where} \qquad \mu_j^\pm=\mu_j^\pm(0).$$

Proof. Since $Q\geqslant 0$, then, due to the perturbation theory, each eigenvalue $\mu_j^\pm(t)$ of $H_t^\pm$ is a continuous piecewise analytic and non-decreasing function of $t\geqslant 0$.

First, we show that $\mu_j^\pm(t)$ is strictly increasing on $\mathbb{R}_{\geqslant 0}$. Indeed, let $t_1>t_2\geqslant 0$. Then, by the identity $H_t^\pm=H_0^\pm+tQ$ and Proposition A.1.*vi*), we have for each j

$$\mu_j^-(t_1)\geqslant\mu_j^-(t_2)+(t_1-t_2)\min_{v\in\mathscr{V}_*}Q(v), \quad \mu_j^+(t_1)\geqslant\mu_j^+(t_2)+(t_1-t_2)\min_{v\in\mathscr{V}_*\setminus\mathscr{V}_{\mathfrak{b}}}Q(v).$$

Since $t_1>t_2$ and $\mathscr{V}_*\subset\operatorname{supp}Q$, these inequalities yield $\mu_j^\pm(t_1)>\mu_j^\pm(t_2)$.

Second, we prove that $\mu_j^{\pm}(\mathbb{R}_{\geqslant 0}) = [\mu_j^{\pm}, +\infty)$ for each j. We rewrite the sequence $Q(v), v \in \mathscr{V}_*$ in the form

$$0 < Q_1^\bullet \leqslant Q_2^\bullet \leqslant \ldots \leqslant Q_p^\bullet,$$

where $Q_j^\bullet = Q(v_j)$, $j \in \mathbb{N}_p$, for some distinct vertices $v_1, v_2, \ldots, v_p \in \mathscr{V}_*$. Using Proposition A.1.*vi*), we obtain

$$\mu_j^-(t) \geqslant tQ_j^\bullet + \mu_1^-, \qquad j \in \mathbb{N}_p.$$

Since $Q_j^\bullet > 0$, $j \in \mathbb{N}_p$, for t large enough the eigenvalue $\mu_j^-(t)$ can be arbitrarily large. The same is true for $\mu_j^+(t)$, since $\mu_j^+(t) \geqslant \mu_j^-(t)$. This and continuity of the functions $\mu_j^{\pm}(t)$ give that $\mu_j^{\pm}(\mathbb{R}_{\geqslant 0}) = [\mu_j^{\pm}, +\infty)$. □

PROPOSITION 4.2. *Let the gap condition (1.14) be fulfilled, $\lambda \in I_k = (\mu_k^+, \mu_{k+1}^-)$, and $(\vartheta, n) \in \mathbb{T}^d \times \mathbb{N}_0$. We assume that Q has maximal support, i.e., $\mathscr{V}_* \subset \operatorname{supp} Q$. Then there exist exactly k values $t_{n,1}(\vartheta), \ldots, t_{n,k}(\vartheta) \in \mathbb{R}_{>0}$ of t such that $\lambda \in \sigma(H_{t,n}(\vartheta))$ and these values satisfy*

$$t_{n,j}(\vartheta) \subset [t_j^+, t_j^-], \qquad t_j^{\pm} = (\mu_j^{\pm})^{-1}(\lambda), \qquad j = 1, \ldots, k. \tag{4.1}$$

Proof. Since $\mathscr{V}_* \subset \operatorname{supp} Q$, then, due to Lemma 4.1, all eigenvalue branches $\mu_j^{\pm}(t)$ of the operator families $H_t^{\pm}$, $t \geqslant 0$, are strictly increasing functions of t and $\mu_j^{\pm}(\mathbb{R}_{\geqslant 0}) = [\mu_j^{\pm}, \infty)$. Then for each $j = 1, \ldots, k$ there exist

- a unique $t_j^- \in \mathbb{R}_{\geqslant 0}$ such that $\mu_j^-(t_j^-) = \lambda$;
- a unique $t_j^+ \in \mathbb{R}_{\geqslant 0}$ such that $\mu_j^+(t_j^+) = \lambda$.

The remaining eigenvalue branches $\mu_j^{\pm}(t)$, $j > k$, lie above λ. Thus, each of the operator families $H_{t,n}^-$ and $H_{t,n}^+$, $t \geqslant 0$, defined by (3.12) has exactly k eigenvalue branches crossing the level λ at the points $t_j^{\pm}$, $j = 1, \ldots, k$. Then, from (3.11) it follows that the operator family $H_{t,n}(\vartheta)$, $t \geqslant 0$, also has exactly k eigenvalue branches crossing the level λ at some points $t_{n,j}(\vartheta)$, $j = 1, \ldots, k$, satisfying the conditions (4.1). □

REMARK 4.3. The dependence of the eigenvalues

$$\lambda_j(t) \equiv \lambda_j(H_{t,n}(\vartheta)), \qquad j = 1, \ldots, 2^{nd_0}(p - r),$$

of the operators $H_{t,n}(\vartheta) = H_{0,n}(\vartheta) + tQ$ on the parameter t for the guided potential Q satisfying the condition $\mathscr{V}_* \subset \operatorname{supp} Q$ is shown in Fig. 3. The horizontal axis represents the parameter t and the vertical axis represents the spectrum of $H_{t,n}(\vartheta)$. For each $\lambda \in I_k$ exactly k eigenvalue branches

$$\lambda_{2^{nd_0}k-(k-1)}(t), \quad \lambda_{2^{nd_0}k-(k-2)}(t), \quad \ldots, \quad \lambda_{2^{nd_0}k}(t)$$

cross the level λ. In the gap I_k each of these eigenvalue branches $\lambda_{2^{nd_0}k-(k-j)}$, $j \in \mathbb{N}_k$, lies between the eigenvalue branches $\mu_j^-(t)$ and $\mu_j^+(t)$ of the Schrödinger operators H_t^- and H_t^+.

PROPOSITION 4.4. *Let the gap condition (1.14) be fulfilled,* $\lambda \in I_k = (\mu_k^+, \mu_{k+1}^-)$ *and* $\vartheta \in \mathbb{T}^d$. *We assume that* $\mathcal{V}_* \subset \operatorname{supp} Q$. *Then the following statements hold true.*

i) There exist k *eigenvalue branches* $\lambda_1^{(k)}(\vartheta,t), \ldots, \lambda_k^{(k)}(\vartheta,t)$ *of the operator family* $H_t(\vartheta)$, $t \geqslant 0$, *crossing the level* λ. *Moreover, in the gap* I_k *each of these branches satisfies*

$$\lambda_j^{(k)}(\vartheta,t) \subset \big[\mu_j^-(t), \mu_j^+(t)\big], \qquad j \in \mathbb{N}_k. \tag{4.2}$$

ii) For each $t \in \mathbb{R}_{>0} \setminus T$, *where*

$$T = \bigcup_{j=1}^{k} \big[t_j^+(\lambda), t_j^-(\lambda)\big], \qquad t_j^\pm = (\mu_j^\pm)^{-1}(\lambda),$$

λ *does not belong to the spectrum of* $H_t(\vartheta)$.

Proof. *i*) By Proposition 4.2, for each $n \in \mathbb{N}_0$ there exist exactly k values $t_{n,1}(\vartheta)$, $\ldots, t_{n,k}(\vartheta) \in \mathbb{R}_{>0}$ of t such that $\lambda \in \sigma\big(H_{t,n}(\vartheta)\big)$ and these values satisfy (4.1). Then for each $j \in \mathbb{N}_k$ there exists a subsequence also denoted by $\big(t_{n,j}(\vartheta)\big)_{n\in\mathbb{N}_0}$ converging to some $t_j(\vartheta) \in [t_j^+, t_j^-]$, where $t_j^\pm = (\mu_j^\pm)^{-1}(\lambda)$. Let $K(\vartheta)$ and $K_n(\vartheta)$, $n \in \mathbb{N}_0$, be defined by (3.15). By Proposition A.1.*v*), for each $n \in \mathbb{N}_0$, the values $\big(-t_{n,j}(\vartheta)\big)^{-1}$, $j \in \mathbb{N}_k$, are eigenvalues of $K_n(\vartheta)$. Since $K_n(\vartheta) \to K(\vartheta)$ in norm and $t_{n,j}(\vartheta) \to t_j(\vartheta)$ as $n \to \infty$, then, by Proposition 3.5.*ii*), $\big(-t_j(\vartheta)\big)^{-1}$, $j \in \mathbb{N}_k$, are eigenvalues of $K(\vartheta)$. This and Proposition A.1.*v*) yield that there exist k values $t_1(\vartheta), \ldots, t_k(\vartheta)$ of t such that $\lambda \in \sigma\big(H_t(\vartheta)\big)$. Thus, there exist k eigenvalue branches $\lambda_j^{(k)}(\vartheta,t)$, $j \in \mathbb{N}_k$, of the operator family $H_t(\vartheta)$, $t \geqslant 0$, crossing the level λ at points $t_j(\vartheta) \in [t_j^+, t_j^-]$. Since λ is any level in the interval I_k, each of these branches satisfies (4.2).

ii) We argue by contradiction. Let $t \in \mathbb{R}_{>0} \setminus T$ and $\lambda \in \sigma\big(H_t(\vartheta)\big)$. Then, by Proposition A.1.*v*), $(-t)^{-1}$ is an eigenvalue of $K(\vartheta)$ and, by Proposition 3.5.*ii*), there exists a sequence $(t_n)_{n\in\mathbb{N}_0}$ such that $(-t_n)^{-1} \in \sigma\big(K_n(\vartheta)\big)$, $n \in \mathbb{N}_0$, and $\lim_{n\to\infty} t_n = t$. This yields that λ is an eigenvalue of $H_{t_n,n}(\vartheta)$ for each $n \in \mathbb{N}_0$. Using Proposition 4.2, we have that $t_n \in T$ for each $n \in \mathbb{N}_0$. Thus, we obtain a contradiction

$$t_n \in T, \quad \forall n \in \mathbb{N}_0, \qquad \text{and} \qquad \lim_{n\to\infty} t_n = t \notin T,$$

which completes the proof. $\square$

Proof of Theorem 2.1. *i*)–*ii*) These items are direct consequences of Proposition 4.4 and the definition (1.11) of the guided band branches and the definition (1.10) of the guided spectrum. $\square$

4.2. Conditions for nonexistence of the guided spectrum

In this subsection we prove Theorem 2.3, which determines sufficient conditions for the guided potentials Q under which the guided spectrum of the guided Schrödinger operators $H_t = H_0 + tQ$, $t > 0$, do not appear in the gaps of the unperturbed operator H_0.

Proof of Theorem 2.3. Since $\operatorname{supp} Q\restriction_{\mathcal{V}_*} \subset \mathcal{V}_{\mathfrak{b}}$, we have $H_t^+ = H_0^+$ by the definition of the operator H_t^+, $t > 0$. Then, using Proposition 3.3, we deduce that for each $(\vartheta,n,t) \in \mathbb{T}^d \times \mathbb{N}_0 \times \mathbb{R}_{\geqslant 0}$ the eigenvalues $\lambda_j\big(H_{t,n}(\vartheta)\big)$ of the operator $H_{t,n}(\vartheta)$ satisfy

$$\lambda_j(H_{0,n}^-) \leqslant \lambda_j(H_{t,n}^-) \leqslant \lambda_j\big(H_{t,n}(\vartheta)\big) \leqslant \lambda_j(H_{0,n}^+), \qquad \forall j = 1,\ldots,2^{nd_0}(p-r), \quad (4.3)$$

where

$$H_{0,n}^{\pm} = \underbrace{H_0^{\pm} \oplus \ldots \oplus H_0^{\pm}}_{2^{nd_0}}. \quad (4.4)$$

The identity (4.4) gives that the spectrum of the operator $H_{0,n}^-$ consists of the eigenvalues μ_m^-, $m \in \mathbb{N}_p$, of H_0^- each of which is repeated 2^{nd_0} times. Similarly, the spectrum of the operator $H_{0,n}^+$ consists of the eigenvalues μ_m^+, $m \in \mathbb{N}_{p-r}$, of H_0^+ each of which is also repeated 2^{nd_0} times. Then, from (4.3) it follows that

$$\mu_m^- \leqslant \lambda_j\big(H_{t,n}(\vartheta)\big) \leqslant \mu_m^+, \qquad m \in \mathbb{N}_{p-r}, \quad j = 2^{nd_0}(m-1)+1,\ldots,2^{nd_0}m.$$

This yields that $I_k = (\mu_k^+, \mu_{k+1}^-)$ is a gap in the spectrum of $H_{t,n}(\vartheta)$ for each $(\vartheta,n,t) \in \mathbb{T}^d \times \mathbb{N}_0 \times \mathbb{R}_{\geqslant 0}$.

For each $(\vartheta,n,t) \in \mathbb{T}^d \times \mathbb{N}_0 \times \mathbb{R}_{\geqslant 0}$ the operators $H_t(\vartheta)$ and $H_{t,n}(\vartheta)$ are bounded and, by Lemma 3.7, $H_{t,n}(\vartheta)$ converges to $H_t(\vartheta)$ strongly as $n \to \infty$. Then, by Proposition A.1.*iv*), I_k is a gap in the spectrum of $H_t(\vartheta)$ for each $(\vartheta,t) \in \mathbb{T}^d \times \mathbb{R}_{\geqslant 0}$. This and the definition (1.10) of the guided spectrum give that I_k is also a gap in the spectrum of H_t for each $t > 0$. ☐

5. Examples

We start this section with the simplest example of the one-dimensional Schrödinger operator with a periodic potential perturbed by a finitely supported potential. Then we consider the guided Schrödinger operator H_t on the hexagonal lattice and the square lattice with 4 vertices in the fundamental domain and describe the guided spectrum of H_t.

5.1. One-dimensional Schrödinger operator

First we consider the unperturbed Schrödinger operator

$$(H_0 f)(n) = 2f(n) - f(n+1) - f(n-1) + W(n)f(n), \qquad n \in \mathbb{Z},$$

acting on $f \in \ell^2(\mathbb{Z})$ with a real p-periodic potential W, $W(n+p) = W(n)$, $n \in \mathbb{Z}$, $p \geqslant 3$. In this case the quotient graph G_* is just the cycle graph with p vertices $1,\ldots,p$, and the fiber Schrödinger operator $\widetilde{H}_0(\vartheta)$, $\vartheta \in \mathbb{T}$, defined by (1.4), (1.5), on G_* is given by the following $(p \times p)$-matrix

$$\widetilde{H}_0(\vartheta) = \begin{pmatrix} 2+w_1 & -1 & 0 & \dots & -e^{-i\vartheta} \\ -1 & 2+w_2 & -1 & \dots & 0 \\ 0 & -1 & 2+w_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -e^{i\vartheta} & 0 & 0 & \dots & 2+w_p \end{pmatrix}, \quad w_n = W(n), \quad n \in \mathbb{N}_p.$$

It is well known that the spectrum of the operator H_0 consists of p bands $\sigma_n = [\lambda_n^-, \lambda_n^+]$, $n \in \mathbb{N}_p$, where $\lambda_1^-, \lambda_2^+, \lambda_3^-, \lambda_4^+, \dots$ are the eigenvalues of the matrix $\widetilde{H}_0(0)$ and $\lambda_1^+, \lambda_2^-, \lambda_3^+, \lambda_4^-, \dots$ are the eigenvalues of $\widetilde{H}_0(\pi)$. These bands are separated by gaps $(\lambda_n^+, \lambda_{n+1}^-)$. Some of the gaps may be degenerate, i.e. $\lambda_n^+ = \lambda_{n+1}^-$.

Now we describe the operators $H_0^\pm$ defined in Subsection 1.3. The quotient graph G_* has only one bridge-edge $(p,1)$. The graph G_*^-, obtained from G_* by deleting this bridge, is the path on p vertices $1,\dots,p$. Then we need to choose the vertex set $\mathscr{V}_\mathfrak{b} \subset \{1,\dots,p\}$. Recall that each bridge of G_* has to be incident to at least one vertex of $\mathscr{V}_\mathfrak{b}$ and we have to choose $\mathscr{V}_\mathfrak{b}$ as small as possible. Thus, we can take for $\mathscr{V}_\mathfrak{b}$ the set $\{p\}$.

The Schrödinger operator H_0^- with the potential W on the path graph G_*^- is given by the $(p \times p)$-matrix

$$H_0^- = \begin{pmatrix} 1+w_1 & -1 & 0 & \dots & 0 \\ -1 & 2+w_2 & -1 & \dots & 0 \\ 0 & -1 & 2+w_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1+w_p \end{pmatrix}. \tag{5.1}$$

The operator H_0^+ is the Schrödinger operator with the potential W on the quotient graph G_* with Dirichlet boundary condition at the vertex p. This operator is given by the $(p-1)\times(p-1)$-submatrix of $\widetilde{H}_0(\vartheta)$, which is obtained from $\widetilde{H}_0(\vartheta)$ by deleting the last row and column (corresponding to the vertex p):

$$H_0^+ = \begin{pmatrix} 2+w_1 & -1 & 0 & \dots & 0 \\ -1 & 2+w_2 & -1 & \dots & 0 \\ 0 & -1 & 2+w_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2+w_{p-1} \end{pmatrix}. \tag{5.2}$$

We assume that the potential W is such that the eigenvalues μ_j^-, $j \in \mathbb{N}_p$, of the operator H_0^- and the eigenvalues μ_j^+, $j \in \mathbb{N}_{p-1}$, of H_0^+ satisfy the inequality $\mu_j^+ < \mu_{j+1}^-$ for all $j \in \mathbb{N}_{p-1}$, i.e. the intervals

$$I_j = (\mu_j^+, \mu_{j+1}^-) \neq \varnothing, \qquad j \in \mathbb{N}_{p-1},$$

are spectral gaps of the unperturbed Schrödinger operator H_0.

Next we consider the perturbed Schrödinger operators $H_t = H_0 + tQ$, $t > 0$, on $\ell^2(\mathbb{Z})$ with a finitely supported potential Q:

$$Q(n) = q_n \geqslant 0, \quad n \in \mathbb{N}_p; \qquad Q(n) = 0, \quad \forall n \notin \mathbb{N}_p.$$

For each $t > 0$ the spectrum of H_t consists of the spectrum of the unperturbed operator H_0 and a finite number of eigenvalues with finite multiplicity. In the gaps of H_0 we will approximate the eigenvalues of H_t by eigenvalues of the Schrödinger operators $H_{t,n} = \Delta_n + W + tQ$, $n \in \mathbb{N}_0$, where Δ_n is the Laplacian on the graph $C_n = \mathbb{Z}/(2^n p\mathbb{Z})$. The graph C_n is the cycle with $2^n p$ vertices $1, 2 \ldots, 2^n p$. By Proposition 3.3, for each $(n,t) \in \mathbb{N}_0 \times \mathbb{R}_{\geqslant 0}$ the eigenvalues $\lambda_j(H_{t,n})$ of $H_{t,n}$ satisfy

$$\lambda_j(H_{t,n}^-) \leqslant \lambda_j(H_{t,n}) \leqslant \lambda_j(H_{t,n}^+), \qquad j = 1, \ldots, 2^n(p-1),$$

where the operators $H_{t,n}^\pm$ are defined similarly to $H_t^\pm$ but only on the approximating cycle graphs C_n. More precisely, let C_n^- be the graph obtained from the cycle graph C_n with $2^n p$ vertices $1, 2, \ldots, 2^n p$ by deleting 2^n edges

$$(p, p+1), \quad (2p, 2p+1), \quad (3p, 3p+1), \quad \ldots, \quad (2^n p, 1),$$

i.e. the graph C_n^- consists of 2^n connected components each of which is the path on p vertices. The operator $H_{t,n}^-$ is the Schrödinger operator with the potential $W + tQ$ on the graph C_n^- and is given by the $(2^n p) \times (2^n p)$-matrix

$$H_{t,n}^- = \begin{pmatrix} H_0^- + tQ & \mathbb{O}_p & \ldots & \mathbb{O}_p \\ \mathbb{O}_p & H_0^- & \ldots & \mathbb{O}_p \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{O}_p & \mathbb{O}_p & \ldots & H_0^- \end{pmatrix},$$

where H_0^- is defined by (5.1), and $\mathbb{O}_p$ is the zero $(p \times p)$-matrix. The operator $H_{t,n}^+$ is the Schrödinger operator with the potential $W + tQ$ on the graph C_n with Dirichlet boundary conditions at the vertices $p, 2p, 3p, \ldots, 2^n p$ and is given by the $(2^n(p-1) \times 2^n(p-1))$-matrix

$$H_{t,n}^+ = \begin{pmatrix} H_0^+ + tQ & \mathbb{O}_{p-1} & \ldots & \mathbb{O}_{p-1} \\ \mathbb{O}_{p-1} & H_0^+ & \ldots & \mathbb{O}_{p-1} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{O}_{p-1} & \mathbb{O}_{p-1} & \ldots & H_0^+ \end{pmatrix},$$

where H_0^+ is defined by (5.2).

Let $\operatorname{supp} Q = \{1, 2, \ldots, p\}$. Then, by Lemma 4.1, all eigenvalue branches $\mu_j^\pm(t)$ of the operator families $H_t^\pm = H_0^\pm + tQ$, $t \geqslant 0$, are strictly increasing functions of t and

$$\mu_j^\pm(\mathbb{R}_{\geqslant 0}) = [\mu_j^\pm, +\infty), \qquad \text{where} \qquad \mu_j^\pm = \mu_j^\pm(0).$$

By Theorem 2.1.*i*), in each gap I_k, $k \in \mathbb{N}_{p-1}$ of the unperturbed operator H_0 there exist k eigenvalue branches $\lambda_j^{(k)}(t)$, $j = 1, \ldots, k$. In the interval I_k each of these branches

$\lambda_j^{(k)}(t)$ lies between the eigenvalue branches $\mu_j^-(t)$ and $\mu_j^+(t)$ of the Schrödinger operators H_t^- and H_t^+.

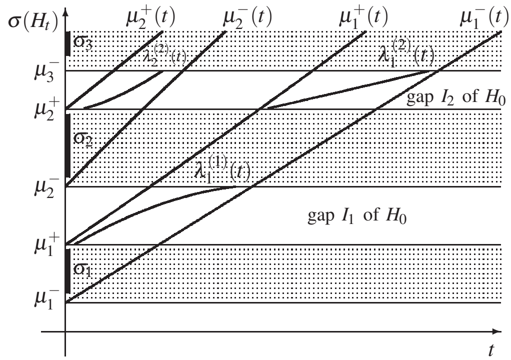

Figure 4: The numerically obtained dependence (for $p=3$, $w_n=n$ and $q_n=n$, $n=1,2,3$) of the eigenvalues $\lambda_1^{(1)}(t)$, $\lambda_1^{(2)}(t)$, $\lambda_2^{(2)}(t)$ of H_t on t (the eigenvalue branches) in the gaps I_1 and I_2 of H_0.

For example, if $p=3$ and $w_n=n$, $n=1,2,3$, then the spectrum of the unperturbed Schrödinger operator H_0 on $\mathbb{Z}$ has the form

$$\sigma(H_0)=\sigma_1\cup\sigma_2\cup\sigma_3=[\lambda_1^-,\lambda_1^+]\cup[\lambda_2^-,\lambda_2^+]\cup[\lambda_3^-,\lambda_3^+],$$

where

$$\lambda_1^-=\lambda_1(0)\approx 1{,}79;\qquad \lambda_2^-=\lambda_2(\pi)\approx 3{,}46;\qquad \lambda_3^-=\lambda_3(0)\approx 5{,}68;$$
$$\lambda_1^+=\lambda_1(\pi)\approx 2{,}32;\qquad \lambda_2^+=\lambda_2(0)\approx 4{,}54;\qquad \lambda_3^+=\lambda_3(\pi)\approx 6{,}21.$$

Eigenvalues μ_j^-, $j\in\mathbb{N}_3$, of H_0^- and μ_1^+,μ_2^+ of H_0^+ are given by

$$\mu_1^-\approx 1{,}52;\qquad \mu_2^-\approx 3{,}31;\qquad \mu_3^-\approx 5{,}17;\qquad \mu_1^+\approx 2{,}38;\qquad \mu_2^+\approx 4{,}62.$$

Thus, $\mu_1^+<\mu_2^-$ and $\mu_2^+<\mu_3^-$, and the intervals

$$I_1=(\mu_1^+,\mu_2^-)\neq\varnothing\qquad\text{and}\qquad I_2=(\mu_2^+,\mu_3^-)\neq\varnothing$$

are gaps in the spectrum of the unperturbed Schrödinger operator H_0 on $\mathbb{Z}$. The dependence of the eigenvalues of the perturbed Schrödinger operator $H_t=H_0+tQ$ on t in the gaps I_1 and I_2 obtained numerically as $Q(n)=n$, $n=1,2,3$, is schematically shown in Fig. 4.

5.2. Hexagonal lattice

We consider the hexagonal lattice $\mathbf{G}=(\mathscr{V},\mathscr{E})$, shown in Fig. 5*a*. The periods of $\mathbf{G}$ are the vectors a_1, a_2. The vertex set and the edge set are given by

$$\mathscr{V}=\mathbb{Z}^2\cup\big(\mathbb{Z}^2+\big(\tfrac{1}{3},\tfrac{1}{3}\big)\big),$$

$$\mathscr{E} = \Big\{ \big(\mathrm{m}, \mathrm{m} + \big(\tfrac{1}{3}, \tfrac{1}{3}\big)\big), \big(\mathrm{m}, \mathrm{m} + \big(-\tfrac{2}{3}, \tfrac{1}{3}\big)\big), \big(\mathrm{m}, \mathrm{m} + \big(\tfrac{1}{3}, -\tfrac{2}{3}\big)\big) \quad \forall \mathrm{m} \in \mathbb{Z}^2 \Big\}.$$

Recall that the coordinates of all vertices are taken with respect to the basis a_1, a_2. The quotient graph $\mathbf{G}_* = (\mathscr{V}_*, \mathscr{E}_*)$, where $\mathscr{V}_* = \{v_1, v_2\}$, consists of two vertices, *multiple* edges $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, all with initial and terminal vertices v_1 and v_2, respectively (Fig. 5*b*) with indices $\widetilde{\tau}(\mathbf{e}_1) = (0,0)$, $\widetilde{\tau}(\mathbf{e}_2) = (1,0)$, $\widetilde{\tau}(\mathbf{e}_3) = (0,1)$.

First we consider the unperturbed Schrödinger operator $H_0 = \Delta + W$ with a periodic (non-constant) potential W on $\mathbf{G}$. Without loss of generality (add a constant to W if necessary) we may assume that

$$W(v_1) = w, \qquad W(v_2) = -w, \qquad w > 0.$$

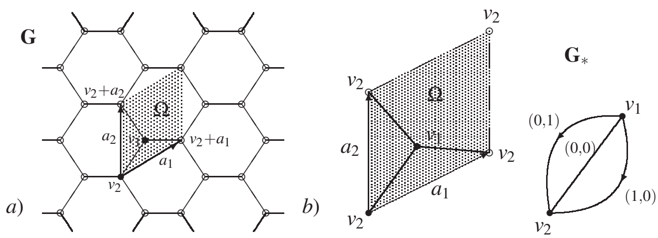

Figure 5: *a*) The hexagonal lattice $\mathbf{G}$; *b*) the quotient graph $\mathbf{G}_*$ with edge indices.

The fiber Schrödinger operator $\widetilde{H}_0(\vartheta)$, $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{T}^2$, defined by (1.4), (1.5), on $\mathbf{G}_*$ has the form

$$\widetilde{H}_0(\vartheta) = \begin{pmatrix} 3+w & -b(\vartheta) \\ -\overline{b}(\vartheta) & 3-w \end{pmatrix}, \qquad b(\vartheta) = 1 + e^{-i\vartheta_1} + e^{-i\vartheta_2}.$$

The eigenvalues of $\widetilde{H}_0(\vartheta)$ are given by

$$\lambda_n(\vartheta) = 3 + (-1)^n \sqrt{w^2 + |b(\vartheta)|^2}, \qquad n = 1, 2.$$

Then the spectrum of the unperturbed Schrödinger operator H_0 on $\mathbf{G}$ has the form (see Fig. 6)

$$\sigma(H_0) = \sigma_1 \cup \sigma_2 = [\lambda_1^-, \lambda_1^+] \cup [\lambda_2^-, \lambda_2^+],$$

where

$$\lambda_1^- = \lambda_1(0) = 3 - \sqrt{9+w^2}, \qquad \lambda_1^+ = \lambda_1\big(\tfrac{2\pi}{3}, -\tfrac{2\pi}{3}\big) = 3 - w,$$
$$\lambda_2^- = \lambda_2\big(\tfrac{2\pi}{3}, -\tfrac{2\pi}{3}\big) = 3 + w, \qquad \lambda_2^+ = \lambda_2(0) = 3 + \sqrt{9+w^2}.$$

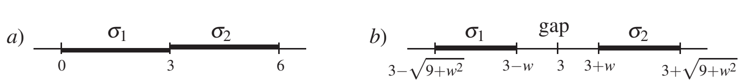

Figure 6: *a*) The spectrum of Δ; *b*) the spectrum of $H_0 = \Delta + W$.

Thus, the periodic potential W opens the gap $(3-w, 3+w)$ in the spectrum of the unperturbed Schrödinger operator $H_0 = \Delta + W$ on $\mathbf{G}$.

Now we show that for $w > 3/4$ the gap condition (1.14) is fulfilled. Let $\mathscr{V}_\mathfrak{b} = \{v_1\}$. Then the operators $H_0^\pm$ defined in Subsection 1.3 have the form

$$H_0^- = \begin{pmatrix} 1+w & -1 \\ -1 & 1-w \end{pmatrix}, \qquad H_0^+ = (3-w). \tag{5.3}$$

The eigenvalues μ_1^-, μ_2^- of H_0^- and μ_1^+ of H_0^+ are given by

$$\mu_j^- = 1 + (-1)^j \sqrt{1+w^2}, \quad j = 1,2, \qquad \mu_1^+ = 3-w.$$

If $w > 3/4$, then $\mu_1^+ < \mu_2^-$ and the interval $I_1 = (\mu_1^+, \mu_2^-) \neq \varnothing$ is a gap in the spectrum of the unperturbed operator H_0. The gap I_1 is not maximal, since it is strictly contained in the maximal gap $(\lambda_1^+, \lambda_2^-)$. More precisely, $\mu_1^+ = \lambda_1^+$, but $\mu_2^- < \lambda_2^-$ (see also Fig.8.b).

Note that if we choose $\mathscr{V}_\mathfrak{b} = \{v_2\}$, then $I_1 = \varnothing$ (for positive w).

Next we consider the perturbed Schrödinger operator $H_t = H_0 + tQ$, $t > 0$, with the guided potential Q satisfying the conditions

$$\operatorname{supp} Q \subset \mathbb{R} \times [0,1), \qquad Q(v + a_1) = Q(v), \qquad \forall v \in \mathscr{V},$$

see Fig. 8*a*.

Let Γ be the lattice generated by the vector a_1, and Γ_0 be the lattice generated by a_2. Due to Proposition 1.3, the operator H_t, $t > 0$, has the decomposition (1.8)–(1.9) into a constant fiber direct integral, where the fiber Schrödinger operator $H_t(\vartheta) = \Delta(\vartheta) + W + tQ$, $\vartheta \in \mathbb{T} = (-\pi, \pi]$, acts on the infinite fundamental graph $C = (\mathscr{V}_\mathrm{c}, \mathscr{E}_\mathrm{c}) = \mathbf{G}/\Gamma$ shown in Fig. 7*a* and the potential Q has finite support:

$$Q(v_j) = q_j \geqslant 0, \quad j = 1,2, \qquad Q(v) = 0, \quad \forall v \in \mathscr{V}_\mathrm{c} \setminus \{v_1, v_2\}.$$

The sequence of the approximating graphs $C_n = C/(2^n \Gamma_0)$, $n \in \mathbb{N}_0$, of the discrete cylinder C is shown in Fig. 7*b*.

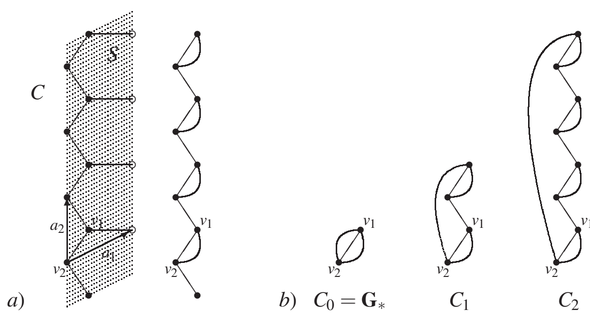

Figure 7: *a*) The discrete cylinder $C = \mathbf{G}/\Gamma$ (the vertices on the left and right side of the strip $S = [0,1) \times \mathbb{R}$ are identified); *b*) the sequence of its approximating graphs $C_0 = \mathbf{G}_*, C_1, C_2, \ldots$.

For each $(\vartheta,n,t)\in\mathbb{T}\times\mathbb{N}_0\times\mathbb{R}_{\geqslant 0}$ the operator $H_{t,n}(\vartheta)$ defined by (3.6), (3.7), acts on the finite graph $C_n=(\mathscr{V}_n,\mathscr{E}_n)$ and, by Proposition 3.3, its eigenvalues $\lambda_j\big(H_{t,n}(\vartheta)\big)$ satisfy

$$\lambda_j(H_{t,n}^-)\leqslant\lambda_j\big(H_{t,n}(\vartheta)\big)\leqslant\lambda_j(H_{t,n}^+),\qquad j=1,\dots,2^n,$$

where

$$H_{t,n}^{\pm}=H_t^{\pm}\oplus\underbrace{H_0^{\pm}\oplus\ldots\oplus H_0^{\pm}}_{2^n-1},\qquad H_t^{\pm}=H_0^{\pm}+tQ,$$

and $H_0^{\pm}$ are given by (5.3).

We assume that $w>3/4$, i.e., the interval $I_1=(\mu_1^+,\mu_2^-)\neq\varnothing$ is a gap in the spectrum of H_0, and let $\lambda\in I_1$.

Case 1. Let $\operatorname{supp}Q\restriction_\Omega=\{v_1,v_2\}$, see Fig. 8*a*. Then, by Theorem 2.1.*i*), there exists a guided band branch $\mathfrak{s}_1(t)$ of the operator family $H_t=H_0+tQ$, $t>0$, crossing the level λ, and in I_1 this branch satisfies

$$\mathfrak{s}_1(t)\subset\big[\mu_1^-(t),\mu_1^+(t)\big],$$

where $\mu_1^{\pm}(t)$ are the eigenvalue branches of the operator families $H_t^{\pm}$, $t\geqslant 0$. The dependence of the guided spectrum of H_t on t in the gap I_1 is shown in Fig. 8*b*. Note that in this case both eigenvalue branches $\mu_1^{\pm}(t)$ cross the level μ_2^-.

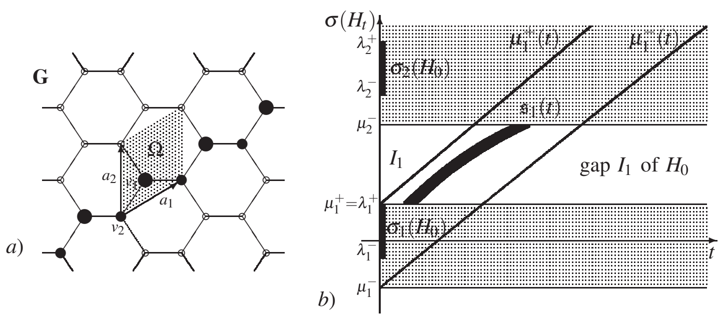

Figure 8: *a*) The hexagonal lattice $\mathbf{G}$; the fundamental cell Ω is shaded; the support of the guided potential Q is shown by black vertices; *b*) the numerically obtained dependence (for $w=2$, $q_1=3$, $q_2=1$) of the guided spectrum of H_t on t (the guided band branch $\mathfrak{s}_1(t)$) in the gap I_1.

Case 2. Let $\operatorname{supp}Q\restriction_\Omega=\{v_1\}$, see Fig. 9*a*. Since $\{v_1\}=\mathscr{V}_{\mathfrak{b}}$, then, by Theorem 2.3, for each $t>0$ the guided Schrödinger operator $H_t=H_0+tQ$ has no guided spectrum in the gap I_1 of H_0, i.e., I_1 is also a gap of H_t for each $t>0$ (see Fig. 9*b*). Note that in this case the eigenvalue branch $\mu_1^+(t)$ is constant. Thus, neither $\mu_1^+(t)$ nor $\mu_1^-(t)$ appears in the gap I_1.

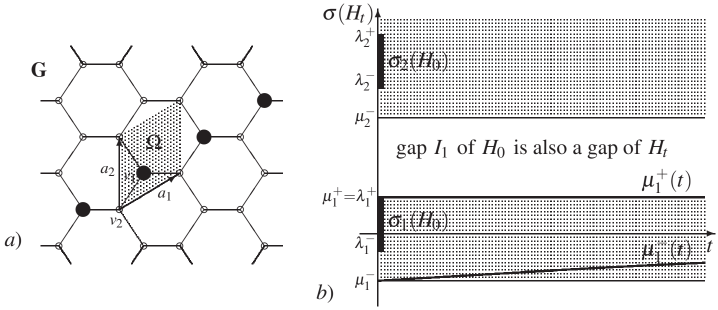

Figure 9: *a*) The hexagonal lattice $\mathbf{G}$; the fundamental cell Ω is shaded; the support of the guided potential Q is shown by black vertices; *b*) for each $t > 0$ there is no guided spectrum of H_t in I_1 ($w = 2$, $q_1 = 3$, $q_2 = 0$).

Case 3. Finally, let $\operatorname{supp} Q\restriction_\Omega = \{v_2\}$, see Fig. 10*a*. The dependence of the guided spectrum of H_t on t in the gap I_1 is shown in Fig. 10*b*. There exists exactly one guided band branch $\mathfrak{s}_1(t)$ of the operator family $H_t = H_0 + tQ$, $t > 0$, crossing the level λ, and in I_1 this branch also satisfies

$$\mathfrak{s}_1(t) \subset \left[\mu_1^-(t), \mu_1^+(t)\right].$$

In this case only the eigenvalue branch $\mu_1^+(t)$ crosses the level μ_2^-. The eigenvalue branch $\mu_1^-(t)$ stays below this level.

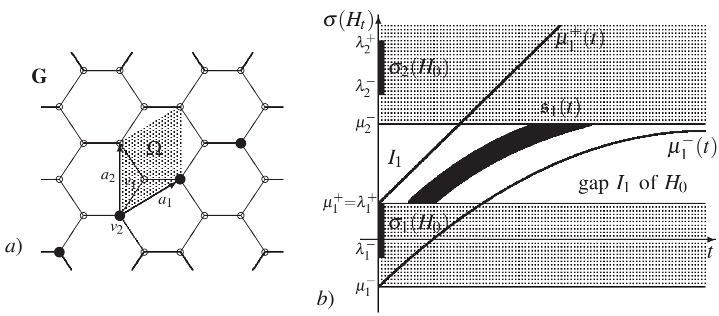

Figure 10: *a*) The hexagonal lattice $\mathbf{G}$; the fundamental cell Ω is shaded; the support of the guided potential Q is shown by black vertices; *b*) the numerically obtained dependence (for $w = 2$, $q_1 = 0$, $q_2 = 1$) of the guided spectrum of H_t on t (the guided band branch $\mathfrak{s}_1(t)$) in the gap I_1.

REMARK 5.1. i) Case 3 is not covered by Theorems 2.1 and 2.3. The guided spectrum of H_t in the gap I_1 and the eigenvalue branches $\mu_1^\pm(t)$ of $H_t^\pm$ schematically shown in Fig.8*b* – Fig.10*b* were obtained numerically as $w = 2$, $q_1 = 3$ and $q_2 = 1$ in Case 1, $q_1 = 3$, $q_2 = 0$ in Case 2, and $q_1 = 0$, $q_2 = 1$ in Case 3.

ii) At the first glance, Case 2 and Case 3 can be seen to be equivalent by some symmetry, so how can the guided spectrum be so different in these cases. This difference is explained by the presence of a periodic potential W. For example, if we change the value w of the periodic potential by $-w$ in Case 3, we obtain the similar behavior of the guided spectrum as in Case 2.

5.3. Square lattice

We consider the square lattice $\mathbb{L}^2$ with periods a_1, a_2, see Fig. 11*a*. The quotient graph $\mathbb{L}^2_* = (\mathscr{V}_*, \mathscr{E}_*)$, where $\mathscr{V}_* = \{v_1, v_2, v_3, v_4\}$, consists of 4 vertices and 8 edges (Fig. 11*b*).

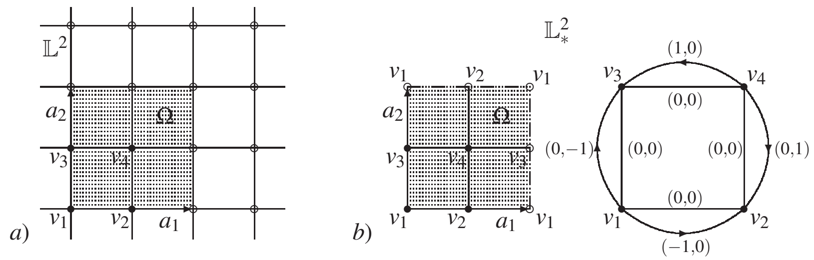

Figure 11: *a*) The square lattice $\mathbb{L}^2$; the fundamental cell Ω is shaded; *b*) the quotient graph $\mathbb{L}^2_*$ with edge indices.

Let $\mathscr{V}_\mathfrak{b} = \{v_1, v_4\}$. Then the operators $H_0^\pm$ defined in Subsection 1.3 have the form

$$H_0^- = \begin{pmatrix} 2+w_1 & -1 & -1 & 0 \\ -1 & 2+w_2 & 0 & -1 \\ -1 & 0 & 2+w_3 & -1 \\ 0 & -1 & -1 & 2+w_4 \end{pmatrix}, \quad H_0^+ = \begin{pmatrix} 4+w_2 & 0 \\ 0 & 4+w_3 \end{pmatrix},$$

where $w_j = W(v_j)$, $j \in \mathbb{N}_4$. Let

$$w_1 = 8, \qquad w_2 = 0, \qquad w_3 = 4, \qquad w_4 = 10.$$

Then the eigenvalues μ_j^-, $j \in \mathbb{N}_4$, of H_0^- and μ_1^+, μ_2^+ of H_0^+ are given by

$$\mu_1^- \approx 1{,}77; \quad \mu_2^- \approx 5{,}65; \quad \mu_3^- \approx 10{,}30; \quad \mu_4^- \approx 12{,}29; \quad \mu_1^+ = 4; \quad \mu_2^+ = 8.$$

Thus, $\mu_1^+ < \mu_2^-$ and $\mu_2^+ < \mu_3^-$, and the intervals

$$I_1 = (\mu_1^+, \mu_2^-) \neq \varnothing \qquad \text{and} \qquad I_2 = (\mu_2^+, \mu_3^-) \neq \varnothing$$

are gaps in the spectrum of the unperturbed Schrödinger operator $H_0 = \Delta + W$ on $\mathbb{L}^2$. Let $\lambda_1 \in I_1$ and $\lambda_2 \in I_2$.

Case 1. We consider a guided potential Q on $\mathbb{L}^2$ such that $\operatorname{supp} Q\upharpoonright_\Omega = \mathscr{V}_*$, see Fig. 12*a*. Then, by Theorem 2.1.*i*), there exist a guided band branch $\mathfrak{s}_1^{(1)}(t)$ of the operator family $H_t = H_0 + tQ$, $t > 0$, crossing the level λ_1, and guided band branches $\mathfrak{s}_1^{(2)}(t)$ and $\mathfrak{s}_2^{(2)}(t)$ of H_t, $t > 0$, crossing the level λ_2, and in I_1 and I_2 these branches satisfy

$$\mathfrak{s}_1^{(1)}(t) \subset \big[\mu_1^-(t), \mu_1^+(t)\big], \qquad \mathfrak{s}_j^{(2)}(t) \subset \big[\mu_j^-(t), \mu_j^+(t)\big], \qquad j = 1,2,$$

where $\mu_j^\pm(t)$ are the eigenvalue branches of the operator families $H_t^\pm$, $t \geqslant 0$. The dependence of the guided spectrum of H_t on t in the gaps I_1 and I_2 is shown in Fig. 12*b*.

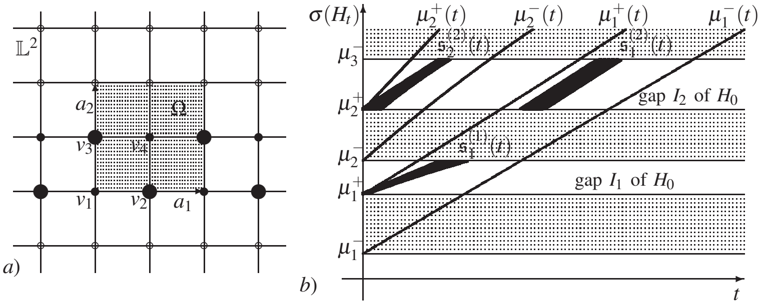

Figure 12: *a*) The square lattice $\mathbb{L}^2$; the fundamental cell Ω is shaded; the support of the guided potential Q is shown by black vertices; *b*) the numerically obtained dependence (for $w_1 = 8$, $w_2 = 0$, $w_3 = 4$, $w_4 = 10$ and $q_n = n$, $n \in \mathbb{N}_4$) of the guided spectrum of H_t on t (the guided band branches) in the gaps I_1 and I_2.

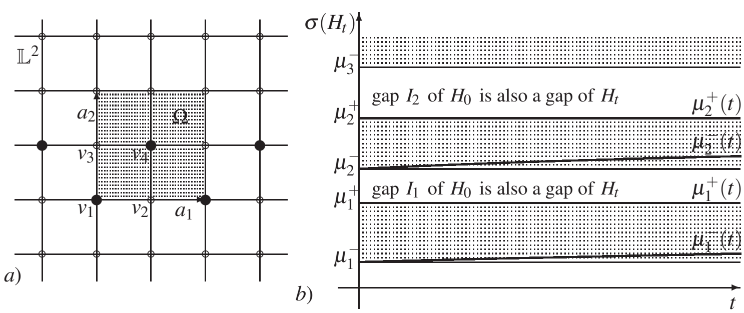

Figure 13: *a*) The square lattice $\mathbb{L}^2$; the fundamental cell Ω is shaded; the support of the guided potential Q is shown by black vertices; *b*) for each $t > 0$ there is no guided spectrum of H_t in I_1 and I_2 ($w_1 = 8$, $w_2 = 0$, $w_3 = 4$, $w_4 = 10$ and $q_1 = 1$, $q_2 = q_3 = 0$, $q_4 = 4$).

Case 2. Let $\operatorname{supp} Q\restriction_{\Omega} = \{v_1, v_4\}$, see Fig. 13*a*. Since $\mathscr{V}_{\mathfrak{b}} = \{v_1, v_4\}$, then, by Theorem 2.3, for each $t > 0$ the guided Schrödinger operator $H_t = H_0 + tQ$ has no guided spectrum in the gaps I_1 and I_2 of H_0, i.e., I_1 and I_2 are also gaps of H_t for each $t > 0$ (see Fig. 13*b*). Note that in this case the eigenvalue branches $\mu_1^+(t)$ and $\mu_2^+(t)$ are constant. Thus, no eigenvalue branches $\mu_j^\pm(t)$ of $H_t^\pm$ appear in the gaps I_1 and I_2.

Case 3. Finally, let $\operatorname{supp} Q\restriction_{\Omega} = \{v_2, v_3\}$, see Fig. 14*a*. The dependence of the guided spectrum of H_t on t in the gaps I_1 and I_2 is shown in Fig. 14*b*. There exist exactly one guided band branch $\mathfrak{s}_1^{(1)}(t)$ of the operator family $H_t = H_0 + tQ$, $t > 0$, crossing the level λ_1, and two guided band branches $\mathfrak{s}_1^{(2)}(t)$ and $\mathfrak{s}_2^{(2)}(t)$ of H_t, $t > 0$, crossing the level λ_2, and in I_1 and I_2 these branches also satisfy

$$\mathfrak{s}_1^{(1)}(t) \subset \big[\mu_1^-(t), \mu_1^+(t)\big], \qquad \mathfrak{s}_j^{(2)}(t) \subset \big[\mu_j^-(t), \mu_j^+(t)\big], \qquad j = 1, 2.$$

In this case the eigenvalue branch $\mu_1^-(t)$ stays below the level μ_3^-.

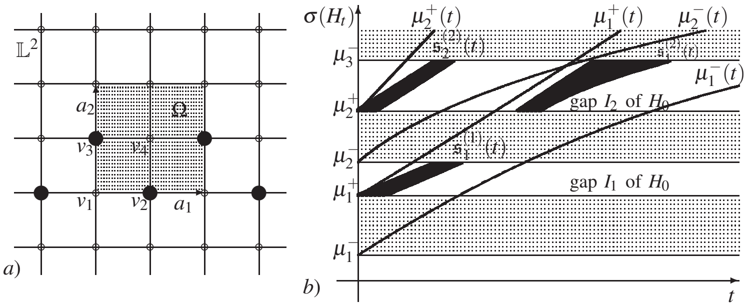

Figure 14: *a*) The square lattice $\mathbb{L}^2$; the fundamental cell Ω is shaded; the support of the guided potential Q is shown by black vertices; *b*) the numerically obtained dependence (for $w_1 = 8$, $w_2 = 0$, $w_3 = 4$, $w_4 = 10$ and $q_1 = q_4 = 0$, $q_2 = 2$, $q_3 = 3$) of the guided spectrum of H_t on t (the guided band branches) in the gaps I_1 and I_2.

REMARK 5.2. Case 3 is not covered by Theorems 2.1 and 2.3. The guided spectrum of H_t in the gaps I_1 and I_2 and the eigenvalue branches $\mu_j^\pm(t)$, $j = 1, 2$, of $H_t^\pm$ schematically shown in Fig.12*b* – Fig.14*b* were obtained numerically as $q_n = n$, $n \in \mathbb{N}_4$, in Case 1; $q_1 = 1$, $q_2 = q_3 = 0$, $q_4 = 4$ in Case 2; and $q_1 = q_4 = 0$, $q_2 = 2$, $q_3 = 3$ in Case 3, where $q_n = Q(v_n)$, $n \in \mathbb{N}_4$.

A. Appendix: well-known properties of self-adjoint operators

In this section we formulate some well-known properties of self-adjoint operators needed to prove our results.

PROPOSITION A.1. *i) Let A_n be a sequence of uniformly bounded operators. If $A_n\psi \to A\psi$ for ψ in some dense subspace, then A_n converges to A strongly (see, Lemma 1.14 in* [39]*, pp.50–51).*

ii) Let A and A_n be bounded operators, and let C be a compact operator. If A_n converges to A strongly, then A_nC converges to AC in norm (see, e.g., Problem 32 in [6]*, p.91).*

iii) Let A and A_n be bounded self-adjoint operators. If A_n converges to A strongly, then $\sigma(A) \subset \lim_{n\to\infty} \sigma(A_n)$. If A_n converges to A in norm, then $\sigma(A) = \lim_{n\to\infty} \sigma(A_n)$ (see, e.g., Theorem 6.38 in [39]*, p.156).*

iv) Let A and A_n be bounded self-adjoint operators and A_n converges to A strongly. Then if $a,b \in \mathbb{R}$, $a<b$, and $(a,b)\cap\sigma(A_n) = \varnothing$ for all n, then $(a,b)\cap\sigma(A) = \varnothing$ (see, e.g., Theorem VIII.24 in [37]*, p.290).*

v) `The Birman-Schwinger principle`*. Let H_0 be a self-adjoint operator, and $\lambda \in \mathbb{R}\setminus\sigma(H_0)$. Suppose $Q \geqslant 0$ is a bounded operator with $Q^{1/2}(H_0-\lambda)^{-1}$ compact. Then the Birman-Schwinger kernel $K := Q^{1/2}(H_0-\lambda)^{-1}Q^{1/2}$ is compact and the following are equivalent:*

1) λ is an eigenvalue of $H_0 - tQ$ of multiplicity m;

2) t^{-1} is an eigenvalue of K of multiplicity m

(see, e.g., Proposition 1.5 in [5]*, p.63).*

vi) Let A, B be self-adjoint $(p\times p)$-matrices. Then for each $n\in\mathbb{N}_p$ we have

$$\lambda_n(A)+\lambda_1(B) \leqslant \lambda_n(A+B) \leqslant \lambda_n(A)+\lambda_p(B),$$

where $\lambda_1(A)\leqslant\ldots\leqslant\lambda_p(A)$ are the eigenvalues of A arranged in non-decreasing order, counting multiplicities (see Theorem 4.3.1 in [22]*, p.181).*

REFERENCES

[1] S. ALAMA, P.A. DEIFT, R. HEMPEL, *Eigenvalue branches of the Schrödinger operator $H-\lambda W$ in a spectral gap of H*, Comm. Math. Phys. **121** (1989), 291–321.

[2] K. ANDO, *Inverse scattering theory for discrete Schrödinger operators on the hexagonal lattice*, Ann. Henri Poincaré **14** (2013), 347–383.

[3] K. ANDO, H. ISOZAKI, H. MORIOKA, *Spectral properties of Schrödinger operators on perturbed lattices*, Ann. Henri Poincaré **17** (2016), 2103–2171.

[4] J. ARAZY, L. ZELENKO, *Finite-dimensional perturbations of self-adjoint operators*, Integral Equations Operator Theory **34**, 2 (1999), 127–164.

[5] M. SH. BIRMAN, *Discrete spectrum in the gaps of a continuous one for perturbations with large coupling constant*, Adv. Soviet Math. **7** (1991), 57–73.

[6] J. BLANK, P. EXNER, M. HAVLICEK, *Hilbert Space Operators in Quantum Physics*, Springer Science and Business Media B.V., 2008.

[7] A. H. CASTRO NETO, F. GUINEA, N. M. R. PERES, K.S. NOVOSELOV, A. GEIM, *The electronic properties of graphene*, Rev. Mod. Phys. **81** (2009), 109–162.

[8] P. A. DEIFT, R.HEMPEL, *On the existence of eigenvalues of the Schrödinger operator $H-\lambda W$ in a gap of $\sigma(H)$*, Comm. Math. Phys. **103** (1986), 461–490.
[9] J. S. FABILA-CARRASCO, F. LLEDÓ, O. POST, *Spectral gaps and discrete magnetic Laplacians*, Linear Algebra Appl. **547**, 15 (2018), 183–216.
[10] R. HALIR, P.J. BOCK, P. CHEBEN, A. ORTEGA-MOÑUX, C. ALONSO-RAMOS, J.H. SCHMID, J. LAPOINTE, D.-X. XU, J. G. WANGUEMERT-PEREZ, I. MOLINA-FERNANDEZ, S. JANZ, *Waveguide sub-wavelength structures: a review of principles and applications*, Laser Photon. Rev. **9** (2015), 25–49.
[11] P. HARRIS, *Carbon nano-tubes and related structure*, Cambridge, Cambridge University Press, 2002.
[12] W.A. HARRISON, *Electronic Structure and the Properties of Solids: The Physics of the Chemical Bond*, Dover Publications, Inc., New York, 1989.
[13] R. HEMPEL, *Eigenvalues in gaps and decoupling by Neumann boundary conditions*, J. Math. Anal. Appl. **169** (1992), 229–259.
[14] R. HEMPEL, *Eigenvalues of Schrödinger operators in gaps of the essential spectrum – an overview*, Contemp. Math. **458** (2008), 393–407.
[15] Y. HIGUCHI, T. MATSUMOTO, O. OGURISU, *On the spectrum of a discrete Laplacian on $\mathbb{Z}$ with finitely supported potential*, Linear and multilinear algebra **59**, 8 (2011), 917–927.
[16] Y. HIGUCHI, Y. NOMURA, *Spectral structure of the Laplacian on a covering graph*, European J. Combin. **30**, 2 (2009), 570–585.
[17] Y. HIGUCHI, T. SHIRAI, *The spectrum of magnetic Schrödinger operators on a graph with periodic structure*, J. Funct. Anal. **169** (1999), 456–480.
[18] Y. HIGUCHI, T. SHIRAI, *A remark on the spectrum of magnetic Laplacian on a graph*, the proceedings of TGT10, Yokohama Math. J. **47** (1999), Special issue, 129–142.
[19] Y. HIGUCHI, T. SHIRAI, *Weak Bloch property for discrete magnetic Schrödinger operators*, Nagoya Math J. **161** (2001), 127–154.
[20] Y. HIGUCHI, T. SHIRAI, *Some spectral and geometric properties for infinite graphs*, AMS Contemp. Math. **347** (2004), 29–56.
[21] F. HIROSHIMA, I. SASAKI, T. SHIRAI, A. SUZUKI, *Note on the spectrum of discrete Schrödinger operators*, J. Math-for-Industry **4** (2012), 105–108.
[22] R. HORN, C. JOHNSON, *Matrix analysis*, Cambridge University Press, 1985.
[23] H. ISOZAKI, E. KOROTYAEV, *Inverse problems, trace formulae for discrete Schrödinger operators*, Ann. Henri Poincaré **13** (2012), 751–788.
[24] S. G. JOHNSON, P. R. VILLENEUVE, S. FAN, J. D.JOANNOPOULOS, *Linear waveguides in photonic crystal slabs*, Phys. Rev. B **62** (2000), 8212–8222.
[25] S. G. JOHNSON, J. D. JOANNOPOULOS, *Photonic crystals. The road from theory to practice*, Springer US, 2002.
[26] E. KOROTYAEV, N. SABUROVA, *Schrödinger operators on periodic discrete graphs*, J. Math. Anal. Appl. **420**, 1 (2014), 576–611.
[27] E. KOROTYAEV, N. SABUROVA, *Spectral band localization for Schrödinger operators on periodic graphs*, Proc. Amer. Math. Soc. **143** (2015), 3951–3967.
[28] E. KOROTYAEV, N. SABUROVA, *Schrödinger operators with guided potentials on periodic graphs*, Proc. Amer. Math. Soc. **145**, 11 (2017), 4869–4883.
[29] E. KOROTYAEV, N. SABUROVA, *Magnetic Schrödinger operators on periodic discrete graphs*, J. Funct. Anal., **272** (2017), 1625–1660.
[30] E. KOROTYAEV, N. SABUROVA, *Invariants for Laplacians on periodic graphs*, Math. Ann., **377** (2020), 723–758.
[31] A. KUTSENKO, *Wave propagation through periodic lattice with defects*, Comput. Mech. **54** (2014), 1559–1568.
[32] A. KUTSENKO, *Algebra of multidimensional periodic operators with defects*, J. Math. Anal. Appl. **428**, 1 (2015), 217–226.
[33] A. KUTSENKO, *Algebra of 2D periodic operators with local and perpendicular defects*, J. Math. Anal. Appl. **442**, 2 (2016), 796–803.
[34] K.S. NOVOSELOV, A.K. GEIM ET AL, *Electric field effect in atomically thin carbon films*, Science **306**, 5696 (2004), 666–669.
[35] G.G. OSHAROVICH, M.V. AYZENBERG-STEPANENKO, *Wave localization in stratified square-cell lattices: The antiplane problem*, J. Sound Vib. **331** (2012), 1378–1397.

[36] V.S. RABINOVICH, S. ROCH, *Essential spectra of difference operators on $\mathbb{Z}^n$-periodic graphs*, J. Phys. A **40**, 33 (2007), 10109–10128.
[37] M. REED, B. SIMON, *Methods of modern mathematical physics, vol.I. Functional analysis*, Academic Press, New York, 1980.
[38] M. REED, B. SIMON, *Methods of modern mathematical physics, vol.IV. Analysis of operators*, Academic Press, New York, 1978.
[39] G. TESCHL, *Mathematical Methods in Quantum Mechanics with Applications to Schrödinger Operators*, American Mathematical Society, Graduate Studies in Mathematics, Volume 99, 2009.

(Received August 6, 2020)

Olaf Post
Fachbereich 4 – Mathematik
Universität Trier
54286 Trier, Germany
e-mail: olaf.post@uni-trier.de

Natalia Saburova
Department of Mathematical Analysis, Algebra and Geometry
Northern (Arctic) Federal University
Severnaya Dvina Emb. 17, Arkhangelsk, 163002, Russia
e-mail: n.saburova@narfu.ru