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SPECTRAL OPTIMIZATION FOR SINGULAR SCHRÖDINGER OPERATORS

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Abstract. For several classes of singular Schrödinger operators which can be formally written as $-\Delta-\alpha\delta(x-\Gamma)$ we discuss the problem of optimization of their principal eigenvalue with respect to the shape of the interaction support Γ.

1. Introduction

Search for a shape that optimizes a given spectral quantity is a trademark question in spectral geometry which has a long history that can be traced back at least to the famous Faber and Krahn proof [24, 27] of the lowest tone conjecture of Lord Rayleigh; this example also illustrates that quite often the optimal shape exhibits a rotational symmetry. It is not just the property of the lowest eigenvalue, as an example one recall the Payne-Pólya-Weinberger inequality the proof of which by Ashbaugh and Benguria was a real *tour de force* [5, 6]. We are not going to describe the history here, however, and refer to the nice lecture series [10].

Our aim in this review is different, we are going to discuss some more recent results concerning optimization of several classes of singular Schrödinger operators which can be formally written as

$$H_{\alpha,\Gamma}=-\Delta-\alpha\delta(x-\Gamma), \quad \alpha>0, \tag{1}$$

where Γ is a manifold or a more general subset of $\mathbb{R}^d$; we shall focus at that at the principal eigenvalues of these operators. There are two main reasons why this problem is of interest. First of all, they are attractive mathematically in view of the relations between spectral properties reflect and the geometry of Γ in a sense wider than the topic of this review. At the same time, they represent an alternative to the conventional theory of quantum graphs [11] over which it has the advantage that the quantum tunneling between edges of such a graph is not neglected – for introduction to such *leaky quantum graphs* and a bibliography we refer to [21, Chap. 10]. Here we will be concerned primarily with situations where Γ is a manifold or a complex of codimension one, however, we shall also say something about more singular interactions, either of δ' type, or those with support of codimension two.

To put our problem in context, let us first mention several related ones. The closest to Faber and Krahn is the question about the shape that optimizes the ground state

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eigenvalue of the *Robin Laplacian* on a region $\Omega \subset \mathrm{R}^d$ which is the self-adjoint operator associated with the quadratic form

$$\psi \mapsto \int_\Omega |\nabla\psi(x)|^2 \mathrm{d}x - \alpha \int_{\partial\Omega} |\psi(s)|^2 \mathrm{d}s$$

on $H^1(\Omega)$. The sign convention is chosen in accordance with the following discussion, and we can naturally exclude the Neumann case, $\alpha = 0$, which is trivial from the present point of view. As long as $\alpha < 0$ the result is similar to that of Faber and Krahn: the principal eigenvalue $\lambda_1^\alpha(\Omega)$ is uniquely minimized among the sets of the same volume by $\lambda_1^\alpha(\mathscr{B})$ where $\mathscr{B}$ is a ball.

The situation is not that simple, though. Bareket conjectured that in the case of an attractive Robin boundary, $\alpha > 0$, the opposite inequality is valid, namely that $\lambda_1^\alpha(\mathscr{B})$ is now *maximal* among the ground state energies for sets of the same volume [8]. This is true for local deformations of a ball, but fails globally: Freitas and Krejčiřík showed that $\lambda_1^\alpha(\Omega) > \lambda_1^\alpha(\mathscr{B})$ may hold if Ω is a *spherical shell* [25]. On the other hand, we note that the analogous inequality does hold if we compare sets of the same *perimeter* [4]. Furthermore, in two dimensions $\lambda_1^\alpha(\Omega) \geqslant \lambda_1^\alpha(\mathscr{B})$ holds if Ω is the exterior of a convex set of the same area/perimeter as $\mathscr{B}$ [28], and under additional geometrical constraints the result extends to non-convex domains and higher dimensions [29].

Moreover, even when the boundary is Dirichlet, the analogue of Faber-Krahn inequality may not be valid because the topology of Ω may change the situation. Let us illustrate this claim on a pair of examples. If we seek extremum among 'fat loops' Ω, i.e. strips of a fixed width built around a loop of fixed length, we find that a circular annulus sharply *maximizes* $\lambda_1^\alpha(\Omega)$ [16]. The analogous result holds for the optimal position of a circular obstacle in a circular cavity: the ground state energy is again *maximal* when the obstacle is placed in the center of the cavity [26].

2. Manifolds without boundary

Before coming to the optimization we have to define properly the operator referring to the formal expression (1) assuming that $\operatorname{codim}\Gamma = 1$. As in the case of Robin billiards, a natural way to define our singular Schrödinger operators is to employ the appropriate quadratic form, namely

$$q_{\delta,\alpha}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \alpha\|\psi|_\Gamma\|^2_{L^2(\Gamma)} \tag{2}$$

with the domain $H^1(\mathbb{R}^d)$ and to use the first representation theorem. The advantage of this approach is that it requires a weak regularity only; it is sufficient to assume that Γ is Lipschitz [9]. If it is a smooth manifold one can easily check that the self-adjoint operator defined in this way, denoted as $H_{\alpha,\Gamma}$ or sometimes also $-\Delta_{\delta,\alpha}$, can alternatively be characterized by boundary conditions: it acts as $-\Delta$ on functions from $H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial\psi}{\partial n}(x)\bigg|_+ - \frac{\partial\psi}{\partial n}(x)\bigg|_- = -\alpha\psi(x);$$

this explains the formal expression as describing the attractive δ-interaction of strength α perpendicular to Γ at the point x. The derivatives here are taken in the same direction; sometimes one uses the convention in which they are taken with respect to the outer normals to the two regions of which Γ is the common boundary, in that case the expression at the left-hand side of the last formula is the sum instead of the difference. We note also that the definition easily extends to the situation where the coupling strength is position dependent characterized by a function $\Gamma \ni x \mapsto \alpha(x)$, however, in this review we consider a constant α only.

Let us now turn to our main task. Consider first the situation where Γ is a *loop* in $\mathbb{R}^d$, $d \geqslant 2$, parametrized by its arc length, i.e. a piecewise differentiable function $\Gamma: [0,L] \to \mathbb{R}^d$ such that $\Gamma(0) = \Gamma(L)$ and $|\dot{\Gamma}(s)| = 1$ for all but finitely many $s \in [0,L]$. Since the definition of $H_{\alpha,\Gamma}$ using the form (2) requires $\operatorname{codim}\Gamma = 1$, we consider loops in the plane, $d = 2$, and denote by $\lambda_1(\alpha,\Gamma)$ the principal eigenvalue of the operator for which we can make the following claim [17]:

THEOREM 1. *For any $\alpha > 0$ and $L > 0$ we have $\lambda_1(\alpha,\Gamma) \leqslant \lambda_1(\alpha,\mathscr{C})$, where $\mathscr{C}$ is a* circle of perimeter L, *the inequality being sharp unless Γ is congruent with $\mathscr{C}$.*

Proof sketch. One employs the generalized Birman-Schwinger principle [12] by which there is one-to-one correspondence between eigenvalues $-\kappa^2$ of $H_{\alpha,\Gamma}$ and solutions to the integral-operator equation

$$\mathscr{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \text{where } \mathscr{R}^{\kappa}_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s) - \Gamma(s')|)$$

on $L^2([0,L])$, where K_0 is the Macdonald function coming from the resolvent kernel of the two-dimensional Laplacian. The question can be reduced to a purely geometric problem, specifically to inequalities on mean values of chords,

$$\int_0^L |\Gamma(s+u) - \Gamma(s)|^p \,\mathrm{d}s \leqslant \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L}\,, \quad p > 0,\ u \in \left(0, \tfrac{1}{2}L\right]\,, \tag{3}$$

which we label as $C_L^p(u)$; the expression at the right-hand side is nothing but the value of the integral for $\Gamma = \mathscr{C}$. They may not hold for all $p > 0$, however, a simple Fourier analysis allows one to demonstrate the following result [17]:

PROPOSITION 1. *$C_L^2(u)$ is valid for any $u \in (0, \frac{1}{2}L]$, and the inequality is strict unless Γ is a planar circle; by convexity the same is true for all $p < 2$.*

Proof. Using a variational argument together with the fact that the function $K_0(\cdot)$ appearing in the resolvent kernel is strictly monotonous and convex the optimization problem for $\mathscr{R}^{\kappa}_{\alpha,\Gamma}$ can be reduced to the inequality $C_L^1(u)$ being thus proved. □

REMARK 1. The inequalities $C_L^p(u)$ hold also for $p \in [-2,0)$, however, in the reversed sense. Taking $p = -1$, for instance, we can infer from here that a charged and ideally flexible loop in the absence of gravity takes a circular form.

The above result also has a discrete analogue. Consider the same loop and place at it *point interactions* at the arc distances $\frac{jL}{N}$, $j=0,\dots,N_1$, in other words, the formal Hamiltonian

$$H^N_{\alpha,\Gamma} = -\Delta + \tilde{\alpha}\sum_{j=0}^{N-1}\delta\left(x-\Gamma\left(\tfrac{jL}{N}\right)\right) \tag{4}$$

in $L^2(\mathbb{R}^d)$, $d=2,3$. The interaction term is more singular than the one in (1) and has to be properly defined. To this aim, one introduces the generalized boundary values which are the coefficients in the expansion of functions from the domain of the adjoint operator H_Y^*, where Y is a shorthand for the interaction support and H_Y is the Laplacian restricted to functions vanishing at the vicinity of the points of Y. It is well known that the said expansions around points $y_j \in Y$ look as follows,

$$\psi(x) = -\frac{1}{2\pi}\log|x-y_j|\,L_0(\psi,y_j) + L_1(\psi,y_j) + \mathcal{O}(|x-y_j|),\ \ d=2, \tag{5a}$$

$$\psi(x) = \frac{1}{4\pi|x-y_j|}L_0(\psi,y_j) + L_1(\psi,y_j) + \mathcal{O}(|x-y_j|),\ \ d=3. \tag{5b}$$

Local self-adjoint extensions which give the meaning to the formal operator (4) are then given by

$$L_1(\psi,y_j) - \alpha L_0(\psi,y_j) = 0, \quad \alpha\in\mathbb{R}, \tag{6}$$

where α, in contrast to the $\tilde{\alpha}$ in (4), is the 'true' parameter characterizing the coupling strength; for details and other properties of point interactions we refer to [2].

We are again interested in the shape of Γ which *maximizes* the ground state energy provided, of course, that the discrete spectrum of $H^N_{\alpha,\Gamma}$ is nonempty. This requirement is nontrivial for $d=3$: there is an α_{crit} depending on the geometry of the interaction support such that $\sigma(H^N_{\alpha,\Gamma}) \neq \emptyset$ holds for $\alpha < \alpha_{\mathrm{crit}}$. In the described situation one can modify the method which led to Theorem 1 to obtain the following result [14]:

THEOREM 2. $\inf\sigma(H^N_{\alpha,\Gamma})$ *is uniquely maximized by an* N*-regular polygon.*

After this interlude let us return to the situation when the interaction support is a loop and consider another more singular case, namely we replace the δ in (1) by δ' interaction [2]. Recall that the latter can be introduced either by boundary condition or using the quadratic form

$$q_{\delta',\beta}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{\beta}\|[\psi]_\Gamma\|^2_{L^2(\Gamma)}$$

defined on $H^1(\mathbb{R}^2\setminus\Gamma)$, where $[\psi]_\Gamma := \psi_+|_\Gamma - \psi_+|_\Gamma$. We have the following result [30]:

THEOREM 3. *For any* $\beta>0$ *we have* $\max_{|\Gamma|=L}\lambda_1^\beta(\Gamma) = \lambda_1^\beta(\mathscr{C})$*, where* $\mathscr{C}$ *is a circle of perimeter* $L>0$ *and the maximum is taken over all* C^2 *smooth loops.*

Concerning the proof, we note that the Birman-Schwinger method does not work in this case, one has to use instead locally orthogonal coordinates in a way similar to those employed in [28] to treat the exterior of a Robin obstacle.

Let us next consider again the interactions defined by (2) and ask about the optimization problem in dimension $d = 3$. The first thing to note is that here the discrete spectrum of $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)$ may be empty if α is small enough; as an example, for Γ being a sphere of radius R bound states are known to exist *iff* $\alpha R > 1$, cf. [3]. This raises the following question: given the *critical* sphere, $\alpha R = 1$, would its deformations produce a discrete spectrum? A partial answer is the following [15]:

THEOREM 4. *Let Γ_ε by a deformation of the sphere expressed in spherical coordinates as $r(\theta,\phi) = R(1+\varepsilon\rho(\theta,\phi))$, where ρ is nonzero function of zero mean. If H_{α,Γ_0} is critical, $\sigma_{\rm disc}(H_{\alpha,\Gamma_\varepsilon}) \neq \emptyset$ holds for all nonzero ε small enough.*

REMARK 2. The above result fails to hold globally: if a surface-preserving deformation of a critical surface is elongated enough, the discrete spectrum is empty. In contrast, deformation of a critical surface always produces a nonempty discrete spectrum if it is *capacity preserving*, cf. [15] for details.

3. Cones

Let us involve next some new geometries into the game and investigate singular Schrödinger operators $H_{\alpha,\Gamma}$ having a conical surface as the interaction support Γ. We start with some definitions: let $\mathscr{T} \subset \mathbb{S}^2$ be a C^2-smooth loop on the 2D unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ of length $|\mathscr{T}|$ without self-intersections. We distinguish between circular and non-circular loops. A circle $\mathscr{C}$ on $\mathbb{S}^2$ has, of course, the length $|\mathscr{C}| \leqslant 2\pi$. The C^2-smooth *cone* $\Sigma_R(\mathscr{T}) \subset \mathbb{R}^3$ of radius $R \in (0,\infty]$ having a C^2-smooth loop $\mathscr{T} \subset \mathbb{S}^2$ as its cross-section is

$$\Sigma_R(\mathscr{T}) := \left\{ r\mathscr{T} \in \mathbb{R}^3 \colon r \in [0,R) \right\};$$

it is called finite (or truncated) if $R < \infty$ and infinite otherwise. The cone $\Sigma_R(\mathscr{T})$ is called *circular* if its cross-section $\mathscr{T}$ is a circle and *non-circular* otherwise. An infinite circular cone with the cross-section length 2π is, of course, a plane.

If $R < \infty$ it is easy to check that $\sigma_{\rm ess}(H_{\alpha,\Gamma}) = [0,\infty)$. Since the system is three-dimensional, the discrete spectrum again may or may not exist, and we are interested in the principal eigenvalue $\lambda_1(H_{\alpha,\Gamma})$ for which we have the following result [22]:

THEOREM 5. *Let $\mathscr{C} \subset \mathbb{S}^2$ be a circle and $\mathscr{T} \subset \mathbb{S}^2$ be a C^2-smooth non-circular loop such that $L := |\mathscr{C}| = |\mathscr{T}| \in (0,2\pi]$. Let $\Gamma_R := \Sigma_R(\mathscr{C})$ and $\Lambda_R := \Sigma_R(\mathscr{T})$ be finite cones of radius $R > 0$ with the cross-sections $\mathscr{C}$ and $\mathscr{T}$, respectively; then*

- $\#\sigma_{\rm disc}(H_{\alpha,\Gamma_R}) \geqslant 1$ *holds* iff $\alpha > \alpha_{\rm crit}$ *for a certain value* $\alpha_{\rm crit} = \alpha_{\rm crit}(L,R) > 0$.
- $\#\sigma_{\rm disc}(H_{\alpha,\Lambda_R}) \geqslant 1$ *for all* $\alpha \geqslant \alpha_{\rm crit}$ *(the borderline case* $\alpha = \alpha_{\rm crit}$ *is included) and the spectral isoperimetric inequality*

$$\lambda_1(H_{\alpha,\Lambda_R}) < \lambda_1(H_{\alpha,\Gamma_R})$$

is satisfied for all $\alpha > \alpha_{\rm crit}$.

COROLLARY 1. *Any (fixed-radius, smooth, conical) deformation of a critical circular cone gives rise to a nonempty discrete spectrum of the corresponding* $H_{\alpha,\Gamma}$.

On the other hand, the spectrum changes for infinite cones: in this case we have $\sigma_{\rm ess}(H_{\alpha,\Gamma}) = [-\frac14\alpha^2,\infty)$ and the discrete spectrum is not only nonempty but infinite except in the trivial case of a plane. Moreover, we even know its accumulation rate: for circular cones we have according to [31]

$$\mathscr{N}_{-\frac14\alpha^2-E}(H_{\alpha,\Gamma}) \sim \frac{\cot\theta}{4\pi}\,|\ln E|, \quad E\to 0+,$$

and a similar result with a different constant also holds in the non-circular case [34].

THEOREM 6. *Let* $\mathscr{C}\subset\mathbb{S}^2$ *be a circle and* $\mathscr{T}\subset\mathbb{S}^2$ *be a* C^2*-smooth non-circular loop such that* $L:=|\mathscr{C}|=|\mathscr{T}|\in(0,2\pi)$. *Let* $\Gamma_\infty := \Sigma_\infty(\mathscr{C})$ *and* $\Lambda_\infty := \Sigma_\infty(\mathscr{T})$ *be infinite cones with cross-sections* $\mathscr{C}$ *and* $\mathscr{T}$, *respectively; then for any* $\alpha>0$ *we have*

- $\#\sigma_{\rm disc}(H_{\alpha,\Gamma_\infty})\cap(-\infty,-\frac14\alpha^2)\geqslant 1$
- *the spectral isoperimetric inequality* $\lambda_1(H_{\alpha,\Lambda_\infty})\leqslant\lambda_1(H_{\alpha,\Gamma_\infty})$ *is valid.*

Sketch of the proof. Let us start with Theorem 5. The strategy is to employ the generalized Birman-Schwinger principle in combination with a minimization result about the energy of knots, cf. [17] and an earlier paper [1]. The former has been used already in proof of Theorem 1; it can be written as

$$\dim\ker\big(H_{\alpha,\Sigma}+\kappa^2\big) = \dim\ker\big(I-\alpha S_\Sigma(\kappa)\big)$$

for any $\kappa>0$, where

$$\big(S_\Sigma(\kappa)\psi\big)(x) := \int_\Sigma G_\kappa(x-y)\,\psi(y)\,\mathrm{d}\sigma(y)$$

and $G_\kappa(\cdot)$ is Green's function of the Laplacian in $\mathbb{R}^3$ at energy $-\kappa^2$. This implies, in particular, the following equivalences:

- $\#\sigma_{\rm disc}(H_{\alpha,\Sigma})\cap(-\infty,-\kappa^2)\geqslant 1$ *iff* $\mu_\Sigma(\kappa)>\alpha^{-1}$, where $\mu_\Sigma(\kappa)>0$ is the largest eigenvalue of $S_\Sigma(\kappa)$
- $\lambda_1(H_{\alpha,\Sigma}) = -\kappa^2$ *iff* $\mu_\Sigma(\kappa)=\alpha^{-1}$.

We also note that the eigenvalue $\mu_\Sigma(\kappa)$ is simple and the corresponding eigenfunction can be chosen positive. To proceed, we need a suitable parametrization of the cone. We begin with arc-length parametrization of the cross section, $\tau:\,[0,L]\to\mathbb{S}^2$ with $|\dot\tau|\equiv 1$ and put

$$\sigma:\,[0,R)\times[0,L]\to\mathbb{R}^3, \quad \sigma(r,s):=r\tau(s); \tag{7}$$

this defines natural co-ordinates (r,s) on Σ_R. One can check easily the following claim:

PROPOSITION 2. *Let $\mathscr{C} \subset \mathbb{S}^2$ be a circle and $\Gamma_R := \Sigma_R(\mathscr{C})$. Then the eigenfunction corresponding to the largest eigenvalue of the Birman-Schwinger operator $S_{\Gamma_R}(\kappa)$ is rotationally invariant, i.e. it depends on the distance from the tip of the cone only.*

The next part of the argument is geometric and employs an inequality reminiscent of (3) known from [32, 33] and other sources: for a C^2-smooth loop $\mathscr{T} \subset \mathbb{S}^2$ we define the functional

$$\Phi_f[\mathscr{T}] := \int_0^L \int_0^L f(|\tau(s) - \tau(t)|^2)\, \mathrm{d}s\, \mathrm{d}t\,.$$

PROPOSITION 3. *Let $f \in C([0,\infty);\mathbb{R})$ be convex and decreasing. Let further $\mathscr{C} \subset \mathbb{S}^2$ be a circle and $\mathscr{T} \subset \mathbb{S}^2$ be a C^2-smooth non-circular loop such that $|\mathscr{T}| = |\mathscr{C}| = L$ for some $L \in (0, 2\pi]$. Then the following isoperimetric inequality holds,*

$$\Phi_f[\mathscr{C}] < \Phi_f[\mathscr{T}]\,.$$

In particular, the above proposition holds with the function

$$f(x) := \frac{e^{-a\sqrt{bx+c}}}{\sqrt{bx+c}}\,,$$

which is convex and decreasing for any positive a,b,c. Comparing the Birman-Schwinger operators for the circular and non-circular cones with the use of the indicated parametrization we employ such isoperimetric inequalities with

$$a(r,r') := \kappa, \quad b(r,r') := rr', \quad c(r,r') := (r-r')^2;$$

we have to exclude the situations where $r = 0$, $r' = 0$ or $r = r'$, but this is a zero measure set. This yields the claim of Theorem 5. As it holds for any radius $R > 0$, Theorem 6 is proved by taking the limit $R \to \infty$. The Birman-Schwinger analysis can be also performed on infinite cones directly making it possible to show that also in this case the inequality is sharp unless Λ_∞ and Γ_∞ are congruent. □

4. Stars

Let us return to the two-dimensional situation and suppose that $\Gamma \subset \mathbb{R}^2$ is a planar graph. To be specific, we consider *star graphs* $\Sigma_N = \Sigma_N(L) \subset \mathbb{R}^2$, which have $N \geqslant 2$ edges of length $L \in (0,\infty]$ each, enumerated in the clockwise manner. They are characterized by the angles $\phi = \phi(\Sigma_N) = \{\phi_1, \phi_2, \ldots, \phi_N\}$ between the neighboring edges, $\phi_n \in (0, 2\pi)$ for all $n \in \{1, \ldots, N\}$ and $\sum_{n=1}^N \phi_n = 2\pi$. By Γ_N we denote the star graph with maximum symmetry, in other words, $\phi = \phi(\Gamma_N) = \left\{\frac{2\pi}{N}, \frac{2\pi}{N}, \ldots, \frac{2\pi}{N}\right\}$.

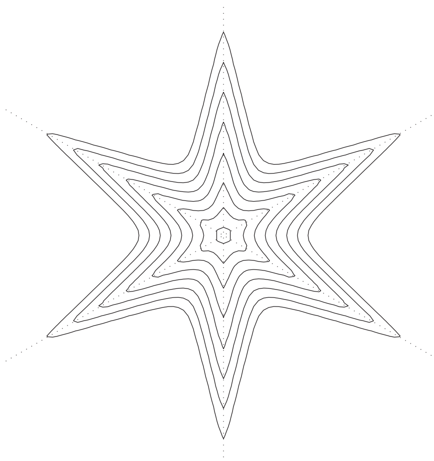
Given $\alpha > 0$, we ask again about the spectral threshold of the operator H_{α,Σ_N}. It is easy to see that $\sigma_{\mathrm{ess}}(H_{\alpha,\Sigma_N}) = [0,\infty)$ if $L < \infty$ and with the set $\sigma_{\mathrm{ess}}(H_{\alpha,\Sigma_N}) = [-\frac{1}{4}\alpha^2, \infty)$ if $L = \infty$. Since the system is two-dimensional we have $\sigma_{\mathrm{disc}}(H_{\alpha,\Sigma_N}) \neq \emptyset$ if $L < \infty$, and the same is true also for an infinite star, cf. [18] or [21, Example 10.2.1]. For the lowest eigenvalue we have the following result [23]:

THEOREM 7. *For any $\alpha > 0$ we have the relation*

$$\max_{\Sigma_N(L)} \lambda_1^\alpha\left(\Sigma_N(L)\right) = \lambda_1^\alpha\left(\Gamma_N(L)\right),$$

where the maximum is taken over all star graphs with $N \geqslant 2$ edges of a given length $L \in (0,\infty]$, the equality being achieved iff *Σ_N and Γ_N are congruent.*

Sketch of the proof. We again employ Birman-Schwinger principle. Writing the corresponding operators for $L < \infty$, one can interchange integration over the variables parametrizing the edges and summation over the edges. Next we use the symmetry of the principal eigenfunction of $H_{\alpha,\Gamma_N(L)}$ sketched in the picture below



and a discrete analogue of the inequality (3) analogous to that employed in the proof of Theorem 2. To establish the relation for $L = \infty$ one uses the strong resolvent convergence which gives, in particular,

$$\lim_{L\to\infty} \lambda_1^\alpha(\Sigma_N(L)) = \lambda_1^\alpha(\Sigma_N(\infty))$$

and the analogous relation for symmetric stars, alternatively the corresponding Birman-Schwinger operators can be compared directly. □

Let us next consider the analogous problem in three dimensions which is much harder; the two-dimensional one might have been technically nontrivial, but the result was easy to guess. The interaction support will be now an equilateral star in $\mathbb{R}^3$, i.e. a complex $\Gamma \equiv \Gamma_N$ of a 'sea urchin' shape with N 'pins', finite or semi-infinite. The first question is, of course, how to define the corresponding operator $H_{\alpha,\Gamma}$. Without going into details, for which we refer to [19], we claim that it acts as the Laplacian on functions that are H^2 outside Γ and at the points of the star one imposes boundary conditions which look like (6) with (5a) in the planes perpendicular to the edges of Γ.

Before proceeding further, let us recall some related problems. The oldest analogue coming to mind is the *Thomson problem* about an optimal distribution of N point charges on the surface of a sphere [36]. Despite the question is more than century old, a rigorous solution is known in some case only, for instance, the case $N=5$ was solved only recently [35]. Various generalizations of this problem to other 'potentials' and higher dimensions triggered numerous mathematical investigations in algebraic combinatorics, cf. [7, 13] an references therein.

We shall make use of some of these results for configurations of N points $\{x_i\}_{i=1}^N$ on the unit sphere S^2. They are said to form an M*-spherical design* if for any polynomial $x \mapsto p(x)$ on $\mathbb{R}^3$ of total degree M the averages over the sphere and over the configuration are the same, in other words, one has $\int_{S^2} p(x)\mathrm{d}x = \frac{1}{N}\sum_i^N p(x_i)$. Let further m be the number of *different* inner products between distinct points of $\{x_i\}_{i=1}^N$. They form a *sharp configuration* if it is $2m-1$ spherical design. By [13] a sharp configuration is *universally optimal* if it minimizes *any* potential energy $f:[0,4]\to\mathbb{R}$ which is completely monotonous, i.e. it satisfies $(-1)^k f^{(k)} \geqslant 0$ for all $k \geqslant 0$. In three dimensions there are five such sharp configurations, namely

- $N=2$, *antipodal points*
- $N=3$, *simplex* with inner product $-1/2$,
- $N=4$, *tetrahedron* – simplex with inner product $-1/3$,
- $N=6$, *octahedron* – cross polytope with inner products $-1,0$,
- $N=12$, *icosahedron* with inner products $-1,\pm 1/\sqrt{5}$.

We note that the configurations corresponding to the two remaining Platonic solids, the cube and the dodekahedron, do not qualify for universality; we recall that they do not represent Thomson's problem solutions either.

One may wonder how to make use of the mentioned minimization problem results. The answer is, as in the previous cases, that maximization of the ground state eigenvalue is equivalent to minimization of the (norm of) the corresponding Birman-Schwinger operator. The quoted result of [13] implies easily the following claim:

LEMMA 1. *Consider an N-arm star with edges of length $L\in(0,\infty]$ determined by unit vectors $\{\bar\gamma_i\}_{i=1}^N$, and let $\{\bar\sigma_i\}_{i=1}^N$ correspond to a sharp-configuration star. Then we have*

$$\sum_{i\neq j} T_{\kappa;s,t}(|\bar\gamma_i-\bar\gamma_j|^2) \geqslant \sum_{i\neq j} T_{\kappa;s,t}(|\bar\sigma_i-\bar\sigma_j|^2)$$

for any $s,t\in[0,L]$ and $T_{\kappa;s,t}(x) := \frac{\mathrm{e}^{-\kappa\sqrt{a+bx}}}{4\pi\sqrt{a+bx}}$ with $a=(s-t)^2$ and $b=st$. Moreover, the inequality is sharp unless the two star graphs are congruent.

We combine this with the fact that the eigenfunction referring to the largest eigenvalue of the Birman-Schwinger operator for a sharp-configuration star has the maximum symmetry being thus of the form $\tilde{f}_\sigma = (f_\sigma, \dots, f_\sigma) \in \bigoplus_{j=1}^N L^2([0,L])$. Then

$$\sup Q_{\kappa,\gamma} \geqslant (Q_{\kappa,\gamma}\tilde{f}_\sigma, \tilde{f}_\sigma) \geqslant \sup Q_{\kappa,\sigma}$$

holds according to the lemma, which allows us to make the following conclusion [20]:

THEOREM 8. *Assume that $N \in \{2,3,4,6,12\}$, then the ground state energy of the N-arm leaky star assumes the unique maximum for $\gamma = \sigma$, where σ corresponds to the appropriate sharp configuration listed above.*

5. Concluding remarks

At least a part of the results included into this survey is of a recent date which shows that this area is active and far from being ready to be sealed and put aside. Indeed, various related questions remain open. Some are of a technical nature coming from the fact that we did not strive here for the weakest assumptions, others questions are deeper and more interesting, for instance

- optimization with a *non-constant coupling strength* α. In this case we lose one important ingredient of our argument, the ground state symmetry. Put like that, however, the question seems to be too broad and one would likely need to fix more narrow classes of Γ and α to get a meaningful problem
- while for planar loops there is a similarity between the δ and δ' interaction, the situation for star graphs is more complicated. The optimal configuration for a δ' *star* is easy to guess when the vertex degree is even and the full symmetry is compatible with a switching sign of the wave function, but not at all for stars with an *odd number of edges*
- there is no need to stress how beautiful, and at the same time difficult is the three-dimensional star graphs optimization problem discussed in the previous section, and there is no doubt that any result for the number of edges different from those five listed in Theorem 8 would be of interest. Note that in contrast to Thomson's problem where the potential coupling appears as a multiplicative constant and, as a result, the solution is scale invariant, the 'potential' in Lemma 1 is more complicated. If the star edges are semi-infinite the problem is still scale invariant, while for a finite star the optimal configuration depends in general of the value of the coupling constant α
- last but not least, one is interested in result going beyond the principal eigenvalue level. It is possible to ask about comparison of the first two eigenvalues in the spirit of Payne-Pólya-Weinberger-Ashbaugh-Benguria, whether the nodal line of the second eigenvalue corresponding to a planar loop always intersects Γ, what can one say about the higher eigenvalues, etc.

We leave this, and more, in the hands of our kind reader and believe that he or she would find a pleasure in thinking about such problems.

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