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ON THE NUMBER OF ISOLATED EIGENVALUES OF A PAIR OF PARTICLES ON THE HALF–LINE

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Abstract. In this note we consider a pair of particles moving on the half-line $\mathbb{R}_+$ with the pairing induced by a hard-wall potential. This model was first introduced in [12] and later applied to investigate condensation of electron pairs in a quantum wire [11, 10]. For this, a detailed spectral analysis proved necessary and as a part of this it was shown in [10] that, in a special case, the discrete spectrum of the Hamiltonian consists of a single eigenvalue only. It is the aim of this note to prove that this is generally the case.

1. Introduction

In this note we consider an interacting system of two particles with the positive half-line $\mathbb{R}_+ = (0,\infty)$ as one-particle configuration space. More explicitly, the Hamiltonian shall be given by

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + v(|x-y|) \tag{1}$$

with hard-wall potential, $d > 0$,

$$v(x) := \begin{cases} 0 & \text{for} \quad x < d, \\ \infty & \text{else.} \end{cases} \tag{2}$$

Note that, through the potential v, the two particles actually form a pair with spatial extension characterised by $d > 0$. The two-particle model with Hamiltonian (1) and potential (2) was introduced in [12]. Its investigation grew out of studying many-particle quantum chaos on quantum graphs [3, 4] taking into account recent results in theoretical physics [14]. More generally, due to the technical advances in the last decades and especially in there realm of nanotechnology, it has become pivotal to study the properties of interacting particle systems in one dimension which may differ greatly from those of systems in higher dimension [7, 8]. Also, since the pairing of electrons (Cooper pairs) in metals is the key mechanism in the formation of the superconducting phase in type-I superconductors [5, 1], an investigation of the Hamiltonian (1) is also interesting from a solid-state physics point of view.

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In this note we are interested in spectral properties of the Hamiltonian H. More explicitly, we are interested in characterising the discrete part of the spectrum. It was the key observation in [12] that the discrete spectrum of H is non-trivial, i.e., there exist eigenvalues below the botton of the essential spectrum. Since this is not the case if one changed the one-particle configuration space to be the whole real line $\mathbb{R}$, the existence of a discrete spectrum is directly linked to the geometry of the one- and two-particle configuration space. Implementing exchange symmetry, the authors of [10] were able to show that the discrete spectrum actually exists of one eigenvalue only. The main purpose of this note is to show that exchange symmetry is indeed not necessary and that the discrete spectrum always consists of one eigenvalue only. Finally, we want to draw attention to the recent paper [15] in which spectral properties of H (up to a scaling factor of $1/\sqrt{2}$ in the potential) for a large class of interaction potentials $v:\mathbb{R}_+\to\mathbb{R}$ where studied. More explicitly, it was assumed that

(i) $v\in L^1_{loc}(\mathbb{R}_+)$ and $\max\{-v,0\}\in L^{\infty}(\mathbb{R}_+)$,

(ii) The one-particle operator $h:=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+v(x)$ is such that $\inf\sigma(h)=\varepsilon_0$ is an isolated eigenvalue,

(iii) $\varepsilon_0<\liminf_{x\to\infty}v(x):=v_\infty$.

The authors found that the essential spectrum is given by the interval $[\varepsilon_0,\infty)$ and - most importantly - that the discrete spectrum is non-empty and contains only finitely many eigenvalues. However, no bounds on the number of isolated eigenvalues could be derived. It is clear that our potential (2) violates condition (i) but it does informally fulfil conditions (ii) and (iii). In addition, the result regarding the essential spectrum carries over: as shown in [12], the essential spectrum of H with potential (2) is given by the interval $\left[\frac{\pi^2}{2d^2},\infty\right)$.

2. The model and main results

Due to the formal nature of the interaction potential (2), H cannot be realised as a self-adjoint operator on $L^2(\mathbb{R}^2_+)$. However, we see that this choice for v means that the two-particle configuration space is actually given by

$$\Omega:=\{(x,y)\in\mathbb{R}^2_+ : |x-y|<d\}. \tag{3}$$

Based on Ω we then introduce the Hilbert spaces $L^2(\Omega):=L^2_0(\Omega)$ as well as

$$\begin{aligned} L^2_s(\Omega)&:=\left\{\varphi\in L^2(\Omega):\varphi(x,y)=\varphi(y,x)\right\},\\ L^2_a(\Omega)&:=\left\{\varphi\in L^2(\Omega):\varphi(x,y)=-\varphi(y,x)\right\}. \end{aligned} \tag{4}$$

When describing two dinstinguishable particles one focusses on $L^2(\Omega)$ while the versions $L^2_{s/a}(\Omega)$ are used if one implements exchange symmetry between the two particles. For example, in [10] the authors considered a pair of two electrons with same spin which implies that one has to work on $L^2_a(\Omega)$ since electrons are fermions. Contrary to this, in [11] the electrons were assumed to be of opposite spin which then required to work on $L^2_s(\Omega)$ instead.

Now, on any of those Hilbert spaces, H is rigorously realised via its associated (quadratic) form, $j \in \{0,s,a\}$,

$$q_j[\varphi] := \int_\Omega |\nabla\varphi|^2 \mathrm{d}x \tag{5}$$

with form domain $\mathscr{D}_j = \{\varphi \in H^1(\Omega) : \varphi \in L^2_j(\Omega) \text{ and } \varphi|_{\partial\Omega_D} = 0\}$ where

$$\partial\Omega_D := \{(x,y) \in \mathbb{R}^2_+ : |x-y| = d\}. \tag{6}$$

Note here that $q_j[\cdot]$ obviously is a closed positive form with a dense form domain. Also note that we write H_j for the self-adjoint operator associated with $q_j[\cdot]$ [2].

REMARK 1. It is clear that the self-adjoint operator associated with $q_j[\cdot]$ is nothing else than a version of the two-dimensional Laplacian $-\Delta$ [9].

In order to formulate our main result we recall the following statement which was proved in [12, 10, 11]. We denote by $\sigma_d(\cdot)$ the discrete spectrum.

THEOREM 1. *For every* $j \in \{0,s,a\}$ *one has*

$$\sigma_d(H_j) \neq \emptyset.$$

It is our goal in this note to prove the following result.

THEOREM 2. *For every* $j \in \{0,s,a\}$ *one has*

$$\sigma_d(H_j) = \{E_j\}$$

with some $E_j \geqslant 0$ *which is an eigenvalue of multiplicity one. In other words, the discrete spectrum consists of one eigenvalue only.*

3. Proof of the main result

In this section we establish a proof of Theorem 2. We note that this was already proved for $j = a$ in [10] but for the sake of completeness we also include a proof thereof in this note.

In a first step we establish an auxiliary result. For this we define the domain

$$\widetilde{\Omega} := \{(x,y) \in \mathbb{R}^2 : |x-y| < d\} \cup \{(x,y) \in \mathbb{R}^2 : |x+y| < d\} \tag{7}$$

and introduce on $L^2(\widetilde{\Omega})$ the two-dimensional Dirichlet Laplacian denoted as $-\widetilde{\Delta}_d^{(D)}$. Note that the quadratic form associated with $-\widetilde{\Delta}_d^{(D)}$ is given by

$$\widetilde{q}_d[\varphi] = \int_{\widetilde{\Omega}} |\nabla\varphi|^2 \, \mathrm{d}x$$

with form domain $\widetilde{\mathscr{D}}_d := \{\varphi \in H^1(\widetilde{\Omega}) : \varphi|_{\partial\widetilde{\Omega}} = 0\}$.

THEOREM 3. *The discrete spectrum of the self-adjoint operator* $-\widetilde{\Delta}_d^{(D)}$ *consists of exactly one eigenvalue of multiplicity one.*

Proof. Since the Laplacian is invariant under rotations as well as translations, we may prove the statement considering the Dirichlet Laplacian on a rotated version of $\widetilde{\Omega}$. Namely, we consider the Dirichlet Laplacian on the "cross-shaped" domain

$$\begin{aligned}\Omega_0 :=&\{(x,y)\in\mathbb{R}^2 : -\infty < y < \infty, -d/\sqrt{2} < x < +d/\sqrt{2}\}\\ &\cup\{(x,y)\in\mathbb{R}^2 : -\infty < x < \infty, -d/\sqrt{2} < y < +d/\sqrt{2}\}.\end{aligned}$$

We denote this operator by $-\Delta_D^{(0)}$. We then employ a bracketing argument and for this we introduce the direct sum of Laplacians $-\Delta_1 \oplus -\Delta_2$; here, $-\Delta_1$ is the two-dimensional Laplacian defined on the bounded domain (square)

$$\Omega_1 := (-d/\sqrt{2}, +d/\sqrt{2}) \times (-d/\sqrt{2}, +d/\sqrt{2})$$

subjected to Neumann boundary conditions along $\partial\Omega_1$. $-\Delta_2$ denotes the two-dimensional Laplacian over the domain

$$\Omega_2 := (\Omega_0 \setminus \Omega_1)^\circ$$

with Neumann boundary conditions along the boundary segments adjacent to Ω_1 and Dirichlet boundary conditions elsewhere. Most importantly, in terms of operators we obtain the inequality

$$-\Delta_1 \oplus -\Delta_2 \leqslant -\Delta_D^{(0)}$$

which implies

$$N(-\Delta_D^{(0)}, E) \leqslant N(-\Delta_1 \oplus -\Delta_2, E)$$

where $N(\cdot, E)$ is the standard counting function, counting all eigenvalues up to energy $E < \inf\sigma_{ess}(-\Delta_1 \oplus -\Delta_2)$. Here, $\sigma_{ess}(\cdot)$ denotes the essential spectrum.

From the definition of Ω_2 it readily follows that

$$N(-\Delta_1 \oplus -\Delta_2, E) = N(-\Delta_1, E)$$

whenever $E < \inf\sigma_{ess}(-\Delta_1 \oplus -\Delta_2)$. The reason for this is that $\inf\sigma_{ess}(-\Delta_1 \oplus -\Delta_2) = \inf\sigma_{ess}(-\Delta_2) = \inf\sigma(-\Delta_2) = \pi^2/(2d^2)$; see [6, 10] for more details (note that the spectrum of $-\Delta_1$ is purely discrete and Ω_2 consists of four rectangular parts for which a separation of variables can be employed to determine the (essential) spectrum directly).

Now, in order to study $N(-\Delta_1, E)$ we take advantage of the fact that Ω_1 is a square. Hence, we can employ a separation of variables which allows us to determine the eigenvalues of $-\Delta_1$ explicitly. Namely,

$$\sigma_d(-\Delta_1) = \left\{0, \frac{\pi^2}{2d^2}, \frac{\pi^2}{d^2}, \dots\right\}.$$

Hence, it follows that $N(-\Delta_1, E) = 1$ for all $E < \frac{\pi^2}{2d^2}$ which proves the statement taking into account that $\inf \sigma_{ess}(-\Delta_D^{(0)}) = \frac{\pi^2}{2d^2}$, see [10].

Now we are in position to prove Theorem 2.

Proof of Theorem 2. We first consider the cases $j \in \{0, s\}$: We introduce the (injective) linear map

$$I_j : \mathscr{D}_j \to \widetilde{\mathscr{D}}_d ,$$

where $I_j\varphi$ is constructed as follows: one takes $\varphi \in \mathscr{D}_j$ and then reflects it across the y-axis. This new function (consisting of the original φ and the new reflected part) is then reflected another time across the x-axis, finally yielding an element of $\widetilde{\mathscr{D}}_d$. Now, from the min-max principle we then conclude that, $n \in \mathbb{N}$,

$$\begin{aligned}
\mu_n(H_j) &= \inf_{W_n \subset \mathscr{D}_j} \sup_{0 \neq \varphi \in W_n} \frac{q_j[\varphi]}{\|\varphi\|^2_{L^2(\Omega)}} \\
&\geqslant \inf_{W_n \subset I_j \mathscr{D}_j} \sup_{0 \neq I_j\varphi \in W_n} \frac{\widetilde{q}_d[I_j\varphi]}{\|I_j\varphi\|^2_{L^2(\widetilde{\Omega})}} \\
&\geqslant \inf_{W_n \subset \widetilde{\mathscr{D}}_d} \sup_{0 \neq \varphi \in W_n} \frac{\widetilde{q}_d[\varphi]}{\|\varphi\|^2_{L^2(\widetilde{\Omega})}} = \mu_n(-\widetilde{\Delta}_d^{(D)}),
\end{aligned}$$

where $\mu_n(\cdot)$ denotes the n-th "min-max eigenvalue" [2, 13]. Also, W_n refers to n-dimensional subspaces. From Theorem 3 it follows that $\mu_1(-\widetilde{\Delta}_d^{(D)})$ is the only eigenvalue in the discrete spectrum and $\mu_n(-\widetilde{\Delta}_d^{(D)}) = \inf \sigma_{ess}(-\widetilde{\Delta}_d^{(D)}) = \pi^2/(2d^2)$ for $n > 1$. Hence, by Theorem 1 we conclude that only $\mu_1(H_j) < \pi^2/(2d^2)$ which yields the statement since both essential spectra start at $\pi^2/(2d^2)$ as shown in [12, 11].

In a next step we consider the case $j = a$: Again we want to make use of the min-max principle and hence introduce the (injective) linear map

$$I_a : \mathscr{D}_a \to \widetilde{\mathscr{D}}_{d/2}$$

which acts as follows: to obtain $I_a\varphi$ one first restricts $\varphi \in \mathscr{D}_a$ to

$$\{(x,y) \in \mathbb{R}^2_+ : |x-y| < d \text{ and } y > x\}.$$

This restriction is then reflected across the axis $x = 0$ and then both segments are translated in the negative y-direction by $d/2$. Finally, we can extend this (translated) function by zero to obtain an element of $\widetilde{\mathscr{D}}_{d/2}$. Now, employing the min-max principle shows that

$$\begin{aligned}
\mu_n(H_a) &= \inf_{W_n \subset \mathscr{D}_a} \sup_{0 \neq \varphi \in W_n} \frac{q_a[\varphi]}{\|\varphi\|^2_{L^2(\Omega)}} \\
&\geqslant \inf_{W_n \subset I_a \mathscr{D}_j} \sup_{0 \neq I_a\varphi \in W_n} \frac{\widetilde{q}_{d/2}[I_a\varphi]}{\|I_a\varphi\|^2_{L^2(\widetilde{\Omega})}} \\
&\geqslant \inf_{W_n \subset \widetilde{\mathscr{D}}_d} \sup_{0 \neq \varphi \in W_n} \frac{\widetilde{q}_{d/2}[\varphi]}{\|\varphi\|^2_{L^2(\widetilde{\Omega})}} = \mu_n(-\widetilde{\Delta}_{d/2}^{(D)}) .
\end{aligned}$$

For $n > 1$, $\mu_n(-\widetilde{\Delta}^{(D)}_{d/2}) = \inf \sigma_{ess}(-\widetilde{\Delta}^{(D)}_{d/2}) = \frac{2\pi^2}{d^2}$ and since it was shown in [10] that $\inf \sigma_{ess}(H_a) = \frac{2\pi^2}{d^2}$, we conclude the statement.

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