

ON THE NOTION OF EFFECTIVE IMPEDANCE

ANNA MURANOVA

Abstract. It is known that electrical networks with resistors are related to the Laplace operator and random walk on weighted graphs. In this paper we consider more general electrical networks with coils, capacitors, and resistors. We give two mathematical models of such networks: complex-weighted graphs and graphs with weights from the ordered field of rational functions. The notion of effective impedance in both approaches is defined.

1. Introduction

It was shown in [3] and [7] that there is a tight relation between electrical networks with resistors and weighted graphs. Ohm's and Kirchhoff's laws imply that the voltage in the network is a solution of the Dirichlet problem for the discrete Laplace operator on the weighted graph. Due to the maximum principle, the solution of the Dirichlet problem in this case exists and is unique (see, for example, [5]). Hence, this provides a mathematical justification of the notion of effective resistance as the inverse energy of the solution of the Dirichlet problem.

Consider now an electrical network of alternating current, that consists of impedances (i.e. resistors, capacitors, and coils). In this case one rewrites Ohm's and Kirchhoff's laws in the complex form (see e.g. [4], [6], [8]) and obtains the Dirichlet problem with complex-valued coefficients. Maximum principle does not exist in this case, and solution of the Dirichlet problem may not exist or may be not unique, which creates difficulties in definition of the effective impedance. In this paper we propose two approaches of overcoming this difficulty.

In the first approach we show that, in the case of multiple solutions, all they have the same energy and, therefore, the effective impedance is well-defined. In the case of absence of solution the effective impedance is set to be 0.

In the second approach, we consider the impedance of each edge as a rational function of the parameter $\lambda = i\omega$, where ω is the frequency of the current (see [1]), and use the fact, that rational functions of λ form an ordered field (see [9]). Fortunately, the maximum principle holds for the Laplace operator with weight from that field, which allows to solve uniquely the Dirichlet problem and, hence, to define the effective impedance as a rational function on λ .

The two notions of effective impedances coincide if the Dirichlet problem of the first approach has a unique solution. Otherwise, the question of identity of the two effective impedances remains open.

Mathematics subject classification (2010): 05C22, 05C25, 34B45, 39A12, 12J15.

Keywords and phrases: Weighted graphs, electrical network, Laplace operator, ordered field of rational functions.

This research was supported by IRTG 2235 Bielefeld-Seoul "Searching for the regular in the irregular: Analysis of singular and random systems".

2. Graphs with complex-valued weight

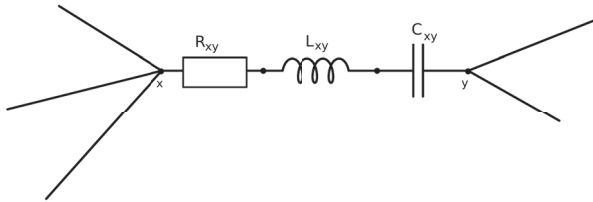
Let (V, E) be a connected graph, where V is a set of vertices and E is a set of (unoriented) edges. Unless otherwise is said, the set V is always assumed finite.

Assume that each edge xy is equipped with a resistance R_{xy} , inductance L_{xy} , and capacitance C_{xy} , where $R_{xy}, L_{xy} \in [0, +\infty)$ and $C_{xy} \in (0, +\infty]$, which correspond to physical resistor, inductor (coil), and capacitor. Let $a_0, a_1 \in V$ be two vertices, to which an external periodic voltage of frequency $\omega > 0$ is applied.

The *impedance* of the edge xy is

$$z_{xy} = R_{xy} + L_{xy}i\omega + \frac{1}{C_{xy}i\omega}. \tag{1}$$

Note that $\text{Re} z_{xy} \geq 0$.



By complex Ohm’s and Kirchoff’s laws the complex voltage $v : V \rightarrow \mathbb{C}$ satisfies the following *Dirichlet problem*:

$$\begin{cases} \sum_{y:y \sim x} \frac{v(y)-v(x)}{z_{xy}} = 0 \text{ on } V \setminus \{a_0, a_1\}, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases} \tag{2}$$

Note that here and further in notations \sum_y means $\sum_{y \in V}$.

The physical meaning of $v(x)$ is that it is a (complex-valued) amplitude of the voltage at the node x , while the actual alternating voltage at time t is equal to $\text{Re}(v(x)e^{i\omega t})$.

It will be convenient for us to use the inverse capacity:

$$D_{xy} = \frac{1}{C_{xy}} \in [0, +\infty),$$

as well as the *admittance* ρ_{xy} :

$$\rho_{xy} = \frac{1}{z_{xy}} = \frac{i\omega}{L_{xy}(i\omega)^2 + R_{xy}(i\omega) + D_{xy}} = \frac{R_{xy}\omega^2 + i(-L_{xy}\omega^3 + D_{xy}\omega)}{(D_{xy} - L_{xy}\omega^2)^2 + R_{xy}^2\omega^2} \tag{3}$$

$$= \frac{\lambda}{L_{xy}\lambda^2 + R_{xy}\lambda + D_{xy}}, \tag{4}$$

where $\lambda = i\omega$ is pure imaginary, $\text{Im } \lambda > 0$ (i.e. $-i\lambda > 0$). We always assume that for any edge

$$R_{xy}^2 + L_{xy}^2 + D_{xy}^2 \neq 0.$$

Note that we can consider ρ as a function from $V \times V$ to \mathbb{C} by setting $\rho_{xy} = 0$, if xy is not an edge. We refer to the structure $\Gamma = ((V, E), \{\rho_{xy}\}, a_0, a_1)$ as an (*electrical*) *network*.

DEFINITION 1. The *weighted Laplacian* Δ_ρ is the operator, defined as follows: for any function $f : V \rightarrow \mathbb{C}$

$$\Delta_\rho f(x) = \sum_{y: y \sim x} (f(y) - f(x))\rho_{xy} = \sum_{y: y \sim x} (\nabla_{xy} f)\rho_{xy},$$

where

$$\nabla_{xy} f = f(y) - f(x)$$

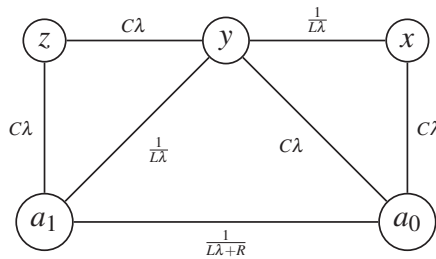
is the *difference operator*.

Therefore, we can rewrite the Dirichlet problem (2) as follows:

$$\begin{cases} \Delta_\rho v(x) = 0 \text{ on } V \setminus \{a_0, a_1\}, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases} \tag{5}$$

Note that if $|V| = n$, then (5) is a $n \times n$ system of linear equations. The existence and uniqueness of the solution of (5) over \mathbb{C} is not always the case. Indeed, the determinant of the Dirichlet problem (5) is equal to zero, when ω is equal to a natural frequency of the given network (see e.g. [2, p. 601]).

EXAMPLE 2. Consider the network Γ as at the figure below, where admittances are shown on each edge ($R, L, C > 0$).



The Dirichlet problem for this network is

$$\begin{cases} \frac{(v(y)-v(x))}{L\lambda} + (v(a_0) - v(x))C\lambda = 0, \\ (v(z) - v(y))C\lambda + \frac{(v(a_1)-v(y))}{L\lambda} + (v(a_0) - v(y))C\lambda + \frac{(v(x)-v(y))}{L\lambda} = 0, \\ (v(a_1) - v(z))C\lambda + (v(y) - v(z))C\lambda = 0, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases} \tag{6}$$

The determinant of this linear system is

$$D = D(\lambda) = \frac{-C(3LC\lambda^2 + 1)(LC\lambda^2 + 2)}{L^2\lambda}$$

and it has two pure imaginary zeros with positive imaginary part: $\lambda_1 = i\sqrt{\frac{2}{LC}}$, $\lambda_2 = i\sqrt{\frac{1}{3LC}}$. In case $D(\lambda) \neq 0$ the solution of the Dirichlet problem (6) is

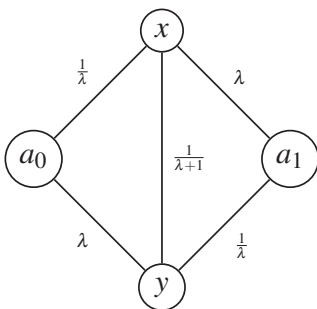
$$v = (v(x), v(y), v(z), v(a_0), v(a_1)) = \left(\frac{1}{3LC\lambda^2 + 1}, \frac{LC\lambda^2 + 1}{3LC\lambda^2 + 1}, \frac{2LC\lambda^2 + 1}{3LC\lambda^2 + 1}, 0, 1 \right).$$

In the case $\lambda = \lambda_1$ the Dirichlet problem (6) has infinitely many solutions

$$v = (-2\tau + 1, 2\tau - 1, \tau, 0, 1), \tau \in \mathbb{C}.$$

In the case $\lambda = \lambda_2$ the Dirichlet problem (6) has no solution.

EXAMPLE 3. Consider the network Γ as at the figure below, where admittances are shown on each edge.



The Dirichlet problem for this network is

$$\begin{cases} \frac{(v(a_0) - v(x))}{\lambda} + \frac{(v(y) - v(x))}{\lambda + 1} + (v(a_1) - v(x))\lambda = 0, \\ (v(a_0) - v(y))\lambda + \frac{(v(x) - v(y))}{\lambda + 1} + \frac{(v(a_1) - v(y))}{\lambda} = 0, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases} \quad (7)$$

The determinant of this linear system is

$$\begin{aligned} D = D(\lambda) &= \frac{\lambda^5 + \lambda^4 + 4\lambda^3 + 2\lambda^2 + 3\lambda + 1}{\lambda^3 + \lambda^2} \\ &= \frac{(\lambda^2 + 1)(\lambda^3 + \lambda^2 + 3\lambda + 1)}{\lambda^2(\lambda + 1)} \end{aligned}$$

and it has one pure imaginary zero, whose imaginary part is positive, $\lambda_0 = i$. In case $D(\lambda) \neq 0$ the solution of the Dirichlet problem (7) is

$$v(x) = \frac{\lambda^3 + \lambda^2 + \lambda}{\lambda^3 + \lambda^2 + 3\lambda + 1},$$

$$v(y) = \frac{2\lambda + 1}{\lambda^3 + \lambda^2 + 3\lambda + 1}.$$

In the case $\lambda = i$ the Dirichlet problem (7) has infinitely many solutions

$$v(x) = \tau - 1 + i,$$

$$v(y) = \tau, \tau \in \mathbb{C}.$$

DEFINITION 4. Let $v(x)$ be a solution of the Dirichlet problem (5). Define the *effective impedance* of the network Γ by

$$Z_{eff} = \frac{1}{\sum_{x:x \sim a_0} v(x)\rho_{xa_0}}$$

and the *effective admittance* by

$$\mathcal{P}_{eff} = \frac{1}{Z_{eff}} = \sum_{x:x \sim a_0} v(x)\rho_{xa_0}.$$

If (5) has no solution, then set $Z_{eff} = 0$ and $\mathcal{P}_{eff} = \infty$.

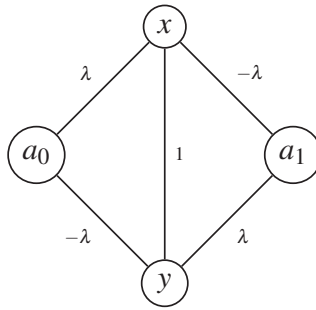
Note that Z_{eff} and \mathcal{P}_{eff} take values in $\mathbb{C} \cup \{\infty\}$. We will prove below (see Theorem 10), that in the case when (5) has multiple solutions, the values Z_{eff} and \mathcal{P}_{eff} are independent of the choice of the solution v .

PROPOSITION 5. For any given network Γ the determinant $D(\lambda)$ of the Dirichlet problem (5) has a finite number of zeros in \mathbb{C} .

Hence, for all $\lambda \in \mathbb{C}$, except for a finite number of values, the Dirichlet problem (5) has a unique solution.

The proof of Proposition 5 will be given in Section 3. Note that the theorem is true just for networks (see Example 6 below).

EXAMPLE 6. Consider the Dirichlet problem for the weighted graph at the figure below, where weights are shown on each edge. Note that here the weights of the edges xa_1 and a_0y are not in the form (4), and, therefore, this is *not a network*.



The Dirichlet problem for this graph is

$$\begin{cases} (v(a_0) - v(x))\lambda + (v(a_1) - v(x))(-\lambda) + (v(y) - v(x)) = 0, \\ (v(a_0) - v(y))(-\lambda) + (v(a_1) - v(y))\lambda + (v(x) - v(y)) = 0 \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases}$$

The determinant of this system is

$$D = D(\lambda) \equiv 0$$

and the Dirichlet problem has infinitely many solutions

$$v(x) = \tau, v(y) = \tau + \lambda, \tau \in \mathbb{C}$$

for any λ .

From physical point of view the effective impedance means that if we replace our entire network by a single edge connecting a_0 and a_1 with the impedance Z_{eff} , then the current in this single-edge network will be the same as in the original one.

LEMMA 7. (Green’s formula) *Let Γ be a network as above and let Ω be a non-empty subset of V . Then for any two functions $f, g : V \rightarrow \mathbb{C}$ the following identity is true:*

$$\sum_{x \in \Omega} \Delta_\rho f(x)g(x) = -\frac{1}{2} \sum_{x,y \in \Omega} (\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy} + \sum_{x \in \Omega} \sum_{y \in V \setminus \Omega} (\nabla_{xy} f)g(x)\rho_{xy}. \tag{8}$$

Proof.

$$\begin{aligned} \sum_{x \in \Omega} \Delta_\rho f(x)g(x) &= \sum_{x \in \Omega} \left(\sum_{y \in V} (f(y) - f(x))\rho_{xy} \right) g(x) \\ &= \sum_{x \in \Omega} \sum_{y \in V} (f(y) - f(x))g(x)\rho_{xy} \end{aligned}$$

$$\begin{aligned} &= \sum_{x \in \Omega} \sum_{y \in \Omega} (f(y) - f(x))g(x)\rho_{xy} + \sum_{x \in \Omega} \sum_{y \in V \setminus \Omega} (f(y) - f(x))g(x)\rho_{xy} \\ &= \sum_{y \in \Omega} \sum_{x \in \Omega} (f(x) - f(y))g(y)\rho_{xy} + \sum_{x \in \Omega} \sum_{y \in V \setminus \Omega} (\nabla_{xy} f)g(x)\rho_{xy}, \end{aligned}$$

where in the last line we have switched notation of the variables x and y in the first sum. Adding together the last two lines and dividing by 2, we obtain (8).

If $\Omega = V$, then $V \setminus \Omega$ is empty so that the last term in (8) vanishes, and we obtain

$$\sum_{x \in V} \Delta_\rho f(x)g(x) = -\frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy}. \tag{9}$$

COROLLARY 8. For any function $f : V \rightarrow \mathbb{C}$,

$$\sum_{x \in V} \Delta_\rho f(x) = 0. \tag{10}$$

Proof. Apply (9) for $g \equiv 1$.

LEMMA 9. For any solution v of the Dirichlet problem (5) we have

$$\sum_{x:x \sim a_0} v(x)\rho_{xa_0} = \Delta_\rho v(a_0) = -\Delta_\rho v(a_1) = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v)(\nabla_{xy} u)\rho_{xy}, \tag{11}$$

where $u : V \rightarrow \mathbb{C}$ is any function such that $u(a_0) = 0$ and $u(a_1) = 1$.

Proof. Using $v(a_0) = 0$, we have

$$\Delta_\rho v(a_0) = \sum_{x:x \sim a_0} (v(x) - v(a_0))\rho_{xa_0} = \sum_{x:x \sim a_0} v(x)\rho_{xa_0}$$

which proves the first identity in (11). Since by (10)

$$\sum_{x \in V} \Delta_\rho f(x) = 0$$

and $\Delta_\rho v(x) = 0$ for all $x \in V \setminus \{a_0, a_1\}$, we obtain

$$\Delta_\rho v(a_0) + \Delta_\rho v(a_1) = 0$$

whence the second identity in (11) follows. Finally, to prove the third identity, we apply the Green’s formula (9) and obtain

$$\frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v)(\nabla_{xy} u)\rho_{xy} = -\sum_x \Delta_\rho v(x)u(x) = -\Delta v(a_1),$$

because $\Delta_\rho v(x) = 0$ for all $x \in V \setminus \{a_0, a_1\}$, while $u(a_0) = 0$ and $u(a_1) = 1$.

THEOREM 10. *The value of the impedance Z_{eff} does not depend on the choice of a solution v of the Dirichlet problem (5). Besides, we have the identity*

$$\frac{1}{2} \sum_{x,y \in V} |\nabla_{xy} v|^2 \rho_{xy} = \mathcal{P}_{eff}. \tag{12}$$

Proof. Let v_1 and v_2 be two solutions of (5). By (11) we have

$$\sum_{x: x \sim a_0} v_1(x) \rho_{xa_0} = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v_1)(\nabla_{xy} v_2) \rho_{xy}$$

and also

$$\sum_{x: x \sim a_0} v_2(x) \rho_{xa_0} = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v_2)(\nabla_{xy} v_1) \rho_{xy},$$

whence the identity

$$\sum_{x: x \sim a_0} v_1(x) \rho_{xa_0} = \sum_{x: x \sim a_0} v_2(x) \rho_{xa_0}$$

follows. Hence, the admittance and impedance are independent of the choice of v . Applying (11) with $u = \bar{v}$, we obtain

$$\frac{1}{2} \sum_{x,y \in V} |\nabla_{xy} v|^2 \rho_{xy} = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v)(\overline{\nabla_{xy} v}) \rho_{xy} = \mathcal{P}_{eff}.$$

By the physical meaning $\text{Re } z_{xy} \geq 0$ and the effective impedance also is expected to have a non-negative real part. We prove this in a following theorem, using Theorem 10.

THEOREM 11. *For any finite network, we have*

$$\text{Re}(Z_{eff}) \geq 0.$$

Moreover, if $\text{Im}(z_{xy}) \leq 0$ for every $xy \in E$ (RC-network), then $\text{Im}(Z_{eff}) \leq 0$ and if $\text{Im}(z_{xy}) \geq 0$ for every $xy \in E$ (RL-network), then $\text{Im}(Z_{eff}) \geq 0$.

Proof. For any $z \in \mathbb{C}$, we have

$$\text{Re } z \geq 0 \Leftrightarrow \text{Re} \left(\frac{1}{z} \right) \geq 0,$$

because if $z = a + bi, a, b \in \mathbb{R}$, then

$$\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i. \tag{13}$$

Therefore, $\text{Re}(Z_{eff}) \geq 0$ is equivalent to $\text{Re}(\mathcal{P}_{eff}) \geq 0$. From the left hand side of (12) it is obvious that $\text{Re}(\mathcal{P}_{eff}) \geq 0$ since $\text{Re}(\rho_{xy}) \geq 0$ for any $xy \in E$ by (3).

We have $\text{Im}(z_{xy}) \leq 0 \Leftrightarrow \text{Im}(\rho_{xy}) \geq 0$ and $\text{Im}(Z_{eff}) \leq 0 \Leftrightarrow \text{Im}(\mathcal{P}_{eff}) \geq 0$ by (13). Due to (12), $\text{Im}(\rho_{xy}) \geq 0$, for any $xy \in E$, implies $\text{Im}(\mathcal{P}_{eff}) \geq 0$. The result for RL-network can be proved analogously.

EXAMPLE 12. The effective impedance for the network from Example 2 is given by.

$$Z_{eff} = \begin{cases} R + i\sqrt{\frac{2L}{C}}, \lambda = i\sqrt{\frac{2}{LC}}, \text{ case of multiple solutions ;} \\ 0, \lambda = i\sqrt{\frac{1}{3LC}}, \text{ case of no solution ;} \\ \frac{3L^2C\lambda^3 + 3RLC\lambda^2 + L\lambda + R}{L^2C^2\lambda^4 + RLC^2\lambda^3 + 5LC\lambda^2 + 2RC\lambda + 1}, \text{ for other } \lambda, \text{ such that } -i\lambda > 0. \end{cases}$$

It is easy to verify, that in this example the effective impedance is a continuous function on $\omega = -i\lambda \in (0, \infty)$.

EXAMPLE 13. The effective impedance for the network from Example 3 is given by

$$Z_{eff} = \begin{cases} \frac{1}{2} - \frac{i}{2}, \lambda = i, \\ \frac{\lambda^3 + \lambda^2 + 3\lambda + 1}{3\lambda^2 + 2\lambda + 1}, \text{ in other cases } (-i\lambda > 0). \end{cases}$$

It is easy to verify, that here the effective impedance is again a continuous function on $\omega = -i\lambda \in (0, \infty)$.

3. Network over an ordered field

Let us consider the admittance ρ_{xy} as a rational function of λ

$$\rho_{xy}(\lambda) = \frac{\lambda}{L_{xy}\lambda^2 + R_{xy}\lambda + D_{xy}} \tag{14}$$

with real coefficients.

Let us denote by $\mathbb{R}(\lambda)$ the set of all rational functions of λ with real coefficients.

DEFINITION 14. [9] Define in $\mathbb{R}(\lambda)$ an order “ \succ ” as follows: for any rational function

$$f(\lambda) = \frac{a_n\lambda^n + \dots + a_1\lambda + a_0}{b_m\lambda^m + \dots + b_1\lambda + b_0} \in \mathbb{R}(\lambda)$$

with $a_n \neq 0, b_m \neq 0$, write

$$f(\lambda) \succ 0, \text{ if } \frac{a_n}{b_m} > 0$$

and

$$f(\lambda) \succ g(\lambda), \text{ if } f(\lambda) - g(\lambda) \succ 0.$$

It is easy to check that \succ is a total order and $(\mathbb{R}(\lambda), \succ)$ is an ordered field (see [9]). Note that this field is non-Archimedean: $\lambda \succ n$ for any $n = \underbrace{1 + \dots + 1}_n$.

Clearly, a rational function $\rho_{xy}(\lambda)$ as in (14) is positive as an element of $(\mathbb{R}(\lambda), \succ)$, where “ \succ ” is defined as in Definition 14.

Let (K, \succ) be an arbitrary ordered field. We say that $k \in K$ is *positive* if $k \succ 0$. For $k_1, k_2 \in K$ we will write

$$k_1 \succeq k_2, \text{ if } k_1 \succ k_2 \text{ or } k_1 = k_2.$$

Moreover, we will write

$$k_1 \prec k_2, \text{ if } k_2 \succ k_1$$

and

$$k_1 \preceq k_2, \text{ if } k_1 \prec k_2 \text{ or } k_1 = k_2.$$

DEFINITION 15. A *network over the ordered field K* is a structure

$$\Gamma = ((V, E), \{\rho_{xy}\}, a_0, a_1),$$

where (V, E) is a connected graph, $\rho : E \rightarrow K$ is a positive function in (K, \succ) (i.e. $\rho_{xy} \succ 0$ for every $xy \in E$), and $a_0, a_1 \in V$ are two fixed vertices.

Note that we can consider ρ as a function from $V \times V$ to K by setting $\rho_{xy} = 0$, if xy is not an edge. Then the weight ρ_{xy} gives rise to a function on vertices as follows:

$$\rho(x) = \sum_y \rho_{xy}, \tag{15}$$

where the notation \sum_y means $\sum_{y \in V}$. Then $\rho(x)$ is called the *weight of a vertex x* . By properties of the ordered field, we have $\rho(x) \succ 0$ for any $x \in V$.

DEFINITION 16. For any function $f : V \rightarrow K$ the *weighted Laplacian Δ_ρ* is defined as

$$\Delta_\rho f(x) = \sum_y (f(y) - f(x))\rho_{xy} = \sum_y (\nabla_{xy} f)\rho_{xy},$$

where

$$\nabla_{xy} f = f(y) - f(x)$$

is the *difference operator*.

From now on we assume that (V, E) is a finite graph and Γ is a network over the ordered field K on this graph.

THEOREM 17. *The following Dirichlet problem:*

$$\begin{cases} \Delta_\rho v(x) = 0 \text{ on } V \setminus \{a_0, a_1\}, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases} \tag{16}$$

where $v : V \rightarrow K$ is an unknown function, has always a unique solution over K .

The key point for the proof of Theorem 17 is the following lemma.

LEMMA 18. (A maximum/minimum principle) *Let B be a non-empty subset of V , such that $V \setminus B$ is also non-empty. Then, for any function $u : V \rightarrow K$, that satisfies $\Delta_\rho u(x) \succeq 0$ (i.e. u is subharmonic) on $V \setminus B$, we have*

$$\max_{V \setminus B} u \preceq \max_B u \tag{17}$$

and for any function $u : V \rightarrow K$, that satisfies $\Delta_\rho u(x) \preceq 0$ (i.e. u is superharmonic) on $V \setminus B$, we have

$$\min_{V \setminus B} u \succeq \min_B u. \tag{18}$$

Proof. It is enough to proof the first claim (then the second claim follows by changing u to $-u$). Set

$$M = \max_{V \setminus B} u,$$

and assume from the contrary, that

$$M \succ \max_B u. \tag{19}$$

Let us consider the set

$$S = \{x \in V : u(x) = M\}.$$

Claim 1. *If $x \in S$, then all neighbors of x also belong to S .*

Observe, that by assumption (19) we have $S \subset V \setminus B$. Therefore, for any $x \in S$ $\Delta_\rho u(x) \succeq 0$, which can be rewritten in the form

$$u(x) \preceq \sum_{y: y \sim x} \frac{\rho_{xy}}{\rho(x)} u(y). \tag{20}$$

By properties of positive elements, we have

$$\frac{\rho_{xy}}{\rho(x)} \succ 0 \text{ for any } y \sim x.$$

Also, for any y we have $u(y) \preceq M$ by the definition of maximum. Therefore,

$$\frac{\rho_{xy}}{\rho(x)} u(y) = \frac{\rho_{xy}}{\rho(x)} M, \text{ if } u(y) = M, \tag{21}$$

$$\text{and } \frac{\rho_{xy}}{\rho(x)} u(y) \prec \frac{\rho_{xy}}{\rho(x)} M, \text{ if } u(y) \prec M, \tag{22}$$

where the last line is true by properties of positive elements. If there exist $y_0 \sim x$ such that $u(y_0) \prec M$, then, summing up all the equalities (21) and inequalities (22), we obtain

$$\sum_{y \sim x} \frac{\rho_{xy}}{\rho(x)} u(y) \prec \sum_{y \sim x} \frac{\rho_{xy}}{\rho(x)} M. \tag{23}$$

But

$$\sum_{y \sim x} \frac{\rho_{xy}}{\rho(x)} M = M = u(x),$$

therefore, (23) is a contradiction with (20).

Claim 2. Let S be a non-empty set of vertices of a connected graph (V, E) such that $x \in S$ implies that all neighbours of x belong to S . Then $S = V$.

Indeed, let $x \in S$ and y be any other vertex. Then by the definition of connected graph, there is a path $\{x_k\}_{k=0}^n$ between x and y , that is,

$$x = x_0 \sim x_1 \sim x_2 \sim \dots \sim x_n = y.$$

Since $x_0 \in S$ and $x_1 \sim x_0$, we obtain $x_1 \in S$. Since $x_2 \sim x_1$, we obtain $x_2 \in S$. By induction, we conclude that all $x_k \in S$, whence $y \in S$.

It follows from two claims that set S must coincide with V , which is not possible since $u(x) \prec M$ for any $x \in B$. This contradiction shows that $M \preceq \max_B u$.

Proof of the Theorem 17. Let us first prove the uniqueness. If we have two solutions v_1 and v_2 of (16), then the difference $v = v_1 - v_2$ satisfies the conditions

$$\begin{cases} \Delta_\rho v(x) = 0 \text{ on } V \setminus \{a_0, a_1\}, \\ v(x) = 0 \text{ on } \{a_0, a_1\}, \end{cases} \tag{24}$$

and, by Lemma 18

$$0 = \max_{\{a_0, a_1\}} v \succeq \max_{V \setminus \{a_0, a_1\}} v \succeq \min_{V \setminus \{a_0, a_1\}} v \succeq \min_{\{a_0, a_1\}} v = 0,$$

whence, $v \equiv 0$ since $v(a_0) = v(a_1) = 0$. Let us now prove the existence of a solution of (16). For any $x \in V \setminus \{a_0, a_1\}$, rewrite the equation $\Delta_\rho v(x) = 0$ in the form

$$\sum_{\substack{y \sim x, \\ y \in V \setminus \{a_0, a_1\}}} \frac{\rho_{xy}}{\rho(x)} v(y) - v(x) = -\frac{\rho_{xa_1}}{\rho(x)} v(a_1) - \frac{\rho_{xa_0}}{\rho(x)} v(a_0). \tag{25}$$

Let us denote by \mathcal{F} the set of all functions v on $V \setminus \{a_0, a_1\}$ with values in $K^{(n-2)}$, where $n = |V|$. Then the left hand side of (25) can be regarded as an operator in this space; let us denote it by Lv , that is

$$Lv(x) = \sum_{\substack{y \sim x, \\ y \in V \setminus \{a_0, a_1\}}} \frac{\rho_{xy}}{\rho(x)} v(y) - v(x), \tag{26}$$

for all $x \in V \setminus \{a_0, a_1\}$. Rewrite the equation (25) in the form $Lv = h$, where h is the right hand side of (25), which is a given function on $V \setminus \{a_0, a_1\}$. Note that \mathcal{F} is a linear space over the field K . Since the family $\{\mathbf{1}_{\{x\}}\}_{x \in V \setminus \{a_0, a_1\}}$ of indicator functions form a basis in \mathcal{F} , we obtain that $\dim \mathcal{F} = n - 2 < \infty$. Hence, the operator $L : \mathcal{F} \rightarrow \mathcal{F}$ is a linear operator in a finitely dimensional space, and the first part of the proof shows that $Lv = 0$ implies $v = 0$ (indeed, just set $v(a_1) = 0$ and $v(a_0) = 0$ in (25)), that is, the operator L is injective. By Linear Algebra, any injective operator acting in the spaces of equal dimensions, must be bijective. Hence, for any $h \in \mathcal{F}$ (in particular, for $h(x) = -\frac{\rho_{xa_1}}{\rho(x)}$), there is a solution, which finishes the proof.

COROLLARY 19. (Proposition 5) *For any given network Γ the determinant $D(\lambda)$ of the Dirichlet problem (5) has a finite number of zeros in \mathbb{C} .*

Proof. For any given network the determinant of the Dirichlet problem (5) is a rational function on λ and, by Theorem 17, it is not constantly zero.

COROLLARY 20. *For the solution $v : V \rightarrow K$ of (16) the following inequality*

$$0_K \preceq v(x) \preceq 1_K \tag{27}$$

is true for any $x \in V$.

Proof. Apply Lemma 18 for $B = \{a_0, a_1\}$.

Now we can define the effective impedance of the network over the ordered field K .

DEFINITION 21. Let v be a solution of the Dirichlet problem (16) for the network. Then define the *effective impedance* by

$$Z_{eff} = \frac{1}{\sum_x v(x)\rho_{xa_0}}, \tag{28}$$

The quantity

$$\mathcal{P}_{eff} = \frac{1}{Z_{eff}} = \sum_x v(x)\rho_{xa_0}$$

is called the *effective admittance*.

Since by Theorem 17 the Dirichlet problem (16) has exactly one solution over the field K , the effective impedance is always well-defined as an element of K .

In particular, when $K = \mathbb{R}(\lambda)$ the effective impedance is always well-defined as a rational function.

Moreover, by (27) we have $\mathcal{P}_{eff} \succeq 0$ and, hence, $Z_{eff} \succeq 0$ in K .

THEOREM 22. (Green’s formula) *Let Γ be a network over the ordered field K with the vertex set V , and let Ω be a non-empty subset of V . Then, for any two functions $f, g : V \rightarrow K$,*

$$\sum_{x \in \Omega} \Delta_\rho f(x)g(x) = -\frac{1}{2} \sum_{x,y \in \Omega} (\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy} + \sum_{x \in \Omega} \sum_{x \in V \setminus \Omega} (\nabla_{xy} f)g(x)\rho_{xy}. \tag{29}$$

If $\Omega = V$, then the last term in (29) vanishes, and we obtain

$$\sum_{x \in V} \Delta_\rho f(x)g(x) = -\frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} f)(\nabla_{xy} g)\rho_{xy} \tag{30}$$

Proof.

$$\begin{aligned} \sum_{x \in \Omega} \Delta_{\rho} f(x) g(x) &= \sum_{x \in \Omega} \left(\sum_{y \in V} (f(y) - f(x)) \rho_{xy} \right) g(x) \\ &= \sum_{x \in \Omega} \sum_{y \in V} (f(y) - f(x)) g(x) \rho_{xy} \\ &= \sum_{x \in \Omega} \sum_{y \in \Omega} (f(y) - f(x)) g(x) \rho_{xy} + \sum_{x \in \Omega} \sum_{y \in V \setminus \Omega} (f(y) - f(x)) g(x) \rho_{xy} \\ &= \sum_{y \in \Omega} \sum_{x \in \Omega} (f(x) - f(y)) g(y) \rho_{xy} + \sum_{x \in \Omega} \sum_{y \in V \setminus \Omega} (\nabla_{xy} f) g(x) \rho_{xy}, \end{aligned}$$

where in the last line we have switched notation of the variables x and y in the first sum. Adding together the last two lines and dividing by 2 (it is possible, since any ordered field has characteristic 0, see [9]), we obtain

$$\sum_{x \in \Omega} \Delta_{\rho} f(x) g(x) = -\frac{1}{2} \sum_{x \in \Omega} \sum_{y \in \Omega} (\nabla_{xy} f) (\nabla_{xy} g) \rho_{xy} + \sum_{x \in \Omega} \sum_{y \in V \setminus \Omega} (\nabla_{xy} f) g(x) \rho_{xy},$$

which was to be proved.

COROLLARY 23. *For any function $f : V \rightarrow K$,*

$$\sum_{x \in V} \Delta_{\rho} f(x) = 0. \tag{31}$$

Proof. Apply (30) for $g \equiv 1$.

LEMMA 24. *For any network we have*

$$\mathcal{P}_{eff} = \Delta_{\rho} v(a_0) = -\Delta_{\rho} v(a_1), \tag{32}$$

where v is the solution of the Dirichlet problem (16).

Proof. Using $v(a_0) = 0$, we obtain

$$\Delta_{\rho} v(a_0) = \sum_{y: y \sim a_0} (v(y) - v(a_0)) \rho_{a_0 y} = \sum_{y: y \sim a_0} v(y) \rho_{a_0 y} = \mathcal{P}_{eff}. \tag{33}$$

The second equality in (32) follows from (31), since v is the solution of the Dirichlet problem (16).

THEOREM 25. *Let v be the solution of the Dirichlet problem (16) for network Γ over the ordered field K . Then*

$$\frac{1}{2} \sum_{x, y \in V} (\nabla_{xy} v)^2 \rho_{xy} = \mathcal{P}_{eff}. \tag{34}$$

Proof. Applying (30) to the left hand side of (34) we obtain

$$\begin{aligned} \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v)^2 \rho_{xy} &= - \sum_{x \in V} \Delta_\rho v(x) v(x) \\ &= - \sum_{x \in V \setminus \{a_0, a_1\}} \Delta_\rho v(x) v(x) - \Delta_\rho v(a_0) v(a_0) + \Delta_\rho v(a_1) v(a_1) \\ &= -\Delta_\rho v(a_1), \end{aligned}$$

since v is the solution of (16). The statement (34) is proved due to Lemma 24.

THEOREM 26. (Dirichlet/Thomson’s principle) *Let v be the solution of the Dirichlet problem (16) for the network Γ over the ordered field K . Then for any other function $f : V \rightarrow K$ such that $f(a_0) = 0$ and $f(a_1) = 1$, the following inequality holds:*

$$\frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v)^2 \rho_{xy} \preceq \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} f)^2 \rho_{xy} \tag{35}$$

Proof. Let $g = f - v$. Then $g(a_0) = g(a_1) = 0$. Therefore,

$$\begin{aligned} \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} f)^2 \rho_{xy} &= \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} (g + v))^2 \rho_{xy} = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} g + \nabla_{xy} v)^2 \rho_{xy} \\ &= \frac{1}{2} \sum_{x,y \in V} ((\nabla_{xy} g)^2 + 2(\nabla_{xy} g)(\nabla_{xy} v) + (\nabla_{xy} v)^2) \rho_{xy} \\ &= \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v)^2 \rho_{xy} + \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} g)^2 \rho_{xy} + \sum_{x,y \in V} (\nabla_{xy} v)(\nabla_{xy} g) \rho_{xy}, \end{aligned}$$

where the last term vanishes by Green’s formula (29), since $g(a_0) = g(a_1) = 0$ and v is the solution of (16) and the second term is greater then zero whenever $g \not\equiv 0$. Therefore, (35) is proved and an equality is attained if and only if $f \equiv v$.

4. Comparison of two definitions of Z_{eff}

Denote by $Z_{eff}^{(1)}(\lambda)$ the effective impedance defined in Section 2, that we from now on will consider as a function of $\lambda = i\omega$.

The effective impedance from Section 3 for the field $K = \mathbb{R}(\lambda)$ will be denoted by $Z_{eff}^{(2)}(\lambda)$. Note that it was already defined as a rational function of λ .

Of course, the arises question is whether

$$Z_{eff}^{(1)}(\lambda) = Z_{eff}^{(2)}(\lambda) \tag{36}$$

for all λ , such that $-i\lambda > 0$.

The unique solution $v(\lambda)$ of the Dirichlet problem (16) can be found by Cramer’s rule applied in the field $\mathbb{R}(\lambda)$. Hence, $v(\lambda)(x)$ is a rational function of λ for any $x \in V$.

Consequently, $Z_{eff}^{(2)}(\lambda)$ is a rational function of λ . Note, that any rational function of $\lambda \in \mathbb{C}$ takes values in $\mathbb{C} \cup \{\infty\}$ (for any $\lambda \in \mathbb{C}$).¹ Consequently, any rational function is continuous with the values in $\mathbb{C} \cup \{\infty\}$. In particular, $Z_{eff}^{(2)}(\lambda)$ is continuous function of $\lambda \in \mathbb{C}$ with values in $\mathbb{C} \cup \{\infty\}$ (including the value ∞).

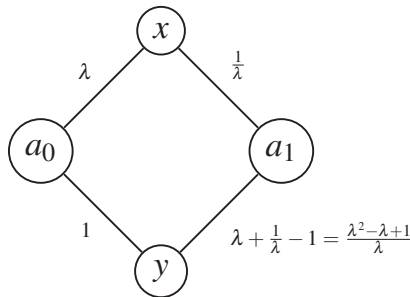
By Cramer’s rule, applied in \mathbb{C} , the function $Z_{eff}^{(1)}(\lambda)$ is also a rational function of λ at all λ , where the determinant $D(\lambda)$ of the Dirichlet problem (5) does not vanish. Moreover, at those λ , where $D(\lambda) \neq 0$ the equality (36) is true by Cramer’s rule.

By Theorem 17, $D(\lambda)$ vanishes only at finitely many values of λ , therefore, the identity (36) will be true for all $-i\lambda > 0$ if we know that $Z_{eff}^{(1)}(\lambda)$ is continuous in λ .

However, it is not obvious for those λ , where (5) has multiple solutions or no solution.

The question should definitely be restricted just to the Dirichlet problem, which arises from electrical networks, and to the case of pure imaginary λ , as the following two examples show.

EXAMPLE 27. Consider the Dirichlet problem for the network at the figure below, where weights are shown on each edge. Note, that here the weight of the edge ya_1 is positive function, but it is *not* in the form (4).



The Dirichlet problem for this network is

$$\begin{cases} (v(a_0) - v(x))\lambda + \frac{v(a_1) - v(x)}{\lambda} = 0, \\ (v(a_0) - v(y)) + (v(a_1) - v(y))(\lambda + \frac{1}{\lambda} - 1) = 0, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases}$$

The determinant of this system is

$$D = D(\lambda) = \frac{1}{\lambda^2}(\lambda^2 + 1)^2$$

¹For any rational function $\frac{P(\lambda)}{Q(\lambda)}$ with polynomials P, Q , one can always assume that P and Q have no common zeros, since otherwise their common linear factor can be cancelled. Hence, one avoids indeterminate form $\frac{0}{0}$.

and its zeros are i and $-i$.

In case $D(\lambda) \neq 0$ the solution of the Dirichlet problem is

$$v(x) = \frac{1}{\lambda^2 + 1}, v(y) = \frac{\lambda^2 - \lambda + 1}{\lambda^2 + 1}.$$

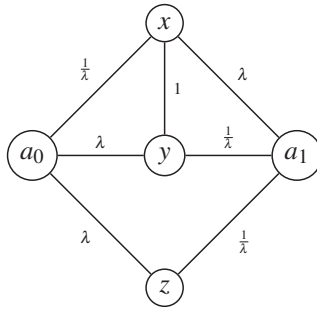
and it has no finite limit as $\lambda \rightarrow i$.

But the effective impedance in this case is $Z_{eff}^{(1)}(\lambda) = 1$. Therefore, $Z_{eff}^{(2)}(\lambda) \equiv 1$. The Dirichlet problem in the case $\lambda = i$ is

$$\begin{cases} (v(a_0) - v(x))i - (v(a_1) - v(x))i = 0, \\ (v(a_0) - v(y)) - (v(a_1) - v(y)) = 0, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases}$$

and, obviously, has no solutions. Therefore, $Z_{eff}^{(1)}(i) = 0$ by definition. Hence $Z_{eff}^{(1)}(\lambda)$ is not continuous at the point $\lambda = i$ and $Z_{eff}^{(1)}(i) \neq Z_{eff}^{(2)}(i)$.

EXAMPLE 28. Consider the Dirichlet problem for the network at the figure below, where admittances are shown on each edge.



The Dirichlet problem for this network is

$$\begin{cases} \frac{(v(a_0) - v(x))}{\lambda} + (v(a_1) - v(x))\lambda + (v(y) - v(x)) = 0, \\ (v(a_0) - v(y))\lambda + \frac{(v(a_1) - v(y))}{\lambda} + (v(x) - v(y)) = 0, \\ (v(a_0) - v(z))\lambda + \frac{(v(a_1) - v(z))}{\lambda} = 0, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases}$$

The determinant of this system is

$$D = D(\lambda) = - \left(\frac{1}{\lambda^2} + \frac{2}{\lambda} + 2 + 2\lambda + \lambda^2 \right) \left(\lambda + \frac{1}{\lambda} \right)$$

$$= -\frac{1}{\lambda^3}(\lambda + 1)^2(\lambda^2 + 1)^2,$$

and it is easy to see, that $\lambda = \pm i$ and $\lambda = -1$ are its zeros.

In case $D(\lambda) \neq 0$ the solution of the Dirichlet problem is

$$v(\lambda) = (v(x), v(y), v(z), v(a_0), v(a_1)) = \left(\frac{\lambda}{1 + \lambda}, \frac{1}{1 + \lambda}, \frac{1}{1 + \lambda^2}, 0, 1 \right).$$

The effective impedance in these cases is

$$Z_{eff}^{(1)}(\lambda) = Z_{eff}^{(2)}(\lambda) = \frac{\lambda^2 + 1}{\lambda^2 + \lambda + 1}. \quad (37)$$

Note that the finite limit of v does not exist when λ goes to i or λ goes to -1 .

The Dirichlet problem in the case $\lambda = i$ is

$$\begin{cases} -(v(a_0) - v(x))i + (v(a_1) - v(x))i + (v(y) - v(x)) = 0, \\ (v(a_0) - v(y))i - (v(a_1) - v(y))i + (v(x) - v(y)) = 0, \\ (v(a_0) - v(z))i - (v(a_1) - v(z))i = 0, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases}$$

and has no solution, which by definition of effective impedance implies $Z_{eff}^{(1)}(i) = 0$. It is easy to see, that $Z_{eff}^{(2)}(i) = 0$. Therefore $Z_{eff}^{(1)}(i) = Z_{eff}^{(2)}(i)$ and $Z_{eff}^{(1)}(\lambda)$ is continuous at the point $\lambda = i$.

The Dirichlet problem in the case $\lambda = -1$ is

$$\begin{cases} -(v(a_0) - v(x)) - (v(a_1) - v(x)) + (v(y) - v(x)) = 0, \\ -(v(a_0) - v(y)) - (v(a_1) - v(y)) + (v(x) - v(y)) = 0, \\ -(v(a_0) - v(z)) - (v(a_1) - v(z)) = 0, \\ v(a_0) = 0, \\ v(a_1) = 1. \end{cases}$$

and it has multiple solutions

$$v = (v(x), v(y), v(z), v(a_0), v(a_1)) = \left(\tau, 1 - \tau, \frac{1}{2}, 0, 1 \right), \tau \in \mathbb{C}.$$

One can calculate, that, $Z_{eff}^{(1)}(-1) = -\frac{2}{3}$. But from (37) follows, that $Z_{eff}^{(1)}(-1) = 2$. Therefore, (36) fails at the point $\lambda = -1$ and $Z_{eff}^{(1)}(\lambda)$ is not continuous at this point.

Acknowledgement. The author thanks her scientific advisor, Professor Alexander Grigor'yan, for helpful discussions on the topic.

REFERENCES

- [1] O. BRUNE, *Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency*, Thesis (Sc. D.). Massachusetts Institute of Technology, Dept. of Electrical Engineering, Massachusetts, 1931.
- [2] CHARLES A. DESOER, ERNEST S. KUH, *Basic circuit theory*, McGraw-Hill Book Company, USA, 1969.
- [3] P.G. DOYLE, J.L. SNELL, *Random walks and electric networks*, Carus Mathematical Monographs 22, Mathematical Association of America, Washington, DC, 1984.
- [4] RICHARD P. FEYNMAN, ROBERT B. LEIGHTON, MATTHEW SANDS, *The Feynman lectures on physics, Volume 2: Mainly Electromagnetism and Matter*, Addison-Wesley publishing company, Reading, Massachusetts, Fourth printing – 1966.
- [5] A. GRIGOR'YAN, *Introduction to Analysis on Graphs*, AMS University Lecture Series, Volume: 71, Providence, Rhode Island, 2018.
- [6] EDWARD HUGHES, *Electrical and electronic technology*, Pearson Education Limited, England, Tenth edition – 2008.
- [7] DAVID A. LEVIN, YUVAL PERES, ELIZABETH L. WILMER, *Markov Chains and Mixing Times*, AMS University Lecture Series, Providence, Rhode Island, 2009.
- [8] ARIEH L. SHENKMAN, *Circuit analysis for power engineering handbook*, Springer Science+Business Media, B. V, USA, 1998.
- [9] B. L. VAN DER WAERDEN, *Algebra, Volume I*, Springer, New York, 2003.

(Received August 10, 2020)

Anna Muranova
5050 Institut für discrete Mathematik
Steyrergasse 30/II, 8010 Graz, Austria
e-mail: anna.muranova@gmail.com