Operators and Matrices
Volume 14, Number 3 (2020), 767–776
doi:10.7153/oam-2020-14-48

BREATHER SOLUTIONS ON DISCRETE NECKLACE GRAPHS

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Abstract. We show the existence of breather solutions in a nonlinear Klein-Gordon system on a discrete graph with periodic junctions. The proof is based on the Theorem of Crandall-Rabinowitz.

1. Introduction

We are interested in the dynamics of some nonlinear lattice differential equations on an infinite discrete graph with periodically ordered junctions. There is a competition between linear decay and the focusing effect of the nonlinearity, which allows for the existence of localized solutions. From a mathematical point of view, existence of real-valued, time-periodic and spatially localized solutions, also known as (discrete) breather solutions, is an interesting topic. Breather solutions in nonlinear PDEs are very rare. Denzler [3] showed that the breathers of the Sine-Gordon equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + \sin(u(x,t)) = 0, \quad x,t \in \mathbb{R},$$

disappear if the nonlinearity is perturbed. The rareness of breathers in PDEs makes it hard to believe that these non-generic, structurally unstable objects describe phenomena in nature. However, the situation is different on lattices and breather solutions come back. MacKay and Aubry [11] constructed breathers in Hamiltonian lattices with anharmonic on-site potentials and weak coupling. In their proof breathers are obtained by continuation from the uncoupled case in which trivial breathers exist. This means that only one oscillator is excited and the others are at rest. With the same technique, the existence of breathers was proved for diatomic Fermi-Pasta-Ulam (FPU) chains, cf. [10]. Aubry et al. [1] have proved the existence of breathers in FPU lattices with frequencies above the phonon spectrum, when the interaction potential V is a strictly convex polynomial of degree 4. These results are obtained via a variational method.

There is an important condition for the existence of breathers, namely that the square of integer multiples of the breather frequency lie in the resolvent set of the linearized right hand side. Since the spectrum of a lattice problem is bounded, there is a good chance to find breather solutions, cf. [4].

Mathematics subject classification (2010): 39A12, 39A28.
Keywords and phrases: Time-periodic, spatially-localized, bifurcation, Klein Gordon, discrete graph.

Paper OaM-14-48

Here, we consider the discrete Klein-Gordon system

$$\begin{aligned}
\partial_t^2 u_j(t) &= f(v_j^+(t)-u_j(t))+f(v_j^-(t)-u_j(t))-h(u_j(t)-w_{j-1}(t))-r_u(u_j(t)),\\
\partial_t^2 v_j^+(t) &= g(w_j(t)-v_j^+(t))-f(v_j^+(t)-u_j(t))-r_v(v_j^+(t)),\\
\partial_t^2 v_j^-(t) &= g(w_j(t)-v_j^-(t))-f(v_j^-(t)-u_j(t))-r_v(v_j^-(t)),\\
\partial_t^2 w_j(t) &= h(u_{j+1}(t)-w_j(t))-g(w_j(t)-v_j^+(t))-g(w_j(t)-v_j^-(t))-r_w(w_j(t)),
\end{aligned} \tag{1}$$

with interaction potentials $f,g,h:\mathbb{R}\to\mathbb{R}$, local potentials $r_u,r_v,r_w:\mathbb{R}\to\mathbb{R}$ and coordinates $u_j,v_j^{\pm},w_j\in\mathbb{R}$, for all $j\in\mathbb{Z}$ and $t\in\mathbb{R}$, on the subsequent discrete graph with periodic branching.

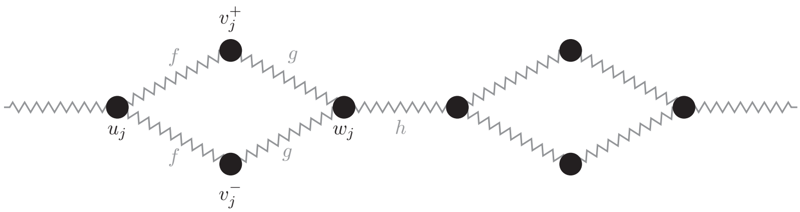

Figure 1: Topology of the discrete necklace graph

The coordinates

$$(u_j,v_j^+,v_j^-,w_j)^T = Z_j\in\mathbb{R}^4, \tag{2}$$

correspond to the horizontal displacement of the mass particles from its equilibrium positions. We assume that all forces vanish at the origin and consider Taylor expansions $f(x)=f_1x+f_2x^2+\ldots$ of the forces. Further, let $f_1,g_1,h_1>0$ and $r_1>0$.

Thus, the main result (cf. Theorem 2) can be stated as follows:

Let $-\omega_0^2=-(f_1+g_1+(r_v)_1)$ be the eigenvalue of the linear part L in (1), which corresponds to the straight line of the spectral picture in Figure 3. Suppose that the non-resonance condition $-m^2\omega_0^2\notin\sigma_{ac}(L)$ is fulfilled for all $m\in\mathbb{N}_0$. Then, there exists a one-parameter family of real-valued solutions that are periodic in time and spatially localized. The constructed breather solutions bifurcate from eigenstates that are localized in a single ring of the periodic graph, cf. Figure 2. Hence, these non-symmetric solutions are strongly localized.

To our knowledge there are no existence results for breathers on discrete graphs with periodic branching so far. In [12], symmetric breathers have been constructed on a metric version of the graph in Figure 1, using spatial dynamics and center manifold reduction.

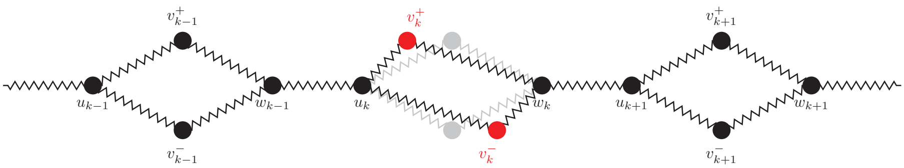

Figure 2: Antisymmetric eigenstate with support in the kth periodicity cell. Only the masses v_k^+ and v_k^- are displaced from their equilibrium positions.

This article is organized as follows. We compute the spectral picture for the Klein-Gordon system in Section 2. In Section 3 we apply the bifurcation theorem of Crandall-Rabinowitz in order to construct the one-parameter family of breathers.

Notation: We equip the vector-valued sequence spaces with the norm

$$\|(Z_j)_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z},\mathbb{R}^4)} = \left(\sum_{j\in\mathbb{Z}} |Z_j|^p_{\mathbb{R}^4}\right)^{\frac{1}{p}} < \infty$$

for $p \in [1,\infty)$.

2. Spectral situation

The linearized discrete Klein Gordon system

$$\begin{pmatrix} \partial_t^2 u_j \\ \partial_t^2 v_j^+ \\ \partial_t^2 v_j^- \\ \partial_t^2 w_j \end{pmatrix} = \begin{pmatrix} f(v_j^+ - u_j) + f(v_j^- - u_j) - h(u_j - w_{j-1}) - r_u u_j \\ g(w_j - v_j^+) - f(v_j^+ - u_j) - r_v v_j^+ \\ g(w_j - v_j^-) - f(v_j^- - u_j) - r_v v_j^- \\ h(u_{j+1} - w_j) - g(w_j - v_j^+) - g(w_j - v_j^-) - r_w w_j \end{pmatrix} \tag{3}$$

is of the form

$$\partial_t^2 Z_j = LZ_j = M^0 Z_j + M^- Z_{j-1} + M^+ Z_{j+1}, \quad j \in \mathbb{Z}, \tag{4}$$

with $Z_j = (u_j, v_j^+, v_j^-, w_j)^T$ and matrices $M^0, M^-, M^+ \in \mathbb{R}^{4\times 4}$ with constant coefficients. Due to its periodic structure, it is solved by so called Bloch waves

$$Z_j(t) = e^{i\omega t} e^{ilj} \tilde{Z}(l), \quad l \in \mathbb{R}, \tag{5}$$

with $\omega \in \mathbb{R}$ and 2π-periodic functions $\tilde{Z}(l) = (\tilde{u}(l), \tilde{v}^+(l), \tilde{v}^-(l), \tilde{w}(l))^T$. This leads to the eigenvalue problem

$$M(l)\tilde{Z}(l) = -\omega^2(l)\tilde{Z}(l), \tag{6}$$

with

$$M(l) := M^0 + e^{-il}M^- + e^{il}M^+ =$$
$$\begin{pmatrix} -(2f_1+h_1+(r_u)_1) & f_1 & f_1 & h_1e^{-il} \\ f_1 & -(g_1+f_1+(r_v)_1) & 0 & g_1 \\ f_1 & 0 & -(g_1+f_1+(r_v)_1) & g_1 \\ h_1e^{il} & g_1 & g_1 & -(h_1+2g_1+(r_w)_1) \end{pmatrix}.$$

Floquet-Bloch theory, cf. [13], implies that the spectrum of L has band gap structure and

$$\sigma(L) = \bigcup_{l\in[-\pi,\pi)} \sigma(M(l)). \tag{7}$$

For a fixed number l, the matrix $M(l)$ has four real eigenvalues. Therefore, $\sigma(L)$ consists of four bands, cf. Figure 3. The spectrum has an absolutely continuous part plus an eigenvalue of infinite multiplicity, which corresponds to the flat spectral band, cf. [9].

The stiffness parameters f_1, g_1 and h_1 determine whether the eigenvalue $-\omega_0^2 = -(f_1+g_1+(r_v)_1)$ is isolated or located within the continuous spectrum. Typical spectral pictures are sketched in Figure 3.

Point spectrum: The eigenspace contains anti-symmetric sequences with respect to the junctions. An eigenbasis $\{E_k\}_{k\in\mathbb{Z}}$ can be chosen compactly supported in the circles. In particular, for a fixed $k\in\mathbb{Z}$, the sequence E_k is defined by

$$v_k^+(t) = \frac{1}{\sqrt{2}} = -v_k^-(t), \quad v_j^\pm = 0\ \forall\, j\neq k,\ u_j = w_j = 0\ \forall\, j, \tag{8}$$

cf. Figure 2.

Continuous spectrum: The (generalized) eigenfunctions corresponding to the absolutely continuous spectrum are symmetric w.r.t. the semi-circles, i.e. $v_j^+ = v_j^-$ for all $j\in\mathbb{Z}$, which we refer to as $\tilde{V}_{sym}$.

Let $p\in\mathbb{N}$, $p\geqslant 2$ be the power of the nonlinearity, which is of the form

$$N^p\begin{pmatrix} u_j \\ v_j^+ \\ v_j^- \\ w_j \end{pmatrix} = \begin{pmatrix} f_p(v_j^+-u_j)^p + f_p(v_j^- - u_j)^p - h_p(u_j-w_{j-1})^p - r_pu_j^p \\ g_p(w_j-v_j^+)^p - f_p(v_j^+-u_j)^p - r_p(v_j^+)^p \\ g_p(w_j-v_j^-)^p - f_p(v_j^- - u_j)^p - r_p(v_j^-)^p \\ h_p(u_{j+1}-w_j)^p - g_p(w_j-v_j^+)^p - g_p(w_j-v_j^-)^p - r_pw_j^p \end{pmatrix}.$$

We decompose the sequence space into its symmetric and anti-symmetric parts,

$$\tilde{V}_{sym} \oplus (\oplus_{k\in\mathbb{Z}}\mathrm{span}\{E_k\}).$$

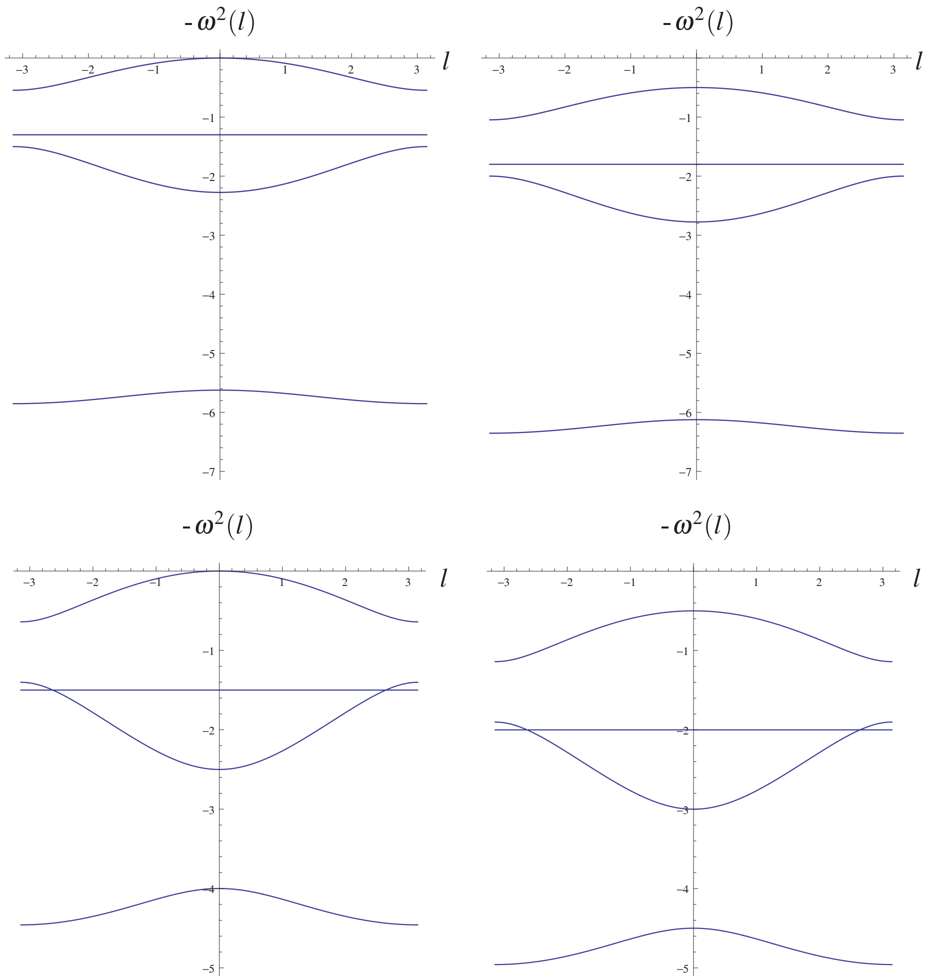

Figure 3: Spectral picture for four examples of parameter sets.
Upper panel: isolated eigenvalue. Left: Vanishing local forces: $f_1 = 1$, $g_1 = 0.3$, $h_1 = 2$; Right: Non-vanishing local forces: $f_1 = 1$, $g_1 = 0.3$, $h_1 = 2$ and $r_1 = 0.5$
Lower panel: embedded eigenvalue. Left: Vanishing local forces: $f_1 = 0.5$, $g_1 = 1$, $h_1 = 1$; Right: Non-vanishing local forces: $f_1 = 0.5$, $g_1 = 1$, $h_1 = 1$ and $r_1 = 0.5$

An explicit computation shows

$$LE_k = -\omega_0^2 E_k, \quad N^p(E_k) \subset \begin{cases} \operatorname{span}\{E_k\}, & \text{for } p \text{ odd}, \\ \tilde{V}_{sym}, & \text{for } p \text{ even}. \end{cases} \tag{9}$$

This means that discrete eigenfunctions that are not present in the initial data will not be excited at any time. Further,

$$Z \in \tilde{V}_{sym} \Rightarrow LZ, N^p(Z) \in \tilde{V}_{sym}, \tag{10}$$

i.e., the subspace $\tilde{V}_{sym}$ is invariant under the actions of L and N^p.

3. Existence of breather solutions for non-vanishing local forces

Discrete breathers arise from the combined effect of nonlinearity and discreteness. We will prove the existence of nontrivial solutions by means of bifurcation analysis. Therefore, we fix an integer $k_0 \in \mathbb{Z}$ and define the time-independent spaces

$$V_{sym} := \left\{ ((u_j, v_j^+, v_j^-, w_j)^T)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{R}^4) \,|\, v_j^+ = v_j^- \text{ for all } j \in \mathbb{Z} \right\}, \tag{11}$$

$$V(k_0) := V_{sym} \oplus \operatorname{span}\{E_{k_0}\}, \tag{12}$$

Let $I = \left[-\frac{\pi}{\omega_0}, \frac{\pi}{\omega_0}\right]$ with $-\omega_0^2$ the eigenvalue of L and define the time-dependent Banach spaces

$$X(k_0) := C^2_{per}(I, V(k_0)), \tag{13}$$

with $\|Z\|_{X(k_0)} := \max_{t\in I} \|Z(t)\|_{\ell^2} + \max_{t\in I} \|\dot{Z}(t)\|_{\ell^2} + \max_{t\in I} \|\ddot{Z}(t)\|_{\ell^2}$ and

$$Y(k_0) := C^0_{per}(I, V(k_0)), \tag{14}$$

with $\|Z\|_{Y(k_0)} := \max_{t\in I} \|Z(t)\|_{\ell^2}$ of periodically extendable functions with values in $V(k_0)$. Moreover, we restrict to even functions in time $X_{even}(k_0) := C^2_{per,even}(I, V(k_0))$ if the power p of the nonlinearity is even and use $X_{odd}(k_0) := C^2_{per,odd}(I, V(k_0))$ if p is odd. We suppress the indices *even* and *odd* until it becomes important. The choice of the Banach spaces is motivated by our interest in (possibly) bifurcating solutions that are real-valued, periodic in time and localized in space. We consider the mapping

$$\begin{aligned} F &: X(k_0) \times \mathbb{R} \to Y(k_0), \\ F(Z,\mu)(t) &= (1+\mu)\ddot{Z}(t) - LZ(t) - N^p(Z)(t). \end{aligned} \tag{15}$$

The mapping F is well-defined due to observations (9), (10) and

$$\|N^p(Z)(t)\|^2_{\ell^2(\mathbb{Z},\mathbb{R}^4)} \leqslant C(f,g,h,r)\|Z(t)\|^{2p}_{\ell^{2p}(\mathbb{Z},\mathbb{R}^4)} \leqslant C(f,g,h,r,p)\|Z(t)\|^{2p}_{\ell^2(\mathbb{Z},\mathbb{R}^4)},$$

with parameter-depending constants C, where we made use of the embedding $\ell^2 \subset \ell^{2p}$. In particular, we will apply the Theorem of Crandall-Rabinowitz (see [8, Theorem I.5.1.]):

THEOREM 1. *We consider a mapping $F : U \times V \to Y$, where $U \times V \subset X \times \mathbb{R}$ is an open neighborhood of $(0,0)$ and X and Y are Banach spaces. Suppose that*

(H0) $F(0,\mu) = 0$ *for all* $\mu \in \mathbb{R}$,

(H1) $F \in C^2(U \times V, Y)$,

(H2) $F(\cdot,0)$ *is a Fredholm operator of index zero with*

$$\dim(\mathrm{Ker}(D_Z F(0,0))) = \mathrm{codim}(\mathrm{Ran}(D_Z F(0,0))) = 1,$$

(H3) Let $E \in X$, $\|E\|_X = 1$ *such that* $\mathrm{span}\{E\} = \mathrm{Ker}(D_Z F(0,0)))$. *Then*

$$[D^2_{\mu Z} F(0,0)](E) \notin \mathrm{Ran}(D_Z F(0,0)).$$

Then there exists a nontrivial branch of solutions described by a C^1-curve

$$\{(Z_s,\mu_s) : s \in (-s_0,s_0), (Z_0,\mu_0) = (0,0)\},$$

which satisfies $F(Z_s,\mu_s) = 0$ locally, and all solutions in a neighborhood of $(0,0)$ are either the trivial solution or on the nontrivial curve.

We verify the hypotheses of the previous theorem for the mapping F with Banach spaces $X(k_0)$ and $Y(k_0)$, defined in (15). First, we observe $F(0,\mu) = 0$ for all $\mu \in \mathbb{R}$, which means that we have a trivial solution branch, i.e. (H0) is fulfilled. Hypothesis (H1) is fulfilled due to the polynomial structure of (1).

We compute the required Frechet derivatives

$$[D_Z F(0,\mu)](H) = (1+\mu)\partial_t^2 H(t) - LH(t), \tag{16}$$

$$[D^2_{\mu Z} F(0,\mu)](H) = \partial_t^2 H(t), \tag{17}$$

for $X(k_0)$ and identify $D^2_{\mu Z} F$ with an element of $\mathscr{L}(X(k_0), Y(k_0))$. Further, let

$$X_{\mathrm{sym,even}} := C^2_{per,even}(I, V_{sym}),\ X_{\mathrm{sym,odd}} := C^2_{per,odd}(I, V_{sym}),$$

and

$$Y_{\mathrm{sym,even}} := C^0_{per,even}(I, V_{sym}),\ Y_{\mathrm{sym,odd}} := C^0_{per,odd}(I, V_{sym}).$$

Again we suppress the indices *even* and *odd* until it becomes important. The simple observation

$$D_Z F(0,0) X_{\mathrm{sym}} \subseteq Y_{\mathrm{sym}}, \tag{18}$$

follows from (10). Let E_{k_0} be the normalized eigenvector of the stationary problem supported in the k_0-th circle with eigenvalue $-\omega_0^2$. We denote the corresponding time-dependent solution of (1) by $E_{k_0}(t) = E_{k_0}\sin(\omega_0 t) \in X(k_0)$.
Obviously, $\mathrm{Ker}(D_Z F(0,0)) = \mathrm{span}\{E_{k_0}(t)\}$.

To verify (H3), we check whether it is true that

$$[D^2_{\mu Z}F(0,0)]E_{k_0}(t) = \partial_t^2 E_{k_0}(t) = -\omega_0^2 E_{k_0}(t) \notin \mathrm{Ran}(D_Z F(0,0)), \tag{19}$$

or equivalently, formulated as a question, does there exist an element $H \in X(k_0)$ such that

$$-\omega_0^2 E_{k_0}(t) = \partial_t^2 H(t) - LH(t).$$

By means of (18), we have $H(t) = \alpha E_{k_0}(t)$ with a constant $\alpha \in \mathbb{R}$. The occurring algebraic equation has no solution w.r.t α, i.e., (H3) is true.
It remains to find conditions under which $\mathrm{codim}(\mathrm{Ran}(D_Z F(0,0))) = 1$. Relation (18) implies $\mathrm{span}\{E_{k_0}(t)\} \nsubseteq [D_Z F(0,0)]X_{\mathrm{sym}}$. Hence,

$$\begin{aligned} &\mathrm{codim}(\mathrm{Ran}(D_Z F(0,0))) = 1 \\ &\quad \Leftrightarrow \quad D_Z F(0,0) \text{ invertible on } X_{\mathrm{sym}} \to Y_{\mathrm{sym}}. \end{aligned} \tag{20}$$

So, it remains to check under which conditions the equation

$$\xi = [D_Z F(0,0)]\eta = (\partial_t^2 - L)\eta \tag{21}$$

will be invertible on the symmetric subspace. In the case of $2\pi/\omega_0$-periodic and odd functions in time, we consider the expansion

$$\xi(t) = \sum_{m\in\mathbb{N}} \xi_m \sin(m\omega_0 t), \quad \eta(t) = \sum_{m\in\mathbb{N}} \eta_m \sin(m\omega_0 t) \tag{22}$$

with ξ_m, $\eta_m \in V_{sym}$ for $m \in \mathbb{N}$. This leads to the time-independent system of equations

$$\xi_m = (-m^2\omega_0^2 - L)\eta_m, \quad m \in \mathbb{N}. \tag{23}$$

Hence, we require that $-m^2\omega_0^2 \notin \sigma_{ac}(L)$ for all $m \in \mathbb{N}$. In particular, F is a (nonlinear) Fredholm operator of index 0. (The index does not depend on (Z,μ), since DF depends continuously on Z and μ, cf. [8, Remark I.2.2.].) In the case of $2\pi/\omega_0$-periodic and even functions in time we replace the sines in (22) by cosines with $m \in \mathbb{N}_0$. This leads to the requirement $-m^2\omega_0^2 \notin \sigma_{ac}(L)$ for all $m \in \mathbb{N}_0$. The spectral picture of a possible set of parameters is sketched in Figure 4.

Thus, Theorem 1 gives the existence of a non-trivial solution branch satisfying

$$(1+\mu_s)\ddot{Z}_s(t) - LZ_s(t) - N^p(Z_s)(t) = 0, \tag{24}$$

with $Z_s \in X(k_0)$ for $s \in (-s_0, s_0)$ with $s_0 > 0$ sufficiently small. A rescaling in time leads to the following theorem.

THEOREM 2. *Let $-\omega_0^2 = -(f_1 + g_1 + (r_v)_1)$ be the eigenvalue of the linear part in (1). Suppose that $-m^2\omega_0^2 \notin \sigma_{ac}(L)$ for all $m \in \mathbb{N}_0$ (non-resonance condition). Then, there exists a one-parameter family of real-valued solutions of (1) that are periodic in time and spatially localized.*

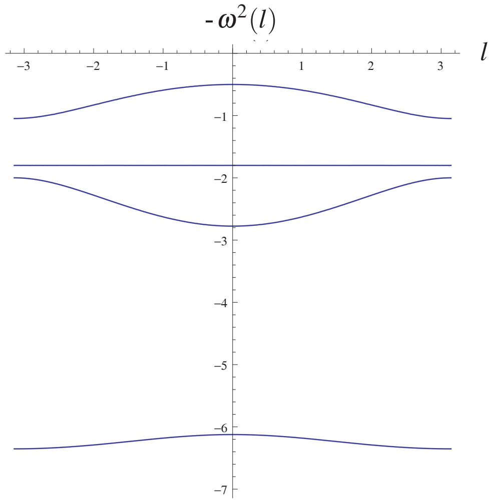

Figure 4: Spectral picture for a set of parameters satisfying the non-resonance condition ($f_1 = 0.3$, $g_1 = 1$, $h_1 = 2$ and $(r_v)_1 = 0.5$).

REMARK 1. To satisfy the non-resonance condition in Theorem 2, the eigenvalue $-\omega_0^2$ has to be isolated, i.e., $-\omega_0^2 \notin \sigma_{ac}(L)$. Moreover, one explicitly verifies that $-\omega_0^2 \in \sigma_{ac}(L)$ for stiffness parameters $f_1 = g_1$. Hence, a symmetry breaking is required to fulfill the assumptions.
In particular, the non-resonance condition in Theorem 2 requires $0 \notin \sigma_{ac}(L)$ for even powers of the nonlinearity. However, the proof does not need this requirement for odd powers. Hence, Theorem 2 can make a statement for vanishing local forces (FPU system) in the case of odd powers of the nonlinearity.

REMARK 2. James et al. showed the existence of breathers on various Fermi-Pasta-Ulam chains in a series of papers [5, 6]. We expect that the ideas can be transferred to construct symmetric breather solutions on our discrete necklace graph with sufficiently small power of the nonlinearity (i.e. an Fermi-Pasta-Ulam chain with three alternating masses and vanishing local forces).

Breathers arising from more than one linear eigenvector may exist, too. There are analytical tools to solve more-dimensional bifurcation equations, but the verification of the assumptions becomes more complicated, cf. [8]. Besides, possible future research topics concern interaction and stability of breathers. For instance, asymptotic stability for time-periodic localized solutions of discrete nonlinear Schrödinger equations on the lattice $\mathbb{Z}$ has been proven in [2, 7].

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(Received September 9, 2020)

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