Operators and Matrices
Volume 14, Number 4 (2020), 777–793
doi:10.7153/oam-2020-14-49

AN EXTENSION OF THE BEURLING–CHEN–HADWIN–SHEN THEOREM FOR NONCOMMUTATIVE HARDY SPACES ASSOCIATED WITH FINITE VON NEUMANN ALGEBRAS

HAIHUI FAN, DON HADWIN AND WENJING LIU

(*Communicated by N.-C. Wong*)

Abstract. In 2015, Yanni Chen, Don Hadwin and Junhao Shen proved a noncommutative version of Beurling's theorems for a continuous unitarily invariant norm α on a tracial von Neumann algebra $(\mathscr{M},\tau)$, where α is $\|\cdot\|_1$-dominating with respect to τ. In the paper, we first define a class of norms $N_\Delta(\mathscr{M},\tau)$ on $\mathscr{M}$, called determinant, normalized, unitarily invariant continuous norms on $\mathscr{M}$. If $\alpha \in N_\Delta(\mathscr{M},\tau)$, then there exists a faithful normal tracial state ρ on $\mathscr{M}$ such that $\rho(x) = \tau(xg)$ for some positive $g \in L^1(\mathscr{Z},\tau)$ and the determinant of g is positive. For every $\alpha \in N_\Delta(\mathscr{M},\tau)$, we study the noncommutative Hardy spaces $H^\alpha(\mathscr{M},\tau)$, then prove that the Chen-Hadwin-Shen theorem holds for $L^\alpha(\mathscr{M},\tau)$. The key ingredients in the proof of our result include a factorization theorem and a density theorem for $L^\alpha(\mathscr{M},\rho)$.

1. Introduction

It has long been of great importance to operator theorist and operator algebraist to study noncommutative Beurling's theorem[1],[4],[5],[7],[10],[16]. We recall some concepts in noncommutative Hardy spaces with finite von Neumann algebras. Given a finite von Neumann algebra $\mathscr{M}$ acting on a Hilbert space H, the set of possibly unbounded closed and densely defined operators on H which are affiliated to $\mathscr{M}$, form a topological algebra where the topology is the (noncommutative) topology of convergence in measure. We denote this algebra by $\widetilde{\mathscr{M}}$. The trace τ extends naturally from $\mathscr{M}$ to the positive operators in $\widetilde{\mathscr{M}}$. The important fact regarding this algebra, is that it is large enough to accommodate all the noncommutative L^p spaces corresponding to $\mathscr{M}$. Specifically, if $1 \leqslant p < \infty$, then we define the space $L^p(\mathscr{M},\tau) = \{x \in \widetilde{\mathscr{M}} : \tau(|x|^p) < \infty\}$, where the ambient norm is given by $\|\cdot\|_p = \tau(\|\cdot\|^p)^{1/p}$. The space $L^\infty(\mathscr{M},\tau)$ is defined to be $\mathscr{M}$ itself. These spaces capture all the usual properties of L^p spaces, with the dual action of L^p on L^q (q conjugate to p) given by $(a,b) \to \tau(ab)$. For any subset S of $\mathscr{M}$, we write $[\mathscr{S}]_p$ for the p-norm closure of $\mathscr{S}$ in $L^p(\mathscr{M},\tau)$, with the understanding that $[\mathscr{S}]_p$ will denote the weak* closure in the case $p = \infty$. W. Arveson [1] introduced a concept of maximal subdiagonal algebra in 1967, also known as a noncommutative H^∞ space, to study the analyticity in operator algebras. Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let $\mathscr{A}$ be a

Mathematics subject classification (2010): 46E20, 30H10, 30J99, 47L10.
Keywords and phrases: Gauge norm, von Neumann algebra, Beurling theorem.

Paper OaM-14-49

weak* closed unital subalgebra of $\mathscr{M}$ and $\mathscr{A}$ is called a finite maximal subalgebra of $\mathscr{M}$ with respect to Φ if (i) $\mathscr{A}+\mathscr{A}^*$ is weak* dense in $\mathscr{M}$; (ii) $\Phi(xy)=\Phi(x)\Phi(y)$ for $\forall x,y\in\mathscr{A}$; (iii) $\tau\circ\Phi=\tau$; and (iv) $\mathscr{D}=\mathscr{A}\cap\mathscr{A}^*$. Such a finite maximal subdiagonal subalgebra $\mathscr{A}$ of $\mathscr{M}$ is also called an H^∞ space of $\mathscr{M}$. For each $1\leqslant p\leqslant\infty$, let H^p be the completion of Arveson's noncommutative H^∞ with respect to $\|\cdot\|_p$. After Arveson's introduction of noncommutative H^p spaces, many researchers obtained Beurling theorems for invariant subspaces in noncommutative H^p spaces (for example, see [2],[5],[7]).

Y. Chen, D. Hadwin, and J. Shen obtained a version of the Blecher-Labuschagne-Beurling invariant subspace theorem on H^∞-right invariant subspace in a noncommutative $L^\alpha(\mathscr{M},\tau)$ space, where α is a normalized unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm.

In this paper, we will extend Chen-Hadwin-Shen's result in [7] by considering drop the condition that α is $\|\cdot\|_1$-dominating. By defining a generalized α norm, we have a version of Chen-Hadwin-Shen's result for noncommutative Hardy spaces.

THEOREM 4.7. *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a determinant, normalized, unitarily invariant, continuous norm on $\mathscr{M}$. Then there exists a faithful normal tracial state ρ on $\mathscr{M}$ such that $\alpha\in N_1(\mathscr{M},\rho)$. Let H^∞ be a finite subdiagonal subalgebra of $\mathscr{M}$ and $\mathscr{D}=H^\infty\cap(H^\infty)^*$. If $\mathscr{W}$ is a closed subspace of $L^\alpha(\mathscr{M},\tau)$ such that $\mathscr{W}H^\infty\subseteq\mathscr{W}$, then there exists a closed subspace $\mathscr{Y}$ of $L^\alpha(\mathscr{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathscr{M}$ such that:*

(1) $u_\lambda^*\mathscr{Y}=0$ *for all* $\lambda\in\Lambda$;

(2) $u_\lambda^*u_\lambda\in\mathscr{D}$ *and* $u_\lambda^*u_\mu=0$ *for all* $\lambda,\mu\in\Lambda$ *with* $\lambda\neq\mu$;

(3) $\mathscr{Y}=[H_0^\infty\mathscr{Y}]_\alpha$;

(4) $\mathscr{W}=\mathscr{Y}\oplus^{col}(\oplus^{col}_{\lambda\in\Lambda}u_\lambda H^\alpha)$.

Many tools used in [7] are no longer available in an arbitrary $L^\alpha(\mathscr{M},\tau)$ space and new techniques need to be invented. First, we need using the Fuglede-Kadison determinant, and inner, outer factorization for noncommutative Hardy spaces, more details seen in [2]. Let Δ be Fuglede-Kadison determinant on $\mathscr{M}$ defined by

$$\Delta(x)=exp(\tau(\log|x|))=exp(\int_0^\infty\log(t)d\nu_{|x|}(t)),$$

where $d\nu_{|x|}(t)$ denotes the probability measure on $\mathbb{R}_+$, Also, the definition of this determinant can be extended to the $*$-algebra $\widetilde{\mathscr{M}}$.
In order to prove our main result of the paper, we first get the following theorem.

THEOREM 2.10. *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, continuous norm on $(\mathscr{M},\tau)$. Then there exists a positive $g\in L^1(\mathscr{Z},\tau)$ such that: (i) $\rho(\cdot)=\tau(\cdot g)$ is a faithful normal tracial state on $\mathscr{M}$; (ii) α is $c\|\cdot\|_{1,\rho}$-dominating, for some $c>0$; (iii), $\rho(x)=\tau(xg)$ for every $x\in L^1(\mathscr{M},\rho)$.*

THEOREM 3.11. *If* $\alpha \in N_\Delta(\mathscr{M},\tau)$*, then there exists a faithful normal tracial state* ρ *such that* $H^\alpha(\mathscr{M},\rho) = H^1(\mathscr{M},\rho) \cap L^\alpha(\mathscr{M},\rho)$.

Then we get a factorization theorem and a density theorem for $L^\alpha(\mathscr{M},\tau)$ to get the main theorem.

THEOREM 4.2. *Suppose* $\alpha \in N_\Delta(\mathscr{M},\tau)$*, there exists a faithful normal tracial state* ρ *on* $\mathscr{M}$ *such that* $\rho(x) = \tau(xg)$ *for some positive* $g \in L^1(\mathscr{Z},\tau)$ *and the determinant of* g *is positive. If* $x \in \mathscr{M}$ *and* $x^{-1} \in L^\alpha(\mathscr{M},\rho)$*, then there are unitary operators* $u_1, u_2 \in \mathscr{M}$ *and* $s_1, s_2 \in H^\infty$ *such that* $x = u_1 s_1 = s_2 u_2$ *and* $s_1^{-1}, s_2^{-1} \in H^\alpha(\mathscr{M},\rho)$.

THEOREM 4.3. *Let* $\alpha \in N_\Delta(\mathscr{M},\tau)$*, then there exists a faithful normal tracial state* ρ *on* $\mathscr{M}$ *such that* $\rho(x) = \tau(xg)$ *for some positive* $g \in L^1(\mathscr{Z},\tau)$ *and the determinant of* g *is positive. Also, if* $\mathscr{W}$ *is a closed subspace of* $L^\alpha(\mathscr{M},\rho)$ *and* $\mathscr{N}$ *is a weak* closed linear subspace of* $\mathscr{M}$ *such that* $\mathscr{W}H^\infty \subset \mathscr{W}$ *and* $\mathscr{N}H^\infty \subset \mathscr{N}$*, then:*

(1) $\mathscr{N} = [\mathscr{N}]_\alpha \cap \mathscr{M}$;

(2) $\mathscr{W} \cap \mathscr{M}$ *is weak* closed in* $\mathscr{M}$;

(3) $\mathscr{W} = [\mathscr{W} \cap \mathscr{M}]_\alpha$;

(4) if $\mathscr{S}$ *is a subspace of* $\mathscr{M}$ *such that* $\mathscr{S}H^\infty \subset \mathscr{S}$ *, then* $[\mathscr{S}]_\alpha = [\overline{\mathscr{S}}^{w*}]_\alpha$ *, where* $\overline{\mathscr{S}}^{w*}$ *is the weak*-closure of* $\mathscr{S}$ *in* $\mathscr{M}$.

The organization of the paper is as follows. In Section 2, we introduce determinant, normalized, unitarily invariant continuous norms. In Section 3, we study the relations between noncommutative Hardy spaces $H^\alpha(\mathscr{M},\rho)$ and $H^\alpha(\mathscr{M},\tau)$. In Section 4, we prove the main result of the paper, a version of Chen-Hadwin-Shen's result for noncommutative Hardy spaces associated with new norm. In Section 5, we get a generalized noncommutative Beurling's theorem for special von Neumann algebras.

2. Determinant, normalized, unitarily invariant continuous norms

Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ, the $\|\cdot\|_p$ is a mapping from $\mathscr{M}$ to $[0,\infty)$ defined by $\|x\|_p = (\tau(|x|^p))^{1/p}$, $\forall x \in \mathscr{M}, 0 < p < \infty$. It is known that $\|\cdot\|_p$ is a norm if $1 \leqslant p < \infty$, and a quasi-norm if $0 < p < 1$.

DEFINITION 2.1. Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Assume $\alpha : \mathscr{M} \to [0,\infty)$ is a norm satisfying:

(1) $\alpha(I) = 1$, i.e., α is normalized;

(2) $\alpha(x) = \alpha(|x|)$ for all $x \in \mathscr{M}$ and $|x| = (x^*x)^{1/2}$, i.e., α is a gauge;

(3) $\alpha(u^*xu) = \alpha(x), u \in \mathscr{U}(\mathscr{M})$ and $x \in \mathscr{M}$, i.e., α is unitarily invariant;

(4) $\lim_{\tau(e)\to 0} \alpha(e) = 0$ as e ranges over the projections in $\mathscr{M}$. i.e., if $\{e_\lambda\}$ is a net of projections in $\mathscr{M}$ and $\tau(e_\lambda) \to 0$, then $\alpha(e_\lambda) \to 0$, which means α is continuous.

Then we call α is a *normalized unitarily invariant continuous norm*. And we denote $N(\mathscr{M},\tau)$ to be the collection of all such norms.

DEFINITION 2.2. We denote by $N_1(\mathscr{M},\tau)$, the collection of all these norms $\alpha : \mathscr{M} \to [0,\infty)$ such that:

(1) $\alpha \in N(\mathscr{M},\tau)$;

(2) $\forall x \in \mathscr{M}$, $\alpha(x) \geqslant c\|x\|_1$, for some $c > 0$.

A norm α in $N_1(\mathscr{M},\tau)$ is called a *normalized, unitarily invariant* $\|\cdot\|_1$*-dominating continuous* norm on $\mathscr{M}$.

DEFINITION 2.3. We denote by $N_\Delta(\mathscr{M},\tau)$, the collection of all these norms $\alpha : \mathscr{M} \to [0,\infty)$ such that:

(1) $\alpha \in N(\mathscr{M},\tau)$;

(2) there exists a positive $g \in L^1(\mathscr{M},\tau)$ such that $\Delta(g) > 0$ and $\alpha(x) \geqslant c\tau(|x|g)$ for some $c > 0$.

A norm α in $N_\Delta(\mathscr{M},\tau)$ is called a *determinant, normalized, unitarily invariant continuous* norm on $\mathscr{M}$.

EXAMPLE 2.4. For the Definition 2.3, if we take $g = 1$, then $\alpha \in N_1(\mathscr{M},\tau)$, i.e., $N_\Delta(\mathscr{M},\tau) \subset N_1(\mathscr{M},\tau)$.

EXAMPLE 2.5. Each p-norm $\|\cdot\|_p$ is in $N(\mathscr{M},\tau)$, $N_1(\mathscr{M},\tau)$, and $N_\Delta(\mathscr{M},\tau)$ for $1 \leqslant p < \infty$.

EXAMPLE 2.6. Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let $E(0,1)$ be a symmetric Banach function space on $(0,1)$ and $E(\tau)$ be the noncommutative Banach function space with a norm $\|\cdot\|_{E(\tau)}$ corresponding to $E(0,1)$ and associated with $(\mathscr{M},\tau)$. If $E(0,1)$ is also order continuous, then the restriction of the norm $\|\cdot\|_{E(\tau)}$ to $\mathscr{M}$ lies in $N(\mathscr{M},\tau)$ and $N_1(\mathscr{M},\tau)$.

In order to prove the first theorem in this paper, we need the following lemmas, the first lemma is proved by H. Fan, D. Hadwin and W. Liu in [9].

LEMMA 2.7. *Suppose (X,Σ,μ) is a probability space and α is a continuous normalized gauge norm on $L^\infty(\mu)$. Then there exists $0 < c < 1$ and a probability measure λ on Σ such that $\lambda \ll \mu$ and $\mu \ll \lambda$, such that α is $c\|\cdot\|_{1,\lambda}$-dominating.*

Before we give the next lemma, we first introduce the property of central valued traces in [15], and introduce a class of determinant, normalized, unitarily invariant continuous norms on finite von Neumann algebras and some interesting examples from this class. In the end of this section, we will obtain our first theorem.

PROPOSITION 2.8. *If $\mathscr{M}$ is a finite von Neumann algebra with the center $\mathscr{Z}$ of $\mathscr{M}$, then there is a unique positive linear mapping φ from $\mathscr{M}$ into $\mathscr{Z}$ such that:*

(1) $\varphi(xy) = \varphi(yx)$, *for each x and y in $\mathscr{M}$;*

(2) $\varphi(z) = z$, *for each z in $\mathscr{Z}$;*

(3) $\varphi(x) > 0$ *if $x > 0$, for x in $\mathscr{M}$;*

(4) $\varphi(zx) = z\varphi(x)$, *for each z in $\mathscr{Z}$ and x in $\mathscr{M}$;*

(5) $\|\varphi(x)\| \leqslant \|x\|$, *for x in $\mathscr{M}$;*

(6) *φ is ultraweakly continuous;*

(7) for any $x \in \mathscr{M}$, $\varphi(x)$ is the unique central element in the norm closure of the convex hull of $\{uxu^*|u \in \mathscr{U}(\mathscr{M})\}$;

(8) every tracial state on $\mathscr{M}$ is of the form $\tau \circ \varphi$ where τ is a state on $\mathscr{Z}$, i.e. every state on the center $\mathscr{Z}$ of $\mathscr{M}$ extends uniquely to a tracial state on $\mathscr{M}$;

(9) φ is faithful.

LEMMA 2.9. *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Suppose $\alpha \in N(\mathscr{M},\tau)$, then the central valued trace φ satisfy $\alpha(\varphi(x)) \leqslant \alpha(x)$, for every $x \in \mathscr{M}$.*

Proof. By proposition 2.8 (7), for any $x \in \mathscr{M}$, the central value trace $\varphi(x)$ is in the norm closure of the convex hull of $\{uxu^*|u \in \mathscr{U}(\mathscr{M})\}$, so there exist a net $\{x_\lambda\}_{\lambda\in\Lambda}$ in the convex hull of $\{uxu^*|u \in \mathscr{U}(\mathscr{M})\}$ such that x_λ converges to $\varphi(x)$. Since α is a continuous norm, $\alpha(x_\lambda - \varphi(x)) \to 0$, i.e., $\alpha(\varphi(x)) = \lim_\lambda \alpha(x_\lambda)$. Since x_λ is in the convex hull of $\{uxu^*|u \in \mathscr{U}(\mathscr{M})\}$, $\alpha(x_\lambda) \leqslant \alpha(x)$. Therefore, $\alpha(\varphi(x)) \leqslant \alpha(x)$. □

THEOREM 2.10. *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, continuous norm on $(\mathscr{M},\tau)$. Then there exists a positive $g \in L^1(\mathscr{Z},\tau)$ such that: (i) $\rho(\cdot) = \tau(\cdot g)$ is a faithful normal tracial state on $\mathscr{M}$; (ii) α is $c\|\cdot\|_{1,\rho}$-dominating for some $c > 0$; (iii) $\rho(x) = \tau(xg)$ for every $x \in L^1(\mathscr{M},\rho)$.*

Proof. Since the center $\mathscr{Z}$ of $\mathscr{M}$ is an abelian von Neumann algebra, there is a compact subset X of $\mathbb{R}$ and a regular Borel probability measure on X such that the mapping π from $\mathscr{Z}$ to $L^\infty(X,\mu)$ is $*$-isomorphic and WOT-homeomorphic. Since α is a continuous normalized unitarily invariant norm on $(\mathscr{M},\tau)$, it is easy to check $\overline{\alpha} = \alpha \circ \pi^{-1}$ satisfying:

(i) $\overline{\alpha}(1) = \alpha \circ \pi^{-1}(1) = \alpha(\pi^{-1}(1)) = \alpha(I) = 1$.

(ii) $\overline{\alpha}(f)=\alpha\circ\pi^{-1}(f)=\alpha(u\pi^{-1}(f))=\alpha(\pi^{-1}(wf))=\alpha(\pi^{-1}(|f|))=\overline{\alpha}(|f|)$, where $|f|=wf, |w|=1$ and there is a unitary u such that $\pi(u)=w$.

(iii) For given borel set $\{E_n\}_{n=1}^{\infty}\subseteq X$, there exists a sequence $\{e_n\}\subseteq\mathscr{Z}$ such that $\pi^{-1}(\chi_{E_n})=e_n$ for every $n\in\mathbb{N}$. If $\mu(E_n)\to 0$, then $\tau(e_n)\to 0$. So $\alpha(e_n)\to 0$ since α is continuous. Thus $\lim_{n\to\infty}\overline{\alpha}(\chi_{E_n})=\lim_{n\to\infty}\alpha\circ\pi^{-1}(\chi_{En})=\lim_{n\to\infty}\alpha(e_n)\to 0$.

Thus $\overline{\alpha}$ is a continuous normalized gauge norm on $L^{\infty}(X,\mu)$.

By the Lemma 2.7, there exists a probability measure λ such that $\lambda\ll\mu$ and $\mu\ll\lambda$ and there exists $c>0$ such that $\forall f\in L^{\infty}(X,\mu)=L^{\infty}(X,\lambda), \overline{\alpha}(f)\geqslant c\|f\|_{1,\lambda}$. Define $\rho_0(x)=\int_X\pi(x)d\lambda$, we check ρ_0 is a faithful normal tracial state on $\mathscr{Z}$.

(1) $\rho_0(I)=\int_X\pi(I)d\lambda=\int_X 1d\lambda=1$.

(2) $\rho_0(xy)=\int_X\pi(xy)d\lambda=\int_X\pi(yx)d\lambda=\rho_0(yx)$.

(3) Since $x_n\to x$ in WOT topology, $\pi(x_n)\to\pi(x)$ in weak* topology, i.e., $\int_X\pi(x_n)d\lambda$ $=\int_X\pi(x_n)gd\mu\to\int_X\pi(x)gd\mu=\int_X\pi(x)d\lambda$. Thus $\rho_0(x_n)\to\rho_0(x)$. Therefore ρ_0 is normal.

(4) For every $x\in\mathscr{Z},\rho_0(x^*x)=\int_X\pi(x^*x)d\lambda=\int_X\pi(x)^2d\lambda=0$, so $\pi(x)^2=0$ and $x=0$, which means ρ_0 is faithful.

Define $\rho=\rho_0\circ\varphi$, now claim that α is $c\|\cdot\|_{1,\rho}$-dominating on $(\mathscr{M},\rho)$. For some constant $c>0$, $\forall x\in\mathscr{Z}$, $\alpha(x)=\overline{\alpha}\circ\pi(x)=\overline{\alpha}(\pi(x))\geqslant c\|\pi(x)\|_{1,\lambda}=c\int_X|\pi(x)|d\lambda=c\int_X\pi(|x|)d\lambda=c\rho(|x|)=c\|x\|_{1,\rho}$. So we have $\alpha(x)\geqslant c\|x\|_{1,\rho},\forall x\in\mathscr{Z}$. Also, we have $\mathscr{M}\xrightarrow{\varphi}\mathscr{Z}\xrightarrow{\rho_0}\mathbb{C}$, where φ is the mapping in proposition 2.8. Then ρ is a state on $\mathscr{M}$, and $\forall x\in\mathscr{M},\alpha(x)\geqslant\alpha(\varphi(x))\geqslant c\|\varphi(x)\|_{1,\rho_0}=c\|\varphi(x)\|_{1,\rho}=c\|x\|_{1,\rho}$. Therefore, there exists a faithful normal tracial state ρ on $\mathscr{M}$ such that α is a $c\|\cdot\|_{1,\rho}$-dominating on $(\mathscr{M},\rho)$.

Since $\rho(x)=\int_X\pi(x)d\lambda=\int_X\pi(x)hd\mu$, where $h=\frac{d\lambda}{d\mu}\in L^1(X,\mu)$, we can choose simple functions $\{h_i\}_{i=1}^{\infty}$ such that $0\leqslant h_1\leqslant h_2\leqslant\cdots$ and $h_n\to h$ as $n\to\infty$. And also we can choose $0\leqslant x_1\leqslant x_2\leqslant\cdots$ in $\mathscr{Z}$ so that $\pi(x_n)=h_n$ for each n. Therefore,

$$\rho(x)=\rho_0(\varphi(x))=\lim_{n\to\infty}\tau(x_n\varphi(x))=\lim_{n\to\infty}\tau(\varphi(x_nx))=\lim_{n\to\infty}\tau(x_nx))=\tau(xg),$$

where $g\in L^1(\mathscr{Z},\tau)$. □

EXAMPLE 2.11. Given any finite von Neumann algebra $\mathscr{M}$ with a faithful normal tracial state τ and $\alpha\in N(\mathscr{M},\tau)$, by Theorem 2.10, there exists a positive $g\in L^1(\mathscr{M},\tau)$ such that $\alpha(x)\geqslant c\tau(|x|g)$ for some $c>0$. If $\Delta(g)>0$, then $\alpha\in N_{\Delta}(\mathscr{M},\tau)$.

3. Noncommutative Hardy spaces

Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Given a von Neumann subalgebra $\mathscr{D}$ of $\mathscr{M}$, a conditional expectation $\Phi\colon \mathscr{M} \to \mathscr{D}$ is a positive linear map satisfying $\Phi(I) = I$ and $\Phi(x_1yx_2) = x_1\Phi(y)x_2$ for all $x_1, x_2 \in \mathscr{D}$ and $y \in \mathscr{M}$. There exists a unique conditional expectation $\Phi_\tau\colon \mathscr{M} \to \mathscr{D}$ satisfying $\tau \circ \Phi_\tau(x) = \tau(x)$ for every $x \in \mathscr{M}$. Now we recall noncommutative classical Hardy spaces H^∞ in [1].

DEFINITION 3.1. Let $\mathscr{A}$ be a weak* closed unital subalgebra of $\mathscr{M}$, and let Φ_τ be the unique faithful normal trace preserving conditional expectation from $\mathscr{M}$ onto the diagonal von Neumann algebra $\mathscr{D} = \mathscr{A} \cap \mathscr{A}^*$. Then $\mathscr{A}$ is called a finite, maximal subdiagonal subalgebra of $\mathscr{M}$ with respect to Φ_τ if:

(1) $\mathscr{A} + \mathscr{A}^*$ is weak* dense in $\mathscr{M}$;

(2) $\Phi_\tau(xy) = \Phi_\tau(x)\Phi_\tau(y)$ for all $x, y \in \mathscr{A}$.

Such $\mathscr{A}$ will be denoted by H^∞, and $\mathscr{A}$ is also called a noncommutative Hardy space.

EXAMPLE 3.2. Let $\mathscr{M} = L^\infty(\mathbb{T}, \mu)$, and $\tau(f) = \int f d\mu$ for all $f \in L^\infty(\mathbb{T}, \mu)$. Let $\mathscr{A} = H^\infty(\mathbb{T}, \mu)$, then $\mathscr{D} = H^\infty(\mathbb{T}, \mu) \cap H^\infty(\mathbb{T}, \mu)^* = \mathbb{C}$. Let Φ_τ be the mapping from $L^\infty(\mathbb{T}, \mu)$ onto $\mathbb{C}$ defined by $\Phi_\tau(f) = \int f d\mu$. Then $H^\infty(\mathbb{T}, \mu)$ is a finite, maximal subdiagonal subalgebra of $L^\infty(\mathbb{T}, \mu)$.

EXAMPLE 3.3. Let $\mathscr{M} = \mathscr{M}_n(\mathbb{C})$ be with the usual trace τ. Let $\mathscr{A}$ be the subalgebra of lower triangular matrices, now $\mathscr{D}$ is the diagonal matrices and Φ_τ is the natural projection onto the diagonal matrices. Then $\mathscr{A}$ is a finite maximal subdiagonal subalgebra of $\mathscr{M}_n(\mathbb{C})$.

Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ, Φ_τ be the conditional expectation and α be a determinant, normalized, unitarily invariant, continuous norm on $\mathscr{M}$. Let $L^\alpha(\mathscr{M}, \tau)$ be the α closure of $\mathscr{M}$,i.e., $L^\alpha(\mathscr{M}, \tau) = [\mathscr{M}]_\alpha$. Similarly, $H^\alpha(\mathscr{M}, \tau) = [H^\infty(\mathscr{M}, \tau)]_\alpha$, $H_0^\infty(\mathscr{M}, \tau) = \ker(\Phi_\tau) \cap H^\infty(\mathscr{M}, \tau)$ and $H_0^\alpha(\mathscr{M}, \tau) = \ker(\Phi_\tau) \cap H^\alpha(\mathscr{M}, \tau)$. If we take $\alpha = \|\cdot\|_p$, then $L^p(\mathscr{M}, \tau) = [\mathscr{M}]_p$, $H^p(\mathscr{M}, \tau) = [H^\infty(\mathscr{M}, \tau)]_p$. Recall ρ is a faithful normal tracial state on $\mathscr{M}$ satisfying all three conditions in Theorem 2.10. We define the noncommutative Hardy spaces $H^1(\mathscr{M}, \rho)$ and $H_0^1(\mathscr{M}, \rho)$ by $H^1(\mathscr{M}, \rho) = \overline{H^\infty(\mathscr{M}, \tau)}^{\|\cdot\|_{1,\rho}}$ and $H_0^1(\mathscr{M}, \rho) = \overline{H_0^\infty(\mathscr{M}, \tau)}^{\|\cdot\|_{1,\rho}}$. In [17], K. S. Satio characterized the noncommutative Hardy spaces $H^p(\mathscr{M}, \tau)$ and $H_0^p(\mathscr{M}, \tau)$. Recall $H^p(\mathscr{M}, \tau) = \{x \in L^p(\mathscr{M}, \tau),\ \tau(xy) = 0, \text{ for all } y \in H_0^\infty\}$ for $1 \leqslant p < \infty$, also we have $H_0^p(\mathscr{M}, \tau) = \{x \in L^p(\mathscr{M}, \tau),\ \tau(xy) = 0, \forall y \in H^\infty\}$. In this paper, we get similar result for noncommutative Hardy spaces $H^p(\mathscr{M}, \rho)$ and $H_0^p(\mathscr{M}, \rho)$ by using the inner-outer factorization and the properties of outer functions in noncommutative Hardy spaces from papers [4] and [5]. Let Δ be Fuglede-Kadison determinant on $\mathscr{M}$ defined by

$$\Delta(x) = exp(\tau(\log|x|)) = exp(\int_0^\infty \log(t) d\nu_{|x|}(t)),$$

where $d\nu_{|x|}(t)$ denotes the probability measure on $\mathbb{R}_+$, Also, the definition of this determinant can be extended to the $*$-algebra $\widetilde{\mathscr{M}}$.

DEFINITION 3.4. Let $1 \leqslant p \leqslant \infty$. An element $x \in H^p(\mathscr{M},\tau)$ is outer if $I \in [xH^p(\mathscr{M},\tau)]_p$, and $x \in H^p(\mathscr{M},\tau)$ is strongly outer if x is outer and $\Delta(x) > 0$. An element u is inner if $u \in H^\infty(\mathscr{M},\tau)$ and u is unitary.

LEMMA 3.5. *(from [5]) If H^∞ is a maximal subdiagonal algebra, then $x \in H^p(\mathscr{M},\tau)$ with $\Delta(x) > 0$ iff $x = uy$ for an inner u and a strongly outer $y \in H^p(\mathscr{M},\tau)$, for $1 \leqslant p \leqslant \infty$. The factorization is unique up to a unitary in $\mathscr{D}$.*

LEMMA 3.6. *(from [5]) Let Φ_τ be the conditional expectation on $\mathscr{M}$. Then $x \in H^p(\mathscr{M},\tau)$ is outer if and only if $\Phi_\tau(x)$ is outer in $L^p(\mathscr{D})$ and $\overline{xH_0^\infty(\mathscr{M},\tau)}^{\|\cdot\|_{p,\tau}} = H_0^p(\mathscr{M},\tau)$, for $1 \leqslant p \leqslant \infty$.*

LEMMA 3.7. *If $\alpha \in N_\Delta(\mathscr{M},\tau)$, then there exits a faithful tracial state ρ and a strongly outer h in $H^1(\mathscr{M},\tau)$ such that $g = |h|$,where g as in Theorem 2.10 and $hH^1(\mathscr{M},\rho) = H^1(\mathscr{M},\tau)$.*

Proof. Since $\alpha \in N_\Delta(\mathscr{M},\tau)$, $\Delta(g) > 0$. By Lemma 3.5, $g = |h|$ for a strongly outer $h \in H^1(\mathscr{M},\tau)$. Let $\rho(x) = \tau(xg)$, $\forall x \in \mathscr{M}$, by Theorem 2.10, ρ is a faithful normal tracial state on $\mathscr{M}$. Then we define $U : L^1(\mathscr{M},\rho) \longrightarrow L^1(\mathscr{M},\tau)$ by $Ux = hx$, which is a surjective isometry:

$$\|U(x)\|_{1,\tau} = \|xg\|_{1,\tau} = \tau(|xg|) = \tau(|x|\,g) = \rho(|x|) = \|x\|_{1,\rho}.$$

Since $g \in gH^1(\mathscr{M},\rho)$ and $H^1(\mathscr{M},\tau) \subseteq H^1(\mathscr{M},\rho)$, $gH^\infty(\mathscr{M},\tau) \subseteq gH^1(\mathscr{M},\rho)$. Since $g = |h|$, $g = vh$, where v is modular. Thus $vhH^\infty(\mathscr{M},\tau) \subseteq gH^1(\mathscr{M},\rho) = vhH^1(\mathscr{M},\rho)$ $= hH^1(\mathscr{M},\rho)$. Since h is a strongly outer in $H^1(\mathscr{M},\tau)$,we have $hH^1(\mathscr{M},\rho) = H^1(\mathscr{M},\tau)$. □

COROLLARY 3.8. *Let Φ_τ be the conditional expectation on $\mathscr{M}$. If $\alpha \in N_\Delta(\mathscr{M},\tau)$, then there exists a faithful normal tracial state ρ such that:*

(1) $H^1(\mathscr{M},\rho) = \{x \in L^1(\mathscr{M},\rho) : \rho(xy) = 0$ *for all* $y \in H_0^\infty\}$;

(2) $H_0^1(\mathscr{M},\rho) = \{x \in L^1(\mathscr{M},\rho) : \rho(xy) = 0$ *for all* $y \in H^\infty\}$;

(3) $H_0^1(\mathscr{M},\rho) = \{x \in H^1(\mathscr{M},\rho) : \Phi_\tau(xh) = 0\}$.

Proof. Since $\alpha \in N_\Delta(\mathscr{M},\tau)$, there exists a positive $g \in L^1(\mathscr{M},\tau)$ and $\Delta(g) > 0$ such that $\alpha(x) \geqslant c\tau(|x|g)$ for some $c > 0$. We define $\rho(x) = \tau(xg), \forall x \in \mathscr{M}$, ρ is a faithful normal tracial state on $\mathscr{M}$. By Lemma 3.7 and $H^1(\mathscr{M},\tau) = \{x \in L^1(\mathscr{M},\tau),$ $\tau(xy) = 0$ for all $y \in H_0^\infty\}$, we have (1). For (2), We know $\overline{H_0^\infty(\mathscr{M},\tau)}^{\|\cdot\|_{1,\rho}} = H_0^1(\mathscr{M},\rho)$, and $hH_0^1(\mathscr{M},\rho) = h\overline{H_0^\infty(\mathscr{M},\tau)}^{\|\cdot\|_{1,\rho}} = \overline{hH_0^\infty(\mathscr{M},\tau)}^{\|\cdot\|_{1,\tau}} = H_0^1(\mathscr{M},\tau)$ since h is outer in $H^1(\mathscr{M},\tau)$. The last statement is clearly by [17]. □

PROPOSITION 3.9. *If $\alpha \in N_\Delta(\mathscr{M},\tau)$, then there exists a faithful normal tracial state ρ such that*

$$H^\alpha(\mathscr{M},\rho) = \{x \in L^\alpha(\mathscr{M},\rho) : \rho(xy) = 0 \textit{ for all } y \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\#\},$$

where $(L^\alpha(\mathscr{M},\rho))^\#$ is the dual space of $L^\alpha(\mathscr{M},\rho)$.

Proof. Since $\alpha \in N_\Delta(\mathscr{M},\tau)$, then there exists a faithful normal tracial state ρ on $\mathscr{M}$ such that $\rho(x) = \tau(xg)$ for some positive $g \in L^1(\mathscr{Z},\tau)$ and the determinant of g is positive, which means $\alpha \in N_1(\mathscr{M},\rho)$. Let $\mathscr{J} = \{x \in L^\alpha(\mathscr{M},\rho) : \rho(xy) = 0$ for all $y \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\#\}$. Suppose $x \in H^\infty(\mathscr{M},\rho)$. If $y \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\# \subseteq H_0^1(\mathscr{M},\rho)$, then it follows from Corollary 3.8 that $\rho(xy) = 0$, for all $x \in \mathscr{J}$, and so $H^\infty(\mathscr{M},\rho) \subseteq \mathscr{J}$. We claim that $\mathscr{J}$ is α-closed in $L^\alpha(\mathscr{M},\rho)$. In fact, suppose $\{x_n\}$ is a sequence in $\mathscr{J}$ and $x \in L^\alpha(\mathscr{M},\rho)$ such that $\alpha(x_n - x) \to 0$. If $y \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\#$, then by the generalized Holder's inequality in[7], we have

$$|\rho(xy) - \rho(x_n y)| = |\rho((x - x_n)y)| \leqslant \alpha(x - x_n)\alpha' \to 0.$$

Which follows that $\rho(xy) = \lim_{n\to\infty} \rho(x_n y) = 0$ for all $y \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\#$. By the definition of $\mathscr{J}$, we know $x \in \mathscr{J}$. Hence $\mathscr{J}$ is closed in $L^\alpha(\mathscr{M},\rho)$. Therefore, $H^\alpha(\mathscr{M},\rho) = [H^\infty(\mathscr{M},\rho)]_\alpha \subseteq \mathscr{J}$.

Next, we show that $H^\alpha(\mathscr{M},\rho) = \mathscr{J}$. Assume, via contradiction, that $H^\alpha(\mathscr{M},\rho) \subsetneq \mathscr{J} \subseteq L^\alpha(\mathscr{M},\rho)$. By the Hahn-Banach theorem, there is a linear functional $\phi \in (L^\alpha(\mathscr{M},\rho))^\#$ and $x \in \mathscr{J}$ such that:

(a) $\phi(x) \neq 0$;

(b) $\phi(y) = 0$ for all $y \in H^\alpha(\mathscr{M},\rho)$.

In the beginning of this proof, we know $\alpha \in N_1(\mathscr{M},\rho)$, which means α is normalized, unitarily invariant $\|\cdot\|_1$-dominating, continuous norm on $(\mathscr{M},\rho)$, it follows from [7] that there exists a $\xi \in (L^\alpha(\mathscr{M},\rho))^\#$ such that

(c) $\phi(z) = \rho(z\xi)$ for all $z \in L^\alpha(\mathscr{M},\rho)$.

Hence from (b) and (c) we can conclude that

(d) $\rho(y\xi) = \phi(y) = 0$ for every $y \in H^\infty(\mathscr{M},\rho) \subseteq H^\alpha(\mathscr{M},\rho) \subseteq L^\alpha(\mathscr{M},\rho)$.

Since $\phi \in (L^\alpha(\mathscr{M},\rho))^\# \subseteq L^1(\mathscr{M},\rho)$, so $\xi \in H_0^1(\mathscr{M},\rho)$, which means $\xi \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\#$. Combining with the fact that $x \in \mathscr{J} = \{x \in L^\alpha(\mathscr{M},\rho) : \rho(xy) = 0, \forall y \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\#\}$, we obtain that $\rho(x\xi) = 0$. Note, again, that $x \in \mathscr{J} \subseteq L^\alpha(\mathscr{M},\rho)$. From (a) and (c), it follows that $\rho(x\xi) = \phi(x) \neq 0$. This is a contradiction. Therefore
$H^\alpha(\mathscr{M},\rho) = \{x \in L^\alpha(\mathscr{M},\rho) : \rho(xy = 0) \text{ for all } y \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\#\}$. □

LEMMA 3.10. *(from [2]) The conditional expectation Φ_τ is multiplicative on Hardy spaces. More precisely, $\Phi_\tau(xy) = \Phi_\tau(x)\Phi_\tau(y)$ for $x \in H^p(\mathscr{M},\tau)$, $y \in H^q(\mathscr{M},\tau)$ and $xy \in H^r(\mathscr{M},\tau)$ with $0 < p,q,r < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

THEOREM 3.11. *If $\alpha \in N_\Delta(\mathscr{M},\tau)$, then there exists a faithful normal tracial state ρ such that $H^\alpha(\mathscr{M},\rho) = H^1(\mathscr{M},\rho) \cap L^\alpha(\mathscr{M},\rho)$.*

Proof. Since $\alpha \in N_\Delta(\mathscr{M},\tau)$, there exists a positive $g \in L^1(\mathscr{M},\tau)$ and $\Delta(g) > 0$ such that $\alpha(x) \geqslant c\tau(|x| \cdot g)$, for some $c > 0$. We define a faithful normal tracial state $\rho(x) = \tau(xg)$, $\forall x \in \mathscr{M}$. Since $\alpha(x) \geqslant c\tau(|x| \cdot g) = c\rho(|x|) = c\|x\|_{1,\rho}$, α is $\|\cdot\|_{1,\rho}$-dominating, so α-convergence implies $\|\cdot\|_{1,\rho}$-convergence, thus $H^\alpha(\mathscr{M},\rho) = \overline{H^\infty(\mathscr{M},\rho)}^\alpha \subseteq \overline{H^\infty(\mathscr{M},\rho)}^{\|\cdot\|_{1,\rho}} = H^1(\mathscr{M},\rho)$. Also, $H^\alpha(\mathscr{M},\rho) = \overline{H^\infty(\mathscr{M},\rho)}^\alpha \subseteq L^\alpha(\mathscr{M},\rho)$. Therefore, $H^\alpha(\mathscr{M},\rho) \subseteq H^1(\mathscr{M},\rho) \cap L^\alpha(\mathscr{M},\rho)$.

To prove $H^1(\mathscr{M},\rho) \cap L^\alpha(\mathscr{M},\rho) \subseteq H^\alpha(\mathscr{M},\rho)$. Suppose $x \in H^1(\mathscr{M},\rho) \cap L^\alpha(\mathscr{M},\rho)$, then $x \in L^\alpha(\mathscr{M},\rho)$. Assume that $y \in H_0^1(\mathscr{M},\rho) \cap (L^\alpha(\mathscr{M},\rho))^\#$. So $\Phi_\tau(hy) = 0$. Note that $hx \in H^1(\mathscr{M},\tau)$, $hy \in H_0^1(\mathscr{M},\tau)$ and $hxhy \in H^1(\mathscr{M},\tau)H_0^1(\mathscr{M},\tau) \subseteq H^{\frac{1}{2}}(\mathscr{M},\tau)$. From Theorem 2.1 in [2], and Lemma 3.10 we know that $\Phi_\tau(hxhy) \in L^{\frac{1}{2}}(\mathscr{D},\tau)$ and $\Phi_\tau(hxhy) = \Phi_\tau(hx)\Phi_\tau(hy) = 0$. Moreover, $x \in L^\alpha(\mathscr{M},\rho)$ and $y \in (L^\alpha(\mathscr{M},\rho))^\#$, from [7], $xy \in L^\alpha(\mathscr{M},\rho) \subseteq L^1(\mathscr{M},\rho)$. So $hxy \in L^1(\mathscr{M},\tau)$, and $\Phi_\tau(hxy)$ is also in $L^1(\mathscr{M},\tau)$. Thus $\rho(xy) = \tau(hxy) = \tau(\Phi_\tau(hxhy)) = \tau(0) = 0$.

Now we check $\Phi_\tau(hxy) = 0$. Since h is strongly outer in $H^1(\mathscr{M},\rho)$, there is a sequence $\{a_n\}$ in H^∞ such that $a_n h \to 1$ in $\|\cdot\|_1$ norm. Therefore, $\|hxyha_n - hxy\|_{\frac{1}{2}} = \|hxy(ha_n - 1)\|_{\frac{1}{2}} \leqslant \|hxy\|_1 \|ha_n - 1\|_1 \to 0$ as $n \to \infty$. And by Theorem 2.1 in [2], $\Phi_\tau(hxyha_n) \to \Phi_\tau(hxy)$. Also, we have $\Phi_\tau(hxyha_n) = \Phi_\tau(hx)\Phi_\tau(hy)\Phi_\tau(a_n) = 0$, so $\Phi_\tau(hxy) = 0$. By the definition of $\mathscr{J}$ in Proposition 3.9, we conclude that $x \in \mathscr{J}$. Therefore $H^1(\mathscr{M},\rho) \cap L^\alpha(\mathscr{M},\rho) \subseteq \mathscr{J} = H^\alpha(\mathscr{M},\rho)$. □

4. Beurling's invariant subspace theorem

In this section, we extend the Chen-Hadwin-Shen theorem for continuous normalized unitarily invariant norms on $(\mathscr{M},\tau)$.

First, we will prove the factorization theorem, in order to do this, we need the following lemma.

LEMMA 4.1. *(from [10]) Let $x \in L^p(\mathscr{M},\tau)$, $p > 0$, then we have:*

(1) $\Delta(x) = \Delta(x^*) = \Delta(|x|)$;

(2) $\Delta(xy) = \Delta(x)\Delta(y) = \Delta(yx)$, *for any* $y \in L^s(\mathscr{M},\tau)$, $s > 0$.

THEOREM 4.2. *Suppose* $\alpha \in N_\Delta(\mathscr{M},\tau)$, *there exists a faithful normal tracial state* ρ *on* $\mathscr{M}$ *such that* $\rho(x) = \tau(xg)$, *for some positive* $g \in L^1(\mathscr{Z},\tau)$ *and the determinant of* g *is positive. If* $x \in \mathscr{M}$ *and* $x^{-1} \in L^\alpha(\mathscr{M},\rho)$, *then there are unitary operators* $u_1, u_2 \in \mathscr{M}$ *and* $s_1, s_2 \in H^\infty$ *such that* $x = u_1 s_1 = s_2 u_2$ *and* $s_1^{-1}, s_2^{-1} \in H^\alpha(\mathscr{M},\rho)$.

Proof. Since $\alpha \in N_\Delta(\mathscr{M},\tau)$, the first statement is clear from Theorem 2.10. Suppose $x \in \mathscr{M}$ with $x^{-1} \in L^\alpha(\mathscr{M},\rho)$. Assume that $x = v|x|$ is the polar decomposition of x in $\mathscr{M}$, where v is a unitary in $\mathscr{M}$ and $|x| \in \mathscr{M}$. Since $log(|x|) \leqslant |x|, log(|h|) - log(|x|) = log(|h||x|^{-1}) \leqslant |h||x|$ and $-\log(|x|) \leqslant |h||x| - log(|h|)$, $|log(|x|)| \leqslant |x| + (|h||x| - log(|h|))$, so $\Delta(|x|) = e^{\tau(\log|x|)} > 0$ and $|x| \in L^1(\mathscr{M})^+$. By Corollary 4.17 in [5], there exists a strongly outer $s \in H^1(\mathscr{M},\tau)$ and $s = u_1|s|$ is the polar decomposition of s such that $|x| = |s|$. Since $|x| \in \mathscr{M}$, $|s| \in \mathscr{M}$, therefore, $s \in \mathscr{M}$ and $s \in H^1(\mathscr{M},\tau)$ implies $s \in H^\infty(\mathscr{M},\tau)$. Also, we have $|x| = u_1^* s$, so $x = vu_1^* s = us$, where $u = vu_1^*$.

Now we check $s^{-1} \in H^\alpha(\mathscr{M},\rho)$. First, $x^{-1} \in L^\alpha(\mathscr{M},\rho) \subseteq L^1(\mathscr{M},\rho)$, so $hx^{-1} \in L^1(\mathscr{M},\tau)$. Since $x^{-1} = |x|^{-1} v^* \in L^\alpha(\mathscr{M},\rho)$, $|x|^{-1} \in L^\alpha(\mathscr{M},\rho) \subseteq L^1(\mathscr{M},\rho)$ and $|h|\,|x|^{-1} \in L^1(\mathscr{M},\tau)$.
$\Delta(|h|\,|x|^{-1}) = \Delta(|h|)\Delta(|x|^{-1}) > 0$ by Lemma 4.1. Then there exists a strongly outer $f \in H^1(\mathscr{M},\tau)$ such that $|h|\,|x|^{-1} = |f|$. Since $H^1(\mathscr{M},\tau) = hH^1(\mathscr{M},\rho)$, there exists $f_1 \in H^1(\mathscr{M},\rho)$ such that $f = hf_1$. Since $\Delta(fs) = \Delta(f)\Delta(s) > 0$, by Lemma 3.5, fs is outer. And $|f|\,|s| = |h|\,|x|^{-1}\,|s|$, so $|h| = |f|\,|s|$. Therefore, $|h| = u_2^* f u_1^* s$, i.e., $h = u_3^* u_2^* f u_1^* s$, $hf_1 = f = u_2 u_3 h s^{-1} u_1$, so $s^{-1} = h^{-1} u_3^* u_2^* h f_1 u_1^* = u_3^* u_2^* f_1 u_1^* \in H^1(\mathscr{M},\rho)$. Also, we know $s^{-1} \in L^\alpha(\mathscr{M},\rho)$. Therefore, by Theorem 3.11, $s^{-1} \in L^\alpha(\mathscr{M},\rho) \cap H^1(\mathscr{M},\rho) = H^\alpha(\mathscr{M},\rho)$. ☐

The following density theorem also plays an important role in the proof of our main result of the paper.

THEOREM 4.3. *Let* $\alpha \in N_\Delta(\mathscr{M},\tau)$, *then there exists a faithful normal tracial state* ρ *on* $\mathscr{M}$ *such that* $\rho(x) = \tau(xg)$ *for some positive* $g \in L^1(\mathscr{Z},\tau)$ *and the determinant of* g *is positive. Also, if* $\mathscr{W}$ *is a closed subspace of* $L^\alpha(\mathscr{M},\rho)$ *and* $\mathscr{N}$ *is a weak* closed linear subspace of* $\mathscr{M}$ *such that* $\mathscr{W}H^\infty \subset \mathscr{W}$ *and* $\mathscr{N}H^\infty \subset \mathscr{N}$, *then:*

(1) $\mathscr{N} = [\mathscr{N}]_\alpha \cap \mathscr{M}$;

(2) $\mathscr{W} \cap \mathscr{M}$ *is weak* closed in* $\mathscr{M}$;

(3) $\mathscr{W} = [\mathscr{W} \cap \mathscr{M}]_\alpha$;

(4) if $\mathscr{S}$ *is a subspace of* $\mathscr{M}$ *such that* $\mathscr{S}H^\infty \subset \mathscr{S}$, *then* $[\mathscr{S}]_\alpha = [\overline{\mathscr{S}}^{w*}]_\alpha$, *where* $\overline{\mathscr{S}}^{w*}$ *is the weak*-closure of* $\mathscr{S}$ *in* $\mathscr{M}$.

Proof. Since $\alpha \in N_\Delta(\mathscr{M},\tau)$, clearly, there exists a faithful normal tracial state ρ on $\mathscr{M}$ by Theorem 2.10. For (1), it is clear that $\mathscr{N} \subseteq [\mathscr{N}]_\alpha \cap \mathscr{M}$. Assume, via contradiction, that $\mathscr{N} \subsetneq [\mathscr{N}]_\alpha \cap \mathscr{M}$. Note that $\mathscr{N}$ is a weak* closed linear subspace

of $\mathscr{M}$ and $L^1(\mathscr{M},\rho)$ is the predual space of $(\mathscr{M},\rho)$. It follows from the Hahn-Banach theorem that there exists a $\xi\in L^1(\mathscr{M},\rho)$ and an $x\in[\mathscr{N}]_\alpha\cap\mathscr{M}$ such that
(a) $\rho(\xi x)\neq 0$ and (b) $\rho(\xi y)=0$ for all $y\in\mathscr{N}$.
We claim that there exists a $z\in\mathscr{M}$ such that
(a$'$) $\rho(zx)\neq 0$ and (b$'$) $\rho(zy)=0$ for all $y\in\mathscr{N}$. Actually assume that $\xi=|\xi^*|v$ is the polar decomposition of $\xi\in L^1(\mathscr{M},\rho)$, where v is a unitary element in $\mathscr{M}$ and $|\xi^*|$ is in $L^1(\mathscr{M},\rho)$ is positive. Let f be a function on $[0,\infty)$ defined by the formula $f(t)=1$ for $0\leqslant t\leqslant 1$ and $f(t)=1/t$ for $t>1$. We define $k=f(|\xi^*|)$ by the functional calculus. Then by the construction of f, we know that $k\in\mathscr{M}$ and $k^{-1}=f^{-1}(|\xi^*|)\in L^1(\mathscr{M},\rho)$. It follows from Theorem 4.2 that there exist a unitary operator $u\in\mathscr{M}$ and $s\in H^\infty$ such that $k=us$ and $s^{-1}\in H^1(\mathscr{M},\rho)$. Therefore, we can further assume that $\{t_n\}_{n=1}^\infty$ is a sequence of elements in H^∞ such that $\|s^{-1}-t_n\|_{1,\rho}$. Observe that:

(i) since s,t_n are in H^∞, for each $y\in\mathscr{N}$ we have that $yt_ns\in\mathscr{N}H^\infty\subseteq\mathscr{N}$ and $\rho(t_ns\xi y)=\rho(\xi yt_ns)=0$;

(ii) we have $s\xi=(u*u)s(|\xi^*|v)=u*(k|\xi^*|)v\in\mathscr{M}$, by the definition of k;

(iii) from (a) and (i), we have $0\neq\rho(\xi x)=\rho(s^{-1}s\xi x)=\lim_{n\to\infty}\rho(t_ns\xi x)$.

Combining (i), (ii) and (iii), we are able to find an $N\in\mathbb{Z}$ such that $z=t_Ns\xi\in\mathscr{M}$ satisfying
(a$'$) $\rho(zx)\neq 0$ and (b$'$) $\rho(zy)=0$ for all $y\in\mathscr{N}$.
Recall that $x\in[\mathscr{N}]_\alpha$. Then there is a sequence $\{x_n\}\subseteq\mathscr{N}$ such that $\alpha(x-x_n)\to 0$. We have $|\rho(zx_n)-\rho(zx)|=|\rho(x-x_n)|\leqslant\|x-x_n\|_{1,\rho}\|z\|\to 0$.
Combining with (b$'$) we conclude that $\rho(zx)=\lim_{n\to\infty}\rho(zx_n)=0$. This contradicts with the result (a$'$). Therefore, $\mathscr{N}=[\mathscr{N}]_\alpha\cap\mathscr{M}$.

For (2), let $\overline{\mathscr{W}\cap\mathscr{M}}^{w*}$ be the weak*-closure of $\mathscr{W}\cap\mathscr{M}$ in $\mathscr{M}$. In order to show that $\mathscr{W}\cap\mathscr{M}=\overline{\mathscr{W}\cap\mathscr{M}}^{w*}$, it suffices to show that $\overline{\mathscr{W}\cap\mathscr{M}}^{w*}\subseteq\mathscr{W}$. Assume, to the contrary, that $\overline{\mathscr{W}\cap\mathscr{M}}^{w*}\nsubseteq\mathscr{W}$. Thus there exists an element x in $\overline{\mathscr{W}\cap\mathscr{M}}^{w*}\subset\mathscr{M}\subseteq L^\alpha(\mathscr{M},\rho)$, but $x\notin\mathscr{W}$. Since $\mathscr{W}$ is a closed subspace of $L^\alpha(\mathscr{M},\rho)$, by the Hahn-Banach theorem, there exists a $\xi\in L^1(\mathscr{M},\rho)$ such that $\rho(\xi x)\neq 0$ and $\rho(\xi y)=0$, for all $y\in\mathscr{W}$. Since $\xi\in L^1(\mathscr{M},\rho)$, the linear mapping $\rho_\xi:\mathscr{M}\to\mathbb{C}$, defined by $\rho_\xi(a)=\rho(\xi a)$, for all $a\in\mathscr{M}$, is weak*-continuous. Note that $x\in\overline{\mathscr{W}\cap\mathscr{M}}^{w*}$ and $\rho(\xi y)=0$ for all $y\in\mathscr{W}$. We know that $\rho(\xi x)=0$, which contradicts with the assumption that $\rho(\xi x)\neq 0$. Hence $\overline{\mathscr{W}\cap\mathscr{M}}^{w*}\subseteq\mathscr{W}$, so $\mathscr{W}\cap\mathscr{M}=\overline{\mathscr{W}\cap\mathscr{M}}^{w*}$.

For (3), since $\mathscr{W}$ is α-closed, it is easy to see $[\mathscr{W}\cap\mathscr{M}]_\alpha\subseteq\mathscr{W}$. Now we assume $[\mathscr{W}\cap\mathscr{M}]_\alpha\subsetneq\mathscr{M}\subseteq L^\alpha(\mathscr{M},\rho)$. By the Hahn-Banach theorem, there exists an $x\in\mathscr{W}$ and $\xi\in L^1(\mathscr{M},\rho)$ such that $\rho(\xi x)\neq 0$ and $\rho(\xi y)=0$, for all $y\in[\mathscr{W}\cap\mathscr{M}]_\alpha$. Let $x=v|x|$ be the polar decomposition of x in $L^\alpha(\mathscr{M},\rho)$, where v is a unitary element in $\mathscr{M}$. Let f be a function on $[0,\infty)$ defined by the formula $f(t)=1$, for $0\leqslant t\leqslant 1$ and $f(t)=1/t$, for $t>1$. We define $k=f(|x|)$ by the functional calculus. Then by the construction of f, we know that $k\in\mathscr{M}$ and $k^{-1}=f^{-1}(|x|)\in L^\alpha(\mathscr{M},\rho)$. It follows from Theorem 4.2 that there exist a unitary operator $u\in\mathscr{M}$ and $s\in H^\infty$ such that $k=su$ and $s^{-1}\in H^\alpha(\mathscr{M},\rho)$. A little computation shows that $|x|k\in\mathscr{M}$ which implies that $xs = xsuu^* = xku^* = v(|x|k)u^* \in \mathscr{M}$. Since $s \in H^\infty$, we know $xs \in \mathscr{W}H^\infty \subseteq \mathscr{W}$ and thus $xs \in \mathscr{W} \cap \mathscr{M}$. Furthermore, note that $(\mathscr{W} \cap \mathscr{M})H^\infty \subseteq \mathscr{W} \cap \mathscr{M}$. Thus, if $t \in H^\infty(\mathscr{M},\rho)$, we see $xst \in \mathscr{W} \cap \mathscr{M}$ and $\rho(\xi xst) = 0$. Since $H^\infty(\mathscr{M},\rho)$ is dense in $H^\alpha(\mathscr{M},\rho)$ and $\xi \in L^1(\mathscr{M},\rho)$, it follows that $\rho(\xi xst) = 0$, for all $t \in H^\alpha(\mathscr{M},\rho)$. Since $s^{-1} \in H^\alpha(\mathscr{M},\rho)$, we see that $\rho(\xi x) = \rho(\xi xss^{-1}) = 0$. This contradicts with the assumption that $\rho(\xi x) \neq 0$. Therefore $\mathscr{W} = [\mathscr{W} \cap \mathscr{M}]_\alpha$.

For (4), assume that $\mathscr{S}$ is a subspace of $\mathscr{M}$ such that $\mathscr{S}H^\infty(\mathscr{M},\rho) \subset \mathscr{S}$ and $\overline{\mathscr{S}}^{w*}$ is weak*-closure of $\mathscr{S}$ in $\mathscr{M}$. Then $[\mathscr{S}]_\alpha H^\infty(\mathscr{M},\rho) \subseteq [\mathscr{S}]_\alpha$. Note that $\mathscr{S} \subseteq [\mathscr{S}]_\alpha \cap \mathscr{M}$. From (2), we know that $[\mathscr{S}]_\alpha \cap \mathscr{M}$ is weak*-closed. Therefore, $\overline{\mathscr{S}}^{w*} \subseteq [\mathscr{S}]_\alpha \cap \mathscr{M}$. Hence $[\overline{\mathscr{S}}^{w*}]_\alpha \subseteq [\mathscr{S}]_\alpha$ and $[\mathscr{S}]_\alpha = [\overline{\mathscr{S}}^{w*}]_\alpha$. $\square$

Before we obtain our main result in the paper, we recall the definitions of internal column sum of a family of subspaces, and the lemma in [4].

DEFINITION 4.4. Let X be a closed subspace of $L^\alpha(\mathscr{M},\tau)$ with $\alpha \in N_\Delta(\mathscr{M},\tau)$. Then X is called an internal column sum of a family of closed subspaces $\{X_\lambda\}_{\lambda\in\Lambda}$ of $L^\alpha(\mathscr{M},\tau)$, denoted by $X = \bigoplus_{\lambda\in\Lambda}^{col} X_\lambda$ if:

(1) $X_\mu^* X_\lambda = \{0\}$, for all distinct $\lambda,\mu \in \Lambda$, and;

(2) $X = [span\{X_\lambda : \lambda \in \Lambda\}]_\alpha$.

DEFINITION 4.5. Let X be a weak*-closed subspace of $\mathscr{M}$ and $\alpha \in N_\Delta(\mathscr{M},\tau)$. Then X is called an internal column sum of a family of weak*-closed subspaces $\{X_\lambda\}_{\lambda\in\Lambda}$ of $L^\alpha(\mathscr{M},\tau)$, denoted by $X = \bigoplus_{\lambda\in\Lambda}^{col} X_\lambda$ if:

(1) $X_\mu^* X_\lambda = \{0\}$ for all distinct $\lambda,\mu \in \Lambda$, and;

(2) $X = \overline{span\{X_\lambda : \lambda \in \Lambda\}}^{w*}$.

LEMMA 4.6. *(from [4]) Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant $\|\cdot\|_{1,\tau}$-dominating continuous norm on $\mathscr{M}$. Let H^∞ be a finite subdiagonal subalgebra of $\mathscr{M}$ and $\mathscr{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathscr{W} \subseteq \mathscr{M}$ is a weak*-closed subspace such that $\mathscr{W}H^\infty \subseteq \mathscr{W}$. Then there exists a weak*-closed subspace $\mathscr{Y}$ of $\mathscr{M}$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathscr{M}$ such that:*

(1) $u_\lambda^* \mathscr{Y} = 0$, for all $\lambda \in \Lambda$;

(2) $u_\lambda^* u_\lambda \in \mathscr{D}$ and $u_\lambda^* u_\mu = 0$, for all $\lambda,\mu \in \Lambda$ with $\lambda \neq \mu$;

(3) $\mathscr{Y} = \overline{H_0^\infty \mathscr{Y}}^{w*}$;

(4) $\mathscr{W} = \mathscr{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} u_\lambda H^\infty)$.

Now we are ready to prove our main result of the paper, an extension of the Chen-Hadwin-Shen theorem for noncommutative Hardy spaces associated with finite von Neumann algebras.

THEOREM 4.7. *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a determinant, normalized, unitarily invariant, continuous norm on $\mathscr{M}$. Then there exists a faithful normal tracial state ρ on $\mathscr{M}$ such that $\alpha \in N_1(\mathscr{M},\rho)$. Let H^∞ be a finite subdiagonal subalgebra of $\mathscr{M}$ and $\mathscr{D} = H^\infty \cap (H^\infty)^*$. If $\mathscr{W}$ is a closed subspace of $L^\alpha(\mathscr{M},\tau)$ such that $\mathscr{W}H^\infty \subseteq \mathscr{W}$, then there exists a closed subspace $\mathscr{Y}$ of $L^\alpha(\mathscr{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathscr{M}$ such that:*

(1) $u_\lambda^*\mathscr{Y} = 0$, for all $\lambda \in \Lambda$;

(2) $u_\lambda^* u_\lambda \in \mathscr{D}$ and $u_\lambda^* u_\mu = 0$, for all $\lambda,\mu \in \Lambda$ with $\lambda \neq \mu$;

(3) $\mathscr{Y} = [H_0^\infty \mathscr{Y}]_\alpha$;

(4) $\mathscr{W} = \mathscr{Y} \oplus^{col} (\oplus^{col}_{\lambda\in\Lambda} u_\lambda H^\alpha)$.

Proof. Suppose $\mathscr{W}$ is a closed subspace of $L^\alpha(\mathscr{M},\tau)$ such that $\mathscr{W}H^\infty \subset \mathscr{W}$. Then it follows from part (2) of the Theorem 4.3 that $\mathscr{W}\cap\mathscr{M}$ is weak* closed in $(\mathscr{M},\tau) = (\mathscr{M},\rho)$, we also notice $L^\infty(\mathscr{M},\tau) = \mathscr{M} = L^\infty(\mathscr{M},\rho), L^\alpha(\mathscr{M},\tau) = L^\alpha(\mathscr{M},\rho)$ and $H^\alpha(\mathscr{M},\tau) = H^\alpha(\mathscr{M},\rho)$. It follows from the Lemma 4.6 that

$$\mathscr{W}\cap\mathscr{M} = \mathscr{Y}_1 \bigoplus^{col} (\bigoplus^{col}_{i\in\mathscr{I}} u_i H^\infty),$$

where $\mathscr{Y}_1$ is a closed subspace of $L^\infty(\mathscr{M},\rho)$ such that $\mathscr{Y}_1 = \overline{\mathscr{Y}_1 H_0^\infty}^{w*}$, and where u_i are partial isometries in $\mathscr{W}\cap\mathscr{M}$ with $u_j^* u_i = 0$ if $i \neq j$ and with $u_i^* u_i \in \mathscr{D}$. Moreover, for each $i, u_i^*\mathscr{Y}_1 = \{0\}$, left multiplication by the $u_i u_i^*$ are contractive projections from $\mathscr{W}\cap\mathscr{M}$ onto the summands $u_i H^\infty$ and left multiplication by $I - \sum_i u_i u_i^*$ is a contractive projection from $\mathscr{W}\cap\mathscr{M}$ onto $\mathscr{Y}_1$.

Let $\mathscr{Y} = [\mathscr{Y}_1]_\alpha$. It is not hard to verify that for each $i, u_i^*\mathscr{M} = \{0\}$. We also claim that $[u_i H^\infty]_\alpha = u_i H^\alpha$. In fact it is obvious that $[u_i H^\infty]_\alpha \supseteq u_i H^\alpha$. We will need only to show that $[u_i H^\infty]_\alpha \subseteq u_i H^\alpha$. Suppose $x \in [u_i H^\infty]_\alpha$, there is a net $\{x_n\}_{n=1}^\infty \subseteq H^\infty$ such that $\alpha(u_i x_n - x) \to 0$. By the choice of u_i, we know that $u_i u \in \mathscr{D} \subseteq H^\infty$, so $u_i u x_n \in H^\infty$, for each $n \geqslant 1$. Combining with the fact that $\alpha(u_i^* u_i x_n - u_i^* x) \leqslant \alpha(u_i x_n - x) \to 0$, we obtain that $u_i^* x \in H^\alpha$. Again from the choice of u_i, we know that $u_i u_i^* u_i x_n = u_i x_n$, for each $n \geqslant 1$. This implies that $x = u_i(u_i^* x) \in u_i H^\alpha$. Thus we conclude that $[u_i H^\infty]_\alpha \subseteq u_i H^\alpha$, so $[u_i H^\infty]_\alpha = u_i H^\alpha$. Now from parts (3) and (4) of the Theorem 4.3 and from the definition of internal column sum, it follows that

$$\mathscr{W} = [\mathscr{W}\cap\mathscr{M}]_\alpha = [\overline{span\{\mathscr{Y}_1, u_i H^\infty : i \in \mathscr{I}\}}^{w*}]_\alpha = [span\{\mathscr{Y}_1, u_i H^\infty : i \in \mathscr{I}\}]_\alpha.$$

$$= [span\{\mathscr{Y}, u_i H^\alpha : i \in \mathscr{I}\}]_\alpha = \mathscr{Y} \bigoplus^{col} (\bigoplus^{col}_{i\in\mathscr{I}} u_i H^\alpha).$$

Next, we will verify that $\mathscr{Y} = [\mathscr{Y} H_0^\infty]_\alpha$. Recall that $\mathscr{Y} = [\mathscr{Y}_1]_\alpha$. It follows from part (1) of the Theorem 4.3, we have

$$[\mathscr{Y}_1 H_0^\infty]_\alpha \cap \mathscr{M} = \overline{\mathscr{Y}_1 H_0^\infty}^{w*} = \mathscr{Y}_1.$$

Hence from part (3) of the Theorem 4.3 we have that

$$\mathscr{Y} \supseteq [\mathscr{Y} H_0^\infty]_\alpha \supseteq [\mathscr{Y}_1 H_0^\infty]_\alpha = [[\mathscr{Y}_1 H_0^\infty]_\alpha \cap \mathscr{M}]_\alpha = [\mathscr{Y}_1]_\alpha = \mathscr{Y}.$$

Thus $\mathscr{Y} = [\mathscr{Y} H_0^\infty]_\alpha$. Moreover, it is not difficult to verify that for each i, left multiplication by the $u_i u_i^*$ are contractive projections from $\mathscr{K}$ onto the summands $u_i H^\alpha$ and left multiplication by $I - \sum_i u_i u_i^*$ is a contractive projection from $\mathscr{W}$ onto $\mathscr{Y}$. Now the proof is completed. □

If we consider α as some specific norms, then we have some corollaries. If we take α be a unitarily invariant, $\|\cdot\|_{1,\tau}$-dominating, continuous norm, then we have Chen-Hadwin-Shen's result in [7].

COROLLARY 4.8. *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, $\|\|_{1,\tau}$-dominating, continuous norm on $\mathscr{M}$. Let H^∞ be a finite subdiagonal subalgebra of $\mathscr{M}$. Let $\mathscr{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathscr{W}$ is a closed subspace of $L^\alpha(\mathscr{M},\tau)$ such that $H^\infty \mathscr{W} \subseteq \mathscr{W}$. Then there exist a closed subspace $\mathscr{Y}$ of $L^\alpha(\mathscr{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathscr{W} \cap \mathscr{M}$ such that:*

(1) $u_\lambda^* \mathscr{Y} = 0$, for all $\lambda \in \Lambda$;

(2) $u_\lambda^* u_\lambda \in \mathscr{D}$ and $u_\lambda^* u_\mu = 0$, for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(3) $\mathscr{Y} = [H_0^\infty \mathscr{Y}]_\alpha$;

(4) $\mathscr{W} = \mathscr{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} H^\alpha u_\lambda)$.

If we take $\alpha = \|\cdot\|_p$, then we have D. Blecher and L. E. Labuschagne's result in [4].

COROLLARY 4.9. *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and H^∞ be a finite subdiagonal subalgebra of $\mathscr{M}$. Let $\mathscr{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathscr{W}$ is a closed subspace of $L^p(\mathscr{M},\tau), 1 \leqslant p \leqslant \infty$ such that $H^\infty \mathscr{W} \subseteq \mathscr{W}$. Then there exist a closed subspace $\mathscr{Y}$ of $L^p(\mathscr{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathscr{W} \cap \mathscr{M}$ such that:*

(1) $u_\lambda^* \mathscr{Y} = 0$, for all $\lambda \in \Lambda$;

(2) $u_\lambda^* u_\mu \in \mathscr{D}$ and $u_\lambda^* u_\mu = 0$, for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(3) $\mathscr{Y} = [H_0^\infty \mathscr{Y}]_p$;

(4) $\mathscr{W} = \mathscr{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} H^p u_\lambda)$.

5. Generalized Beurling theorem for special von Neumann algebras

In Theorem 4.7, if let $\mathscr{M}$ be classical Hardy space on unit circle $\mathbb{T}$ with Haar measure, i.e., $\mathscr{M} = L^\infty(\mathbb{T},\mu)$, $H^\infty = H^\infty(\mathbb{T},\mu)$, then $\mathscr{D} = H^\infty \cap (H^\infty)^* = \mathbb{C}$ and the center $\mathscr{Z}$ of $\mathscr{M} = L^\infty(\mathbb{T},\mu)$ is itself. So $\mathscr{Z} \nsubseteq \mathscr{D} = \mathbb{C}$. However, for a finite von Neumann algebra $\mathscr{M}$ with a faithful normal tracial state τ, let H^∞ be a finite subdiagonal subalgebra of $\mathscr{M}$, $\mathscr{D} = H^\infty \cap (H^\infty)^*$, if the center $\mathscr{Z} \subseteq \mathscr{D}$, then generalized Beurling theorem holds for normalized, unitarily invariant, continuous norms on $(\mathscr{M},\tau)$.

THEOREM 5.1. *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let H^∞ be a finite subdiagonal subalgebra of $\mathscr{M}$, $\mathscr{D} = H^\infty \cap (H^\infty)^*$, and the center $\mathscr{Z} \subseteq \mathscr{D}$. Let α be a normalized, unitarily invariant, continuous norm on $(\mathscr{M},\tau)$. If $\mathscr{W}$ is a closed subspace of $L^\alpha(\mathscr{M},\tau)$ such that $\mathscr{W}H^\infty \subseteq \mathscr{W}$, then there exist a closed subspace $\mathscr{Y}$ of $L^\alpha(\mathscr{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathscr{M}$ such that:*

(1) $u_\lambda^* \mathscr{Y} = 0$, for all $\lambda \in \Lambda$;

(2) $u_\lambda^* u_\lambda \in \mathscr{D}$ and $u_\lambda^* u_\mu = 0$, for all $\lambda,\mu \in \Lambda$ with $\lambda \neq \mu$;

(3) $\mathscr{Y} = [H_0^\infty \mathscr{Y}]_\alpha$;

(4) $\mathscr{W} = \mathscr{Y} \oplus^{col} (\oplus^{col}_{\lambda\in\Lambda} u_\lambda H^\alpha)$.

Proof. By Theorem 2.10, there exist a faithful normal tracial state ρ on $\mathscr{M}$ and a $c > 0$ such that α is a continuous normalized unitarily invariant $c\|\cdot\|_{1,\rho}$-dominating norm on $(\mathscr{M},\rho)$. First, recall the definition of conditional expectation $\Phi_{\mathscr{D},\tau}$. We know that $\Phi_{\mathscr{D},\tau}$ is multiplicative on H^∞. In general, $\Phi_{\mathscr{D},\rho}$ won't be multiplicative on H^∞, however, the condition $\mathscr{Z} \subset \mathscr{D}$ makes sure $\Phi_{\mathscr{D},\rho}$ is multiplicative on H^∞, because we can choose $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots$ in $\mathscr{Z}$ such that, for every $x \in \mathscr{M}$,

$$\rho(x) = \lim_{n\to\infty} \tau(x_n x) = \lim_{n\to\infty} \tau\left(\Phi_{\mathscr{D},\tau}(x_n x)\right).$$

Since $\mathscr{Z} \subset \mathscr{D}$, $\Phi_{\mathscr{D},\tau}(x_n x) = x_n \Phi_{\mathscr{D},\tau}(x)$. Thus

$$\rho(x) = \lim_{n\to\infty} \tau\left(x_n \Phi_{\mathscr{D},\tau}(x)\right) = \rho\left(\Phi_{\mathscr{D},\tau}(x)\right).$$

It follows that $\Phi_{\mathscr{D},\tau} = \Phi_{\mathscr{D},\rho}$. This now reduces to the $c\|\cdot\|_1$-dominating version of the Chen-Hadwin-Shen theorem in [7]. □

REFERENCES

[1] W. B. ARVESON, *Analyticity in operator algebras*, Amer. J. Math. 89 (1967) 578–642.

[2] T. N. BEKJAN AND Q. XU, *Riesz and Szego type factorizations for non-commutative Hardy spaces*, J. Operator Theory, 62 (2009) 215–231.

[3] A. BEURLING, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1949) 239–255.

[4] D. BLECHER AND L. E. LABUSCHAGNE, *A Beurling theorem for non-commutative L^p*, J. Operator Theory, 59 (2008) 29–51.
[5] D. BLECHER AND L. E. LABUSCHAGNE, *Outers for non commutative H^p revisited*, arxiv: 1304.0518v1 [math. OA] (2013).
[6] S. BOCHNER, *Generalized conjugate and analytic functions without expansions*, Proc. Nat. Acad. Sci. U.S.A. 45 (1959) 855–857.
[7] Y. CHEN, D. HADWIN AND J. SHEN, *A non-commutative Beurling's theorem with respect to unitarily invariant norms*, arXiv: 1505.03952v1 [math. OA] (2015).
[8] J. DIXMIER, *Formes linaires sur un anneau d'oprateurs (French)*, Bull. Soc. Math. France, 81 (1953) 9–39.
[9] H. FAN, D. HADWIN AND W. LIU, *An extension of the Chen-Beurling-Helson-Lowdenslager theorem*, arXiv:1611.00357 [math.FA].
[10] U. HAAGERUP AND H. SCHULTZ, *Brown measures of unbounded operators affiliated with a finite von Neumann algebra*, Math. Scand., 100 (2007), 209–263.
[11] P. HALMOS, *Shifts on Hilbert spaces*, J. Reine Angew. Math. 208 (1961) 102–112.
[12] H. HELSON, *Lectures on Invariant Subspaces*, Academic Press, New York-London, 1964.
[13] H. HELSON AND D. LOWDENSLAGER, *Prediction theory and Fourier series in several variables*, Acta Math. 99 (1958) 165–202.
[14] K. HOFFMAN, *Analytic functions and logmodular Banach algebras*, Acta Math. 108 (1962) 271–317.
[15] R. V. KADISON AND J. R. RINGROSE, *Fundamentals of the theory of operator algebras, Vol. II: Advanced theory*, New York etc., Academic Press 1986. XIV.
[16] E. NELSON, *Notes on non-commutative integration*, J. Funct. Anal. 15 (1974) 103–116.
[17] K. S. SAITO, *A note on invariant subspaces for finite maximal subdiagonal algebras*, Proc. Amer. Math. Soc. 77 (1979) 348–352.
[18] T. P. SRINIVASAN, *Simply invariant subspaces*, Bull. Amer. Math. Soc. 69 (1963) 706–709.
[19] M. TAKESAKI, *Theory of operator algebras I*, Springer, 1979.

(Received January 14, 2018)

Haihui Fan
Institute of Information Engineering
Chinese Academy of Science
No. 89 Minzhuang Rd, Haidian District, Beijing 100093, PRC
e-mail: fanhaihui@iie.ac.cn

Don Hadwin
University of New Hampshire
33 academic way, Durham, NH 03824
e-mail: don@unh.edu

Wenjing Liu
University of New Hampshire
33 academic way, Durham, NH 03824
e-mail: wbs4@wildcats.unh.edu