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ON REAL OR INTEGRAL SKEW LAPLACIAN SPECTRUM OF DIGRAPHS

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Abstract. For a simple connected graph G with n vertices and m edges, let $\overrightarrow{G}$ be a digraph obtained by giving an arbitrary direction to the edges of G. In this paper, we consider the skew Laplacian matrix of a digraph $\overrightarrow{G}$ and we obtain the skew Laplacian spectrum of the orientations of a complete bipartite graph, complete split graph and the join of two graphs. We prove that deleting an edge of a Hamiltonian path in a transitive tournament does not effect the skew Laplacian spectrum. We show the existence of various families of skew Laplacian integral digraphs.

1. Introduction

Consider a simple graph G with n vertices and m edges and having the vertex set $V=\{v_1,v_2,\ldots,v_n\}$. Let $\overrightarrow{G}$ be a digraph obtained by assigning arbitrarily a direction to each of the edges of G. The digraph $\overrightarrow{G}$ is called an orientation of G or oriented graph corresponding to G. Also, the graph G is called the underlying graph of $\overrightarrow{G}$. Let $d_i^+=d^+(v_i)$, $d_i^-=d^-(v_i)$ and $d_i=d_i^++d_i^-$, $i=1,2,\ldots,n$, be respectively the out-degree, in-degree and degree of the vertices of $\overrightarrow{G}$. The out-adjacency matrix of the digraph $\overrightarrow{G}$ is the $n\times n$ matrix $A^+=A^+(\overrightarrow{G})=(a_{ij})$, where $a_{ij}=1$, if (v_i,v_j) is an arc and $a_{ij}=0$, otherwise. The in-adjacency matrix of the digraph $\overrightarrow{G}$ is the $n\times n$ matrix $A^-=A^-(\overrightarrow{G})=(a_{ij})$, where $a_{ij}=1$, if (v_j,v_i) is an arc and $a_{ij}=0$, otherwise. We note that $A^-=(A^+)^t$. The skew adjacency matrix of a digraph $\overrightarrow{G}$ is the $n\times n$ matrix $S=S(\overrightarrow{G})=(s_{ij})$, where

$$s_{ij}=\begin{cases}1, & \text{if there is an arc from } v_i \text{ to } v_j,\\ -1, & \text{if there is an arc from } v_j \text{ to } v_i,\\ 0, & \text{otherwise.}\end{cases}$$

Clearly $S(\overrightarrow{G})$ is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. For recent developments on the theory of skew spectrum, we refer to [1, 14].

Let $D^+=D^+(\overrightarrow{G})=diag(d_1^+,d_2^+,\ldots,d_n^+)$, $D^-=D^-(\overrightarrow{G})=diag(d_1^-,d_2^-,\ldots,d_n^-)$ and $D(\overrightarrow{G})=diag(d_1,d_2,\ldots,d_n)$ be respectively, the diagonal matrices of vertex out-degrees, vertex in-degrees and vertex degrees of $\overrightarrow{G}$. Further, let A^+ and A^- be respectively, the out-adjacency and in-adjacency matrices of a digraph $\overrightarrow{G}$. If $S(\overrightarrow{G})$ is

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the skew adjacency matrix of $\overrightarrow{G}$ and $A(G)$ is the adjacency matrix of the underlying graph G of the digraph $\overrightarrow{G}$, then $A(G) = A^+ + A^-$ and $S(\overrightarrow{G}) = A^+ - A^-$. Analogous to the definition of Laplacian matrix of a graph, Cai et al. [4] called the matrix $\widetilde{SL}(\overrightarrow{G}) = \widetilde{D}(\overrightarrow{G}) - S(\overrightarrow{G})$, where $\widetilde{D}(\overrightarrow{G}) = D^+(\overrightarrow{G}) - D^-(\overrightarrow{G})$, as the *skew Laplacian matrix* of the digraph $\overrightarrow{G}$. Clearly the matrix $\widetilde{SL}(\overrightarrow{G})$ is not symmetric and so its eigenvalues need not be real. The characteristic polynomial

$$P_{sl}(\overrightarrow{G},x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

of the matrix $\widetilde{SL}(\overrightarrow{G})$ is called the *skew Laplacian characteristic polynomial* of the digraph $\overrightarrow{G}$. The zeros of the polynomial $P_{sl}(\overrightarrow{G},x)$, that is, the eigenvalues of the matrix $\widetilde{SL}(\overrightarrow{G})$ are the skew Laplacian eigenvalues of the digraph $\overrightarrow{G}$ and are denoted by $\nu_1, \nu_2, \ldots, \nu_n$. The sign of the even cycle $C_k = u_1u_2\ldots u_ku_1$, denoted by $sgn(C_k)$, is defined as $sgn(C_k) = s_{12}s_{23}\ldots s_{k-1k}s_{k1}$, where s_{ij} is the $(i,j)^{th}$ entry of the matrix $\widetilde{SL}$. An even oriented cycle C_k is called evenly-oriented (oddly-oriented) if its sign is positive (negative). If every even cycle in $\overrightarrow{G}$ is evenly-oriented, then $\overrightarrow{G}$ is called evenly-oriented. An even oriented cycle C_{2k} is said to be uniformly oriented if $sgn(C_{2k}) = (-1)^k$. The following observations are immediate from the definition of $\widetilde{SL}$.

THEOREM 1.1. [4]

(i) If $\nu_1, \nu_2, \ldots, \nu_n$ are the eigenvalues of $\widetilde{SL}(\overrightarrow{G})$, then $\sum_{i=1}^{n} \nu_i = 0$.

(ii) 0 is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$ with multiplicity at least p, where p is the number of components of $\overrightarrow{G}$ with all ones vector $(1,1,\ldots,1)$ as the corresponding eigenvector.

(iii) If $P_{sl}(\overrightarrow{G},x) = x^n + \sum_{i=1}^{n} a_ix^{n-i}$ is the skew Laplacian characteristic polynomial of digraph $\overrightarrow{G}$, then $a_1 = 0$, $a_2 = m + \sum_{i<j}(d_i^+ - d_i^-)(d_j^+ - d_j^-)$, $a_n = 0$.

As usual, we denote the complete graph on n vertices by K_n, the complete bipartite graph on $s+t$ vertices by $K_{s,t}$ and the cycle on n vertices by C_n. For other undefined notations and terminology from graphs and spectral graph theory, the readers are referred to [3, 17]. Evidently much research has been done on spectral theory of skew matrices of oriented graphs, see [11, 14, 18, 19, 21], but the research on the skew Laplacian spectrum of a digraph $\overrightarrow{G}$ has recently started and it will be of great interest to develop the theory in this direction. Although the skew Laplacian matrix of a digraph was so defined that it uses the structure of the digraph and at the same time enjoys the same characteristics as possessed by the Laplacian matrix of a graph, it seems the definition of $\widetilde{SL}$ uses the structure of the digraph, but not all the properties of $L(G)$ are possessed by $\widetilde{SL}$. It is well-known that 0 is an eigenvalue of $L(G)$ with multiplicity equal to the number of components of G. In fact, the eigenvalue 0 in the spectrum of $L(G)$ decides the connectedness of the graph G. This need not be true for the matrix $\widetilde{SL}$, as is clear from the following observation, the proof of which follows from Theorem 2.1 in [20].

THEOREM 1.2. *Let G be a bipartite graph and let $\overrightarrow{G}$ be the corresponding digraph of G. If $\overrightarrow{G}$ is an Eulerian digraph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$, then the multiplicity of 0 in the spectrum of $\widetilde{SL}$ is same as the multiplicity of 0 in the spectrum of $A(G)$.*

Let $K_{r,s}$ be the complete bipartite graph with both r and s even. Orient the edges of $K_{r,s}$ in such a way that in the resulting digraph $\overrightarrow{G}$ all the even cycles are oriented uniformly. Since 0 is an adjacency eigenvalue of $K_{r,s}$ of multiplicity $r+s-2$, from Theorem 1.2, it follows that 0 is the skew Laplacian eigenvalue of $\overrightarrow{G}$ of multiplicity $r+s-2$. For some recent papers on skew Laplacian spectrum, we refer to [2, 5, 9, 10].

A graph is said to be adjacency (Laplacian, signless Laplacian) integral if all of its adjacency (Laplacian, signless Laplacian) eigenvalues are integers. Since there is no general characterization (besides the definition) of adjacency (Laplacian, signless Laplacian) integral graphs, the problem of finding (or characterizing) adjacency (Laplacian, signless Laplacian) integral graphs has to be treated in some special classes of graphs. Several papers can be found in the literature on the adjacency (Laplacian, signless Laplacian) integral graphs. For some recent papers, we refer to [6, 7, 8, 12, 13, 15, 16] and the references therein.

As is clear from the definition, the skew Laplacian matrix of a digraph $\overrightarrow{G}$ is not symmetric and so its eigenvalues need not be real. The following problems will be of interest in the theory of matrices which are not symmetric and have real entries.

PROBLEM 1.3. Which digraphs $\overrightarrow{G}$ have all skew Laplacian eigenvalues real.

PROBLEM 1.4. Which digraphs $\overrightarrow{G}$ have all skew Laplacian eigenvalues integers.

Although, like the case in graphs both these problems seem to be difficult for all digraphs in general. However, in case we restrict to a special class of digraphs, we may get an insight of the possible solution of these problems. In this paper, we will focus on the above mentioned problems and show the existence of various families of digraphs having real or integral skew Laplacian spectrum.

We call a digraph $\overrightarrow{G}$ real digraph if all its skew Laplacian eigenvalues are real and a partial real digraph if some of its skew Laplacian eigenvalues are real. A real digraph $\overrightarrow{G}$ is said to be skew Laplacian integral digraph if all its skew Laplacian eigenvalues are integers.

The rest of the paper is organized as follows. In Section 2, we obtain the skew Laplacian spectrum of orientations of complete bipartite graphs. We also show the existence of some families of skew Laplacian integral digraphs. In Section 3, we obtain the skew Laplacian spectrum of transitive tournaments and show that deleting a particular edge does not change the skew Laplacian spectrum. In Section 4, we obtain the skew characteristic polynomial of the orientations of join of two graphs in terms of the skew characteristic polynomial of the parent digraphs. Also, we obtain the skew Laplacian spectrum of orientations of complete split graphs. We also show the existence of some families of skew Laplacian integral digraphs.

2. Skew Laplacian spectrum of oriented complete bipartite graphs

In this section, we obtain the skew Laplacian spectrum of the orientations of a complete bipartite graph. We show the existence of various families of skew Laplacian integral digraphs and skew Laplacian equienergetic digraphs.

A subset U of the vertex set $V(G)$ is said to be an independent set if the subgraph induced by the vertices in U is an empty graph. Let $N_i^+ = N^+(v_i) = \{v_j : v_iv_j \in E(\overrightarrow{G})\}$ and $N_i^- = N^-(v_i) = \{v_j : v_jv_i \in E(\overrightarrow{G})\}$, be respectively, the set of out-neighbours and in-neighbours of the vertex v_i in $\overrightarrow{G}$. Clearly, $N_i^+ \cup N_i^- = N_i$, the neighbourhood set of the vertex v_i and $N_i^+ \cap N_i^- = \emptyset$.

The following lemma gives the information about the skew Laplacian eigenvalues together with the corresponding eigenvectors, when $\overrightarrow{G}$ has an independent set with the same set of neighbours.

LEMMA 2.1. *Let G be a graph of order n having vertex set $V(G)$ and let $\overrightarrow{G}$ be an orientation of G. Let $U = \{v_1, v_2, \ldots, v_k\}$ be an independent subset of the vertex set $V(G)$ having the same set of neighbours in G. If $N^+(v_i)$ is same for all $v_i \in U$ and $N^-(v_i)$ is same for all $v_i \in U$, then $|N^+(v_i)| - |N^-(v_i)|$ is a skew Laplacian eigenvalue of $\overrightarrow{G}$ of multiplicity at least $k-1$ with the corresponding $k-1$ eigenvectors $(1,-1,0,\ldots,0,\ldots,0)^t$, $(1,0,-1,\ldots,0,\ldots,0)^t$, $\ldots$, $(1,0,0,\ldots,-1,\ldots,0)^t$.*

Proof. Let $\overrightarrow{G}$ be an orientation of a graph G having vertex set

$$V(G) = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}.$$

With out loss of generality, let $U = \{v_1, v_2, \ldots, v_k\}$ be an independent set in G and so in $\overrightarrow{G}$. Suppose that all the vertices in U have the same neighbourhood set, say $U' = \{v_{k+1}, v_{k+2}, \ldots, v_s\}$ in G. Let the edges be oriented so that $N^+(v_i)$ is same for all $v_i \in U$ and $N^-(v_i)$ is same for all $v_i \in U$ in $\overrightarrow{G}$. We label the rows and columns of the matrix $\widetilde{SL}(\overrightarrow{G})$ in the same order as in $V(G)$. Let $X = (x_1, x_2, \ldots, x_n)^t$ be an eigenvector corresponding to an eigenvalue ν of $\widetilde{SL}(\overrightarrow{G})$. So $\widetilde{SL}(\overrightarrow{G})X = \nu X$. It can be easily seen that the eigenvalue $|N^+(v_i)| - |N^-(v_i)|$ with corresponding eigenvectors $X_1 = (1,-1,0,\ldots,0,\ldots,0)^t$, $X_2 = (1,0,-1,\ldots,0,\ldots,0)^t$, $\ldots$, $X_{k-1} = (1,0,0,\ldots,-1,\ldots,0)^t$ satisfy this relation. Since these $(k-1)$ eigenvectors are linearly independent, it follows that $|N^+(v_i)| - |N^-(v_i)|$ is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$ with multiplicity at least $k-1$ having the above mentioned $(k-1)$ vectors as corresponding eigenvectors. $\square$

Let M be a complex matrix of order n described in the following block form

$$M = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ldots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix},$$

where the blocks A_{ij} are $n_i \times n_j$ matrices for any $1 \leqslant i, j \leqslant s$ and $n = n_1 + \ldots + n_s$. For $1 \leqslant i, j \leqslant s$, let b_{ij} denote the average row sum of $A_{i,j}$. The quotient matrix $B = (b_{ij})$ is an $s \times s$ matrix whose entries are the average row sums of the blocks A_{ij} of M. If each block A_{ij} of M has constant row sum, the matrix B is called *equitable quotient matrix* of M. We can find a relation between the spectrum of a complex matrix and its *equitable quotient matrix* in the following theorem [22].

THEOREM 2.2. [22] *The eigenvalues of the equitable quotient matrix B are the eigenvalues of the matrix M, where M is the matrix described above.*

Let $V_1 = \{x_1, x_2, \ldots, x_r\}$ and $V_2 = \{y_1, y_2, \ldots, y_s\}$ be the partite sets of $K_{r,s}$, with $n = r + s$. We give different orientations to $K_{r,s}$ one by one. Let $\overrightarrow{H}_1$ be the orientation when all the edges are directed from V_1 to V_2, $\overrightarrow{H}_2$ be the orientation when all the edges are directed from V_2 to V_1, $\overrightarrow{H}_3$ be the orientation when each $x_i \in V_1$ has same out-neighbours $N^+(x_i)$ in V_2, $\overrightarrow{H}_4$ be the orientation when each $y_j \in V_2$ has same out-neighbours $N^+(y_j)$ in V_1. For $V_1 = U_1 \cup U_2 \cup \ldots \cup U_k$, let $\overrightarrow{H}_5$ be the orientation such that $N^+(U_i) = V_2, N^-(U_i) = \emptyset$, for $i = 1, 2, \ldots, t$ and $N^+(U_i) = \emptyset, N^-(U_i) = V_2$, for $i = t+1, \ldots, k$. For $V_1 = U_1 \cup U_2$, $V_2 = U_3 \cup U_4$, let $\overrightarrow{H}_6$ be the orientation such that $N^+(U_1) = U_4, N^-(U_1) = U_3, N^+(U_2) = U_3, N^-(U_2) = U_4$. For $V_1 = U_1 \cup U_2 \cup \ldots \cup U_k$ and $V_2 = W_1 \cup W_2 \cup \ldots \cup W_k$, let $\overrightarrow{H}_7$ be the orientation with $N^+(U_i) = W_i, N^-(U_i) = W_{k+1-i}, N^+(W_1) = U_{k+1-i}, N^-(W_i) = U_i$.

Now, we obtain the skew Laplacian spectrum of the digraphs $\overrightarrow{H}_1$ and $\overrightarrow{H}_2$.

THEOREM 2.3. *The skew Laplacian spectrum of $\overrightarrow{H}_1$ is $\{s-r, 0, s^{[r-1]}, (-r)^{[s-1]}\}$ and the skew Laplacian spectrum of $\overrightarrow{H}_2$ is $\{-(s-r), 0, (-s)^{[r-1]}, r^{[s-1]}\}$.*

Proof. Assume that the edges are oriented in such a way so that all the edges are oriented from V_1 to V_2. Since V_1 is an independent set and the orientation $\overrightarrow{H}_1$ is chosen so that, for all $x_i \in V_1$, we have $N^+(x_i) = V_2$ and $N^-(x_i) = \emptyset$, therefore from Lemma 2.1, it follows that $|N^+(x_i)| - |N^-(x_i)| = |V_2| = s$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_1$ with multiplicity at least $r-1$. Again, V_2 is an independent set and the orientation $\overrightarrow{H}_1$ is chosen so that, for all $y_i \in V_2$, we have $N^+(y_i) = \emptyset$ and $N^-(y_i) = V_1$. From Lemma 2.1, it follows that $|N^+(y_i)| - |N^-(y_i)| = -|V_1| = -r$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_1$ with multiplicity at least $s-1$. Since 0 is always an eigenvalue of $\widetilde{SL}(\overrightarrow{H}_1)$ and $tr(\widetilde{SL}(\overrightarrow{H}_1)) = 0$, it follows that the remaining two skew Laplacian eigenvalues are $0, s-r$. Thus, the skew Laplacian spectrum of $\overrightarrow{H}_1$ is $\{s-r, 0, s^{[r-1]}, (-r)^{[s-1]}\}$, completing the proof of the first part.

The proof of the second part follows by using the fact that $\widetilde{SL}(\overrightarrow{H}_2) = -\widetilde{SL}(\overrightarrow{H}_1)$, see [2]. $\square$

Now, we obtain the skew Laplacian spectrum of the digraphs $\overrightarrow{H}_3$ and $\overrightarrow{H}_4$.

THEOREM 2.4. *The skew Laplacian spectrum of $\overrightarrow{H}_3$ is*

$$\{\nu_1, \nu_2, (2t-s)^{[r-1]}, 0, r^{[s-t-1]}, (-r)^{[t-1]}\},$$

where ν_1 *and* ν_2 *are the zeros of the polynomial* $p(x) = x^2 - (2t-s)x + rs - r^2$ *and* $|N^+(x_i)| = t$. *The skew Laplacian spectrum of* $\overrightarrow{H}_4$ *is*

$$\{\nu_1, \nu_2, (2t-r)^{[s-1]}, 0, s^{[r-t-1]}, (-s)^{[t-1]}\},$$

where ν_1 *and* ν_2 *are the zeros of the polynomial* $p(x) = x^2 - (2t-r)x + rs - s^2$ *and* $|N^+(y_i)| = t$.

Proof. Suppose that the edges are oriented in such a way that all the vertices $x_i \in V_1$ have the same out-neighbourhood set $N^+(x_i)$. With out loss of generality, let $N^+(x_i) = \{y_1, y_2, \ldots, y_t\}$. Then $N^-(x_i) = \{y_{t+1}, y_{t+2}, \ldots, y_s\}$. Since V_1 is an independent set, from Lemma 2.1, it follows that $|N^+(x_i)| - |N^-(x_i)| = t - (s-t) = 2t - s$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_3$ with multiplicity at least $r-1$. Now, $N^+(x_i)$ is an independent set and the orientation $\overrightarrow{H}_3$ is chosen so that for all $y_i \in N^+(x_i)$, we have $N^+(y_i) = \emptyset$, and so $N^-(y_i) = V_1$. From Lemma 2.1, it follows that $|N^+(y_i)| - |N^-(y_i)| = -|V_1| = -r$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_3$ with multiplicity at least $t-1$. Also, $N^-(x_i)$ is an independent set and the orientation $\overrightarrow{H}_3$ is chosen so that, for all $y_i \in N^-(x_i)$, we have $N^+(y_i) = V_1$, and therefore $N^-(y_i) = \emptyset$. From Lemma 2.1, it follows that $|N^+(y_i)| - |N^-(y_i)| = |V_1| = r$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_3$ with multiplicity at least $s-t-1$. Since 0 is always an eigenvalue of $\widetilde{SL}(\overrightarrow{H}_3)$, let $\nu_1, \nu_2, 0$ be the remaining three skew Laplacian eigenvalue of $\overrightarrow{H}_3$. Using the fact that $tr(\widetilde{SL}(\overrightarrow{H}_3)) = 0$, we get $\nu_1 + \nu_2 = 2t - s$. Again, $tr(\widetilde{SL}^2(\overrightarrow{H}_3)) = \sum_{i=1}^{n}(d_i^+ - d_i^-)^2 - 2m$, implying that $\nu_1^2 + \nu_2^2 = (2t-s)^2 + 2r^2 - 2rs$. Using the relation $(\nu_1 + \nu_2)^2 = \nu_1^2 + \nu_2^2 + 2\nu_1\nu_2$, we see that ν_1 and ν_2 are the zeros of the polynomial $p(x) = x^2 - (2t-s)x + rs - r^2$. Thus, the skew Laplacian spectrum of $\overrightarrow{H}_3$ is $\{\nu_1, \nu_2, (2t-s)^{[r-1]}, 0, r^{[s-t-1]}, (-r)^{[t-1]}\}$, where ν_1 and ν_2 are the zeros of the polynomial $p(x) = x^2 - (2t-s)x + rs - r^2$, completing the proof of first part. The second part can be proved in a similar way. $\square$

The next result gives the skew Laplacian spectrum of the digraphs $\overrightarrow{H}_5$ and $\overrightarrow{H}_6$.

THEOREM 2.5. *The skew Laplacian spectrum of* $\overrightarrow{H}_5$ *is*

$$\{\nu_1, \nu_2, s^{[\sum_{i=1}^{t}(|U_i|)-1]}, (-s)^{[\sum_{i=t+1}^{k}(|U_i|)-1]}, (\alpha)^{[s-1]}, 0\},$$

where ν_1, ν_2 *are the zeros of the polynomial* $g(x) = x^2 - (\alpha - s)x - s\big(\alpha + 2|U_k| + 2|U_{k-1}| - 2\sum_{i=t+1}^{k-2}|U_i|\big)$ *and* $\alpha = \sum_{i=t+1}^{k}|U_i| - \sum_{i=1}^{t}|U_i|$. *The skew Laplacian spectrum of* $\overrightarrow{H}_6$ *is*

$$\{\nu_1, \nu_2, \nu_3, (|U_4|-|U_3|)^{[|U_1|-1]}, (|U_3|-|U_4|)^{[|U_2|-1]}, (|U_1|-|U_2|)^{[|U_3|-1]}, (|U_2|-|U_1|)^{[|U_4|-1]}\},$$

where ν_1, ν_2 *and* ν_3 *are the zeros of the polynomial* $p(x) = x^3 - ax^2 + bx - c$ *with* $a = 2(|U_1|-1)(|U_4|-|U_3|)$, $b = \frac{a^2}{2} - \Big[(|U_1|-|U_2|)^2 + (|U_4|-|U_3|)^2 - (|U_1|+|U_2|)(|U_4|+|U_3|)\Big]$, $3c = |U_1||U_3|(|U_3|-|U_1|) + |U_1||U_4|(|U_1|-|U_4|) + |U_2||U_3|(|U_2|-|U_3|)$

$+|U_2||U_4|(|U_4|-|U_2|)-a\Big[2(|U_1|-|U_2|)^2+2(|U_4|-|U_3|)^2-2(|U_1|+|U_2|)(|U_4|+|U_3|)-b\Big]$.

Proof. Let $V_1=U_1\cup U_2\cup\cdots\cup U_k$. Assume that the edges are oriented so that $N^+(U_i)=V_2, N^-(U_i)=\emptyset$, for $i=1,2,\ldots,t$ and $N^+(U_i)=\emptyset, N^-(U_i)=V_2$, for $i=t+1,\ldots,k$. Since U_i, $i=1,2,\ldots,t$, is an independent set, from Lemma 2.1, it follows that $|N^+(U_i)|-|N^-(U_i)|=s$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_5$ with multiplicity at least $\sum_{i=1}^{t}(|U_i|-1)$. Again, U_i is an independent set for $i=t+1,t+2,\ldots,k$, from Lemma 2.1, it follows that $|N^+(U_i)|-|N^-(U_i)|=-s$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_5$ with multiplicity at least $\sum_{i=t+1}^{k}(|U_i|-1)$. Further, V_2 is an independent set, from Lemma 2.1, it follows that $\sum_{i=t+1}^{k}|U_i|-\sum_{i=1}^{t}|U_i|$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_5$ with multiplicity at least $s-1$. This way we have obtained $n-k-1$ skew Laplacian eigenvalues of $\overrightarrow{H}_5$. To find the other eigenvalues, we label the vertices of V_1 first and then the vertices of V_2. Under this labelling the skew Laplacian matrix takes the form

$$\widetilde{SL}(\overrightarrow{H}_5)=\begin{pmatrix} sI_{|U_1|} & \cdots & 0_{|U_1|\times|U_t|} & 0_{|U_1|\times|U_{t+1}|} & \cdots & 0_{|U_1|\times|U_k|} & -J_{|U_1|\times s}\\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0_{|U_t|\times|U_1|} & \cdots & sI_{|U_t|} & 0_{|U_t|\times|U_{t+1}|} & \cdots & 0_{|U_t|\times|U_k|} & -J_{|U_1|\times s}\\ 0_{|U_{t+1}|\times|U_1|} & \cdots & 0_{|U_{t+1}|\times|U_t|} & -sI_{|U_{t+1}|} & \cdots & 0_{|U_{t+1}|\times|U_k|} & J_{|U_1|\times s}\\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0_{|U_k|\times|U_1|} & \cdots & 0_{|U_k|\times|U_t|} & 0_{|U_k|\times|U_{t+1}|} & \cdots & -sI_{|U_k|} & J_{|U_1|\times s}\\ J_{s\times|U_1|} & \cdots & J_{s\times|U_t|} & -J_{s\times|U_{t+1}|} & \cdots & -J_{s\times|U_k|} & B \end{pmatrix},$$

where $B=\Big(\sum_{i=t+1}^{k}|U_i|-\sum_{i=1}^{t}|U_i|\Big)I_s$.

The equitable quotient matrix of $\widetilde{SL}(\overrightarrow{H}_5)$ is

$$M=\begin{pmatrix} s & \cdots & 0 & 0 & \cdots & 0 & -s\\ 0 & \cdots & 0 & 0 & \cdots & 0 & -s\\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & \cdots & s & 0 & \cdots & 0 & -s\\ 0 & \cdots & 0 & -s & \cdots & 0 & s\\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & -s & s\\ |U_1| & \cdots & |U_t| & -|U_{t+1}| & \cdots & -|U_k| & \alpha \end{pmatrix}, \quad \text{where} \quad \alpha=\sum_{i=t+1}^{k}|U_i|\text{-}\sum_{i=1}^{t}|U_i|.$$

Let $P(x,M)=|xI_{k+1}-M|$ be the characteristic polynomial of M. Operating $C_1\to C_1+C_2+\cdots+C_{k+1}$ in $P(x,M)$ and then $C_{k+1}\to C_{k+1}-rC_1$ in the resulting determinant, it can be seen that the characteristic polynomial of M is

$$P(x,M) = x(x-s)^{t-1}\begin{vmatrix} x+s & 0 & \cdots & 0 & -2s \\ 0 & x+s & \cdots & 0 & -2s \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & x+s & -2s \\ -|U_{t+1}| & -|U_{t+2}| & \cdots & -|U_k| & x-\alpha \end{vmatrix}_{k-t}.$$

Now, evaluating along first row repeatedly, we arrive at

$$P(x,M) = x(x-s)^{t-1}(x+s)^{k-t-1}\Big[x^2-(\alpha-s)x-s\big(\alpha+2|U_k|+2|U_{k-1}|-2\sum_{i=t+1}^{k-2}|U_i|\big)\Big].$$

Since, by Theorem 2.2, any eigenvalue of M is an eigenvalue of $\widetilde{SL}(\overrightarrow{H}_5)$, the result follows for the first part.

For the second part, using the fact that [8] $tr(\widetilde{SL}^2) = -2m + \sum_{i=1}^{n}(d_i^+ - d_i^-)^2$,

$$tr(\widetilde{SL}^3) = \sum_{i=1}^{n}(d_i^+ - d_i^-)^3 + 3M_1^-(\overrightarrow{H}_6) - 3M_1^+(\overrightarrow{H}_6) - 6\Big(t^+(\overrightarrow{H}_6) - t^-(\overrightarrow{H}_6)\Big),$$

together with the Newton's identities and proceeding similarly as in the case of $\overrightarrow{H}_5$, we arrive at the result. □

The skew Laplacian spectrum of the digraphs $\overrightarrow{H}_7$ can be computed as follows.

THEOREM 2.6. *The skew Laplacian spectrum of $\overrightarrow{H}_7$ is*

$$\{\nu_1,\nu_2,\ldots\nu_{2k},(|W_i|-|W_{k+1-i}|)^{[|U_i|-1]},(|U_{k+1-i}|-|U_i|)^{[|W_i|-1]}, i=1,2,\ldots,k\},$$

where $\nu_1,\nu_2,\ldots,\nu_{2k}$ are the eigenvalues of the matrix M given by (1).

Proof. Let $V_1 = U_1\cup U_2\cup\ldots\cup U_k$ and $V_2 = W_1\cup W_2\cup\ldots\cup W_k$. Suppose that the edges are oriented so that $N^+(U_i) = W_i$, $N^-(U_i) = W_{k+1-i}$, $N^+(W_1) = U_{k+1-i}$, $N^-(W_i) = U_i$, for $i = 1,2,\ldots,k$. Since U_i, $i = 1,2,\ldots,k$, is an independent set, so from Lemma 2.1, it follows that $|N^+(U_i)| - |N^-(U_i)| = |W_i| - |W_{k+1-i}|$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_7$ with multiplicity at least $|U_i|-1$. Again, W_i is an independent set for $i = 1,2,\ldots,k$, so from Lemma 2.1, it follows that $|N^+(W_i)| - |N^-(W_i)| = |U_{k+1-i}| - |U_i|$ is a skew Laplacian eigenvalue of $\overrightarrow{H}_7$ with multiplicity at least $|W_i|-1$. This way we have obtained $n-2k$ skew Laplacian eigenvalues of $\overrightarrow{H}_7$. To find the other eigenvalues, we label the vertices of V_1 first and then the vertices of V_2. With this labelling the skew Laplacian matrix takes the form

$$\widetilde{SL}(\overrightarrow{H}_7) = \begin{pmatrix} P & Q \\ -Q^t & S \end{pmatrix},$$

where $P = diag(\delta_1 I_{\alpha_1}, \delta_2 I_{\alpha_2}, \ldots, \delta_k I_{\alpha_k})$, $\delta_i = |W_i| - |W_{k+1-i}|$, $\alpha_i = |U_i|$, $S = diag(\gamma_1 I_{\beta_1}, \gamma_2 I_{\beta_2}, \ldots, \gamma_k I_{|\beta_k})$, $\gamma_i = |U_{k+1-i}| - |U_i|$, $\beta_i = |W_i|$, for $i = 1, 2, \ldots, k$ and

$$Q = \begin{pmatrix} -J_{\alpha_1\times\beta_1} & 0_{\alpha_1\times\beta_2} & 0_{\alpha_1\times\beta_3} & \cdots & 0_{\alpha_1\times\beta_{k-1}} & J_{\alpha_1\times\beta_k} \\ 0_{\alpha_2\times\beta_1} & -J_{\alpha_2\times\beta_2} & 0_{\alpha_1\times\beta_3} & \cdots & J_{\alpha_2\times\beta_{k-1}} & 0_{\alpha_2\times\beta_k} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0_{\alpha_{k-1}\times\beta_1} & J_{\alpha_{k-1}\times\beta_2} & 0_{\alpha_{k-1}\times\beta_3} & \cdots & -J_{\alpha_{k-1}\times\beta_{k-1}} & 0_{\alpha_{k-1}\times\beta_k} \\ J_{\alpha_k\times\beta_1} & 0_{\alpha_k\times\beta_2} & 0_{\alpha_k\times\beta_3} & \cdots & 0_{\alpha_k\times\beta_{k-1}} & -J_{\alpha_k\times\beta_k} \end{pmatrix}_k.$$

The equitable quotient matrix of $\widetilde{SL}(\overrightarrow{H}_7)$ is

$$M = \begin{pmatrix} P_1 & Q_1 \\ -Q_1^t & S_1 \end{pmatrix}, \tag{1}$$

where $P_1 = diag(\delta_1, \delta_2, \ldots, \delta_k)$, $S_1 = diag(\gamma_1, \gamma_2, \ldots, \gamma_k)$ and

$$Q_1 = \begin{pmatrix} -\beta_1 & 0 & \cdots & 0 & \beta_k \\ 0 & -\beta_2 & \cdots & \beta_{k-1} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \beta_2 & \cdots & -\beta_{k-1} & 0 \\ \beta_1 & 0 & \cdots & 0 & -\beta_k \end{pmatrix}, \quad Q_1^t = \begin{pmatrix} -\alpha_1 & 0 & \cdots & 0 & \alpha_k \\ 0 & -\alpha_2 & \cdots & \alpha_{k-1} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \alpha_2 & \cdots & -\alpha_{k-1} & 0 \\ \alpha_1 & 0 & \cdots & 0 & -\alpha_k \end{pmatrix}.$$

Since, by Theorem 2.2, the eigenvalues of M are the eigenvalues of $\widetilde{SL}(\overrightarrow{H}_7)$, it follows that the remaining $2k$ eigenvalues are given by the matrix M. □

Let $\overrightarrow{G}$ be an orientation of a complete k-partite graph $K_{r_1, r_2, \ldots, r_k}$. Using the same procedure as in the above theorems, we can obtain the skew Laplacian spectrum of $\overrightarrow{G}$ for various orientations. The following observation is immediate from Theorem 2.3.

THEOREM 2.7. *The digraphs $\overrightarrow{H}_1$ and $\overrightarrow{H}_2$ are skew Laplacian integral digraph.*

The next observation follows from Theorem 2.4.

THEOREM 2.8. *The digraph $\overrightarrow{H}_3$ is skew Laplacian integral digraph, provided $(2|N^+(x_i)| - |V_2|)^2 - 4(|V_1||V_2| - |V_1|^2)$ is a perfect square. In particular, if $|V_1| = |V_2|$, then $\overrightarrow{H}_3$ is always skew Laplacian integral digraph. The digraph $\overrightarrow{H}_4$ is skew Laplacian integral digraph, provided $(2|N^+(y_i)| - |V_1|)^2 - 4(|V_1||V_2| - |V_2|^2)$ is a perfect square. In particular, if $|V_1| = |V_2|$, then $\overrightarrow{H}_4$ is always skew Laplacian integral digraph.*

Now, we have the following result which follows from Theorem 2.5.

THEOREM 2.9. *The digraph $\overrightarrow{H}_5$ is skew Laplacian integral digraph, provided $(\alpha - |V_2|)^2 - 4|V_2|(4|U_{k-1}| + 4|U_k| - |V_1|)$ is a perfect square. The digraph $\overrightarrow{H}_6$ is skew Laplacian integral digraph, provided all the zeros of the polynomial $p(x) = x^3 - ax^2 + bx - c$ with $a = 2(|U_1| - 1)(|U_4| - |U_3|)$, $b = \frac{a^2}{2} - \Big[(|U_1| - |U_2|)^2 + (|U_4| - |U_3|)^2 -$*

$(|U_1|+|U_2|)(|U_4|+|U_3|)\Big]$, $3c=|U_1||U_3|(|U_3|-|U_1|)+|U_1||U_4|(|U_1|-|U_4|)+|U_2||U_3|$ $(|U_2|-|U_3|)+|U_2||U_4|(|U_4|-|U_2|)-a\Big[2(|U_1|-|U_2|)^2+2(|U_4|-|U_3|)^2-2(|U_1|+$ $|U_2|)(|U_4|+|U_3|)-b\Big]$ *are integers.*

The next observation follows from Theorem 2.6.

THEOREM 2.10. *The digraph $\overrightarrow{H}_7$ is a skew Laplacian integral digraph, provided all the eigenvalues of the matrix M are integers.*

3. Skew Laplacian spectrum of transitive tournament

In this section, we obtain the skew Laplacian spectrum of a transitive tournament. We show by deleting a particular edge in a transitive tournament does not alter the skew Laplacian spectrum. Let K_n be a complete graph on n vertices. Any orientation of K_n is said to be a tournament. If $v_i \to v_j$ is an arc in a tournament, the vertex v_i is said to dominate the vertex v_j. For three vertices u, v and w in a tournament, if v dominates u and u dominates w implies v dominates w, for all u,v,w in the tournament, the tournament is said to be a transitive tournament. We denote a transitive tournament of order n by T_n. The following theorem determines the skew Laplacian spectrum of a transitive tournament.

THEOREM 3.1. *The skew Laplacian spectrum of a transitive tournament T_n of order n is equal to $\{\pm(n-2j) : j=1,2,3,\dots\lfloor\frac{n}{2}\rfloor\}$, or $\{0,\pm(n-2j) : j=1,2,\dots,\lfloor\frac{n}{2}\rfloor$, according as n is even or odd.*

Proof. Let T_n be a transitive tournament on n vertices having vertex set $V(T_n)=\{v_1,v_2,\dots,v_n\}$. With out loss of generality, we orient all the edges incident on v_1 in the direction away from v_1, all the edges incident at v_2 in the direction away from v_2, except the edge v_1v_2 which is already oriented, and in general all the edges incident at v_k, $2 \leqslant k \leqslant n$, in the direction away from v_k, except the edges $v_1v_k, v_2v_k, \dots, v_{k-1}v_k$ which are already oriented. If we label the rows and columns of $\widetilde{SL}(T_n)$ in the same order as in $V(T_n)$, then it can be seen that the skew Laplacian characteristic polynomial of T_n is given by

$$P_{sl}(T_n,x)=\begin{vmatrix} x-(n-1) & 1 & 1 & \cdots & 1 & 1 \\ -1 & x-(n-3) & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots & \\ -1 & -1 & -1 & \cdots & x+(n-3) & 1 \\ -1 & -1 & -1 & \cdots & -1 & x+(n-1) \end{vmatrix}.$$

Operating $C_1 \to C_1+C_2+\cdots+C_n$ and then $C_i \to C_i-C_1$, for $i=2,3,\dots,n$, we get

$$P_{sl}(T_n,x) = x\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & x-(n-2) & 0 & \cdots & 0 & 0 \\ 1 & -2 & x-(n-4) & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & -2 & -2 & \cdots & x+(n-4) & 0 \\ 1 & -2 & -2 & \cdots & -2 & x+(n-2) \end{vmatrix}.$$

It is now clear that the skew Laplacian spectrum of T_n is $\{\pm(n-2j) : j = 1,2,3,\dots\lfloor\frac{n}{2}\rfloor\}$, when n is even and equal to $\{0,\pm(n-2j) : j = 1,2,\dots,\lfloor\frac{n}{2}\rfloor$, when n is odd, completing the proof. □

Let $v_1 \to v_2 \to \cdots \to v_n$ be a Hamiltonian path. Further, for $i = 1,2,\dots,n-1$, let $e = v_iv_{i+1}$ be an arc in a transitive tournament T_n. Let $T_n - e$ be the digraph obtained by removing the arc $e = v_iv_{i+1}$ from T_n. The following result gives the skew Laplacian spectrum of digraph $T_n - e$.

THEOREM 3.2. *For digraph $T_n - e$ defined above, the skew Laplacian spectrum is equal to $\{\pm(n-2j) : j = 1,2,3,\dots\lfloor\frac{n}{2}\rfloor\}$, or $\{0,\pm(n-2j) : j = 1,2,\dots,\lfloor\frac{n}{2}\rfloor$, according as n is even or odd.*

Proof. Let T_n be a transitive tournament on n vertices having vertex set $V(T_n) = \{v_1,v_2,\cdots,v_n\}$. With out loss of generality, we orient all the edges incident on v_1 in the direction away from v_1, all the edges incident at v_2 in the direction away from v_2, except the edge v_1v_2 which is already oriented and in general all the edges incident at v_k, $2 \leqslant k \leqslant n$, in the direction away from v_k, except the edges $v_1v_k, v_2v_k, \dots, v_{k-1}v_k$ which are already oriented. Let $T_n - e$ be the digraph obtained by removing the edge $e = v_iv_{i+1}$ from T_n. With out loss of generality, suppose that $e = v_1v_2$. If we label the rows and columns of $\widetilde{SL}(T_n - e)$ in the same order as in $V(T_n)$, it can be seen that the skew Laplacian characteristic polynomial of $T_n - e$ is given by

$$P_{sl}(T_n-e,x) = \begin{vmatrix} x-(n-2) & 0 & 1 & \cdots & 1 & 1 \\ 0 & x-(n-2) & 1 & \cdots & 1 & 1 \\ -1 & -1 & x-(n-5) & \cdots & 1 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ -1 & -1 & -1 & \cdots & x+(n-3) & 1 \\ -1 & -1 & -1 & \cdots & -1 & x+(n-1) \end{vmatrix}.$$

Operating $C_1 \to C_1 + C_2 + \cdots + C_n$ and then $C_i \to C_i - C_1$, for $i = 3,4,\dots,n$, we get

$$P_{sl}(T_n,x) = x\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & x-(n-2) & 0 & \cdots & 0 & 0 \\ 1 & -2 & x-(n-4) & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & -2 & -2 & \cdots & x+(n-4) & 0 \\ 1 & -2 & -2 & \cdots & -2 & x+(n-2) \end{vmatrix}.$$

Clearly the skew Laplacian spectrum of $T_n - e$ is $\{\pm(n-2j) : j = 1,2,3,\dots\lfloor\frac{n}{2}\rfloor\}$, when n is even and equal to $\{0,\pm(n-2j) : j = 1,2,\dots,\lfloor\frac{n}{2}\rfloor$, when n is odd, completing the proof. $\square$

Theorem 3.2 shows that by deleting any arc in a Hamiltonian path of a transitive tournament T_n does not effect the skew Laplacian spectrum. So, the digraphs T_n and $T_n - e$ are always non-isomorphic skew Laplacian cospectral digraphs. Theorems 3.1 and 3.2 together imply the following result.

THEOREM 3.3. *The transitive tournament T_n and the digraph $T_n - e$ obtained from T_n by deleting an arc in a Hamiltonian path are skew Laplacian integral digraphs.*

It is clear that all the skew Laplacian eigenvalues of a transitive tournament T_n are even integers when n is even, and odd integers when n is odd. Moreover, the eigenvalues are symmetric about the origin, a property similar to the property enjoyed by the bipartite graphs with respect to the adjacency spectrum.

4. Skew Laplacian spectrum of join and complete split digraphs

In this section, we obtain the skew characteristic polynomial of the orientations of join of two graphs in terms of the skew characteristic polynomial of the component digraphs. Also, we obtain the skew Laplacian spectrum of the orientations of the complete split graph. We show the existence of some families of skew Laplacian integral digraphs. The *join (complete product)* of G_1 and G_2 is a graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all the edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2. Let $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ be orientations of G_1 and G_2 respectively. Let $\overrightarrow{G} = \overrightarrow{G_1} \to \overrightarrow{G_2}$, be the digraph obtained by taking union of digraphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ and joining each vertex v in $\overrightarrow{G_1}$ with every vertex u in $\overrightarrow{G_2}$ by an arc directed from v to u. It is clear that the underlying graph of $\overrightarrow{G}$ is the join of G_1 and G_2.

Recall that a square matrix is said to be diagonalizable if it is similar to a diagonal matrix. Since the skew Laplacian matrix $\widetilde{SL}(\overrightarrow{G})$ of a digraph is not symmetric, therefore it need not be diagonalizable. For example, the skew Laplacian matrix of the orientations of a k-matching $\overrightarrow{G} = k\overrightarrow{K}_2$ is not diagonalizable, as it is a nilpotent matrix. We call a digraph $\overrightarrow{G}$ diagonalizable if its skew Laplacian matrix is a diagonalizable matrix.

Now, we obtain the skew characteristic polynomial of the digraph $\overrightarrow{G} = \overrightarrow{G_1} \to \overrightarrow{G_2}$ in terms of the skew characteristic polynomial of the digraphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$.

THEOREM 4.1. *Let $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ be diagonalizable digraphs of order n_1 and n_2, respectively. If $\overrightarrow{G} = \overrightarrow{G_1} \to \overrightarrow{G_2}$, then*

$$P_{sl}(\overrightarrow{G},x) = \frac{x(x-n_2+n_1)}{(x+n_1)(x-n_2)} P_{sl}(\overrightarrow{G_1},x-n_2) P_{sl}(\overrightarrow{G_2},x+n_1).$$

Proof. For $i=1,2$, let $\widetilde{SL}(\overrightarrow{G_i})$ be the skew Laplacian matrix and $P_{sl}(\overrightarrow{G_i},x)$ be the skew characteristic polynomial of the digraph $\overrightarrow{G_i}$ having order n_i. Let $\overrightarrow{G}=\overrightarrow{G_1}\to\overrightarrow{G_2}$. With out loss of generality, we can label the vertices of $\overrightarrow{G}$ so that its skew Laplacian matrix can be put into the form

$$\widetilde{SL}(\overrightarrow{G})=\begin{pmatrix} n_2I_{n_1}+\widetilde{SL}(\overrightarrow{G_1}) & -J_{n_1\times n_2} \\ J_{n_2\times n_1} & \widetilde{SL}(\overrightarrow{G_2})-n_1I_{n_2}\end{pmatrix},$$

where $J_{n_1\times n_2}$ is an all one matrix.

It is well known that $e_{n_i}=(1,1,\ldots,1)^t$, the all ones vector of order n_i, is an eigenvector corresponding to eigenvalue 0 of $\widetilde{SL}(\overrightarrow{G_i})$. Let x be a vector orthogonal to e_{n_1}, satisfying $\widetilde{SL}(\overrightarrow{G_1})x=\lambda x$. Taking $X=\begin{pmatrix} x \\ 0\end{pmatrix}$ and using $-J_{n_1\times n_2}x=0$, we have $\widetilde{SL}(\overrightarrow{G})X=(n_2+\lambda)X$. This shows that $n_2+\lambda$ is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$ corresponding to the eigenvalue λ of $\widetilde{SL}(\overrightarrow{G_1})$. Let y be a vector orthogonal to e_{n_2}, satisfying $\widetilde{SL}(\overrightarrow{G_2})x=\lambda x$. Taking $Y=\begin{pmatrix} 0 \\ y\end{pmatrix}$ and using $J_{n_2\times n_1}y=0$, we have $\widetilde{SL}(\overrightarrow{G})Y=(\lambda-n_1)Y$. This shows that $\lambda-n_1$ is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$ corresponding to the eigenvalue λ of $\widetilde{SL}(\overrightarrow{G_2})$. Since the matrices $\widetilde{SL}(\overrightarrow{G_1})$ and $\widetilde{SL}(\overrightarrow{G_2})$ are diagonalizable implies that the multiplicity of the eigenvalue ρ_i of $\widetilde{SL}(\overrightarrow{G_i})$ will be the multiplicity of the eigenvalue $n_j+\rho_i$ of $\widetilde{SL}(\overrightarrow{G})$, where $1\leqslant i\neq j\leqslant 2$. Thus, in this way, we get n_1+n_2-2 eigenvalues of $\widetilde{SL}(\overrightarrow{G})$. The equitable quotient matrix of $\widetilde{SL}(\overrightarrow{G})$ is

$$M=\begin{pmatrix} n_2 & -n_2 \\ n_1 & -n_1\end{pmatrix}.$$

Since the characteristic polynomial of M is $x(x+n_1-n_2)$ and by Theorem 2.2 any eigenvalue of M is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$, the result follows. $\square$

Let $\overrightarrow{G}=\overrightarrow{G_1}\leftarrow\overrightarrow{G_2}$ be the digraph obtained by taking the union of digraphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ and joining each vertex v in $\overrightarrow{G_1}$ with every vertex u in $\overrightarrow{G_2}$ by an arc directed from u to v. Proceeding similarly as in Theorem 4.1, we arrive at the following observation.

THEOREM 4.2. *Let $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ be diagonalizable digraphs of order n_1 and n_2, respectively. If $\overrightarrow{G}=\overrightarrow{G_1}\leftarrow\overrightarrow{G_2}$, then*

$$P_{sl}(\overrightarrow{G},x)=\frac{x(x-n_1+n_2)}{(x+n_2)(x-n_1)}P_{sl}(\overrightarrow{G_1},x+n_2)P_{sl}(\overrightarrow{G_2},x-n_1).$$

Next we construct skew Laplacian integral digraphs from a given pair of skew Laplacian integral digraphs.

THEOREM 4.3. *Let $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ be diagonalizable digraphs of order n_1 and n_2, respectively. Then the digraphs $\overrightarrow{G_1}\to\overrightarrow{G_2}$ and $\overrightarrow{G_1}\leftarrow\overrightarrow{G_2}$ are skew Laplacian integral if and only if both the digraphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are skew Laplacian integral.*

Proof. If $\nu_i, 0$, for $i = 1,2,\ldots,n_1-1$, are the skew Laplacian eigenvalues of G_1, and $\xi_i, 0$, for $i = 1,2,\ldots,n_2-1$, are the skew Laplacian eigenvalues of G_2, then from Theorem 4.1, it is clear that the skew Laplacian eigenvalues of $\overrightarrow{G_1} \to \overrightarrow{G_2}$ are

$$\nu_i + n_2,\ \xi_k - n_1,\ n_2 - n_1,\ 0,\ \ i = 1,2,\ldots,n_1-1,\ k = 1,2,\ldots,n_2-1.$$

Similarly, from Theorem 4.2, the skew Laplacian eigenvalues of $\overrightarrow{G_1} \leftarrow \overrightarrow{G_2}$ are

$$\nu_i - n_2,\ \xi_k + n_1,\ n_1 - n_2,\ 0,\ \ i = 1,2,\ldots,n_1-1,\ k = 1,2,\ldots,n_2-1.$$

The result now follows. $\square$

EXAMPLE 4.4. Let T_r and T_s respectively be transitive tournaments on r and s vertices, with $r+s=n$, where both r and s are odd. Let $T_r - e_i$ and $T_s - e_j$ be the digraphs obtained by deleting the arcs $e_i = v_i v_{i+1}$ and $e_j = u_i u_{i+1}$ respectively from the Hamiltonian paths in T_r and T_s. Since for odd natural number l, the skew Laplacian eigenvalues of the transitive tournaments T_l and $T_i - e$, where e is an arc in a Hamiltonian path in T_l are distinct, it follows that their skew Laplacian matrices are diagonalizable. Consider the digraphs $\overrightarrow{G_1} = T_r \to T_s$, $\overrightarrow{G_2} = T_r \to T_s - e_j$, $\overrightarrow{G_3} = T_r - e_i \to T_s$, $\overrightarrow{G_4} = T_r - e_i \to T_s - e_j$. Using Theorems 3.1, 3.2 and 4.1, it follows that all these digraphs are skew Laplacian integral digraphs.

EXAMPLE 4.5. Let $\overrightarrow{K}_{1,r-1}$ be an orientation of a star on r vertices, when all the edges are directed away or towards the root vertex v_1 and let T_s be a transitive tournament on s vertices with $r+s=n$, where s is odd. It is clear from Theorem 2.3 that the skew Laplacian matrix of $\overrightarrow{K}_{1,r-1}$ is a diagonalizable matrix.Now, using Theorems 2.3, 3.1, 3.2 and 4.1, it follows that each of the digraphs $\overrightarrow{G_1} = \overrightarrow{K}_{1,r-1} \to T_s$, $\overrightarrow{G_2} = \overrightarrow{K}_{1,r-1} \to T_s - e_j$, $\overrightarrow{G_3} = \overrightarrow{K}_{1,r-1} \leftarrow T_s$, $\overrightarrow{G_4} = \overrightarrow{K}_{1,r-1} \leftarrow T_s - e_j$ are skew Laplacian integral digraphs.

EXAMPLE 4.6. Let T_{r_1}, T_{r_2} and T_{r_3} be transitive tournaments respectively on r_1, r_2 and r_3 vertices with $r_1+r_2+r_3=n$, where r_i is odd for $i=1,2,3$. Let $T_{r_1} - e_i$, $T_{r_2} - e_j$ and $T_{r_3} - e_k$ be the digraphs obtained by deleting the arcs $e_i = v_i v_{i+1}$, $e_j = u_i u_{i+1}$ and $e_k = w_k w_{k+1}$ from the Hamiltonian paths respectively in T_{r_1}, T_{r_2} and T_{r_3}. Consider the digraphs $\overrightarrow{G_1} = T_{r_1} \to (T_{r_2} \cup T_{r_3})$, $\overrightarrow{G_2} = T_{r_1} \to (T_{r_2} - e_j \cup T_{r_3})$, $\overrightarrow{G_3} = T_{r_1} \to (T_{r_2} \cup T_{r_3} - e_k)$, $\overrightarrow{G_4} = T_{r_1} \to (T_{r_2} - e_j \cup T_{r_3} - e_k)$, $\overrightarrow{G_5} = T_{r_1} - e_i \to (T_{r_2} - e_j \cup T_{r_3} - e_k)$, $\overrightarrow{G_6} = T_{r_1} - e_i \to (T_{r_2} \cup T_{r_3} - e_k)$, $\overrightarrow{G_7} = T_{r_1} - e_i \to (T_{r_2} - e_j \cup T_{r_3})$, $\overrightarrow{G_8} = T_{r_1} - e_i \to (T_{r_2} \cup T_{r_3})$. Using Theorems 3.1, 3.2, 4.1 and 4.2, it follows that each of these digraphs are skew Laplacian integral digraphs.

If K_r is the complete graph on r vertices and $\overline{K}_s$ is an empty graph on s vertices with $r+s=n$, the graph $C(r,s) = K_r \vee \overline{K}_s$ is called the complete split graph. The following theorem gives the skew Laplacian spectrum of some orientations of $C(r,s)$.

THEOREM 4.7. *Let $\overrightarrow{G}$ be an orientation of the complete split graph $C(r,s)$ and let $\nu_1, \nu_2, \ldots, \nu_{r-1}, 0$ be the skew Laplacian eigenvalues of K_r.*

(1). If $\overrightarrow{G}$ is obtained by orienting the edges in K_r in such a way that its skew Laplacian matrix is diagonalizable and the edges between $K_r \vee \overline{K}_s$ directed from K_r to $\overline{K}_s$, then the skew Laplacian spectrum of $\overrightarrow{G}$ is $\{\nu_i + s, (-r)^{[s-1]}, s-r, 0 : \ i = 1,2,\ldots,r-1\}$.

(2). If $\overrightarrow{G}$ is obtained by orienting the edges in K_r in such a way that its skew Laplacian matrix is diagonalizable and the edges between $K_r \vee \overline{K}_s$ directed from $\overline{K}_s$ to K_r, then the skew Laplacian spectrum of $\overrightarrow{G}$ is $\{\nu_i - s, r^{[s-1]}, r-s, 0 : \ i = 1,2,\ldots,r-1\}$.

(3). If $V(\overline{K}_s) = U_1 \cup U_2$ and $N^+(U_1) = V(K_r), N^-(U_1) = \emptyset, N^+(U_2) = \emptyset, N^-(U_2) = V(K_r)$, then the skew Laplacian spectrum of $\overrightarrow{G}$ is $\{\nu_i + (|U_2| - |U_1|), r^{[|U_1|-1]}, (-r)^{[|U_2|-1]}, 0, x_1, x_2 : i = 1,2,\ldots,r-1\}$, *where x_1, x_2 are the zeros of the polynomial* $g(x) = x^2 - (|U_2| - |U_1|)x - r^2$.

(4). If $V(\overline{K}_s) = U_1 \cup U_2 \cup U_3 \cup \cdots \cup U_k$ and $N^+(U_i) = V(K_r), N^-(U_i) = \emptyset$, for $i = 1,2,\ldots,t$ and $N^+(U_i) = \emptyset, N^-(U_i) = V(K_r)$, for $i = t+1,\ldots,k$, then the skew Laplacian spectrum of $\overrightarrow{G}$ is $\{\nu_i + \sum_{i=t+1}^{k} |U_i| - \sum_{i=1}^{t} |U_i|, r^{[\sum_{i=1}^{k} |U_i| - 1]}, (-r)^{[\sum_{i=t+1}^{k} |U_i| - 1]}, 0, x_1, x_2 : i = 1,2,\ldots,r-1\}$, *where x_1, x_2 are the zeros of the polynomial* $g(x) = x^2 - (\alpha - r)x - r\big(\alpha + 2|U_k| + 2|U_{k-1}| - 2\sum_{i=t+1}^{k-2} |U_i|\big)$, $\alpha = \sum_{i=t+1}^{k} |U_i| - \sum_{i=1}^{t} |U_i|$.

Proof. Proofs of part 1 and 2 follow from Theorems 4.1 and 4.2 and the fact that all the skew Laplacian eigenvalues of $\overline{K}_s$ are zeros.

(3). Suppose the edges in K_r be oriented in such a way that its skew Laplacian matrix is diagonalizable. Let $V(\overline{K}_s) = U_1 \cup U_2$. With out loss of generality, we orient the edges between K_r and $\overline{K}_s$ in such a way that $N^+(U_1) = V(K_r)$, $N^-(U_1) = \emptyset$, $N^+(U_2) = \emptyset, N^-(U_2) = V(K_r)$. Since U_1 is an independent set, from Lemma 2.1, it follows that $|N^+(U_1)| - |N^-(U_2)| = r$ is a skew Laplacian eigenvalue of $\overrightarrow{G}$ with multiplicity at least $|U_1| - 1$. Also, U_2 is an independent set, from Lemma 2.1, it follows that $|N^+(U_2)| - |N^-(U_2)| = -r$ is a skew Laplacian eigenvalue of $\overrightarrow{G}$ with multiplicity at least $|U_2| - 1$. To find the other eigenvalues, we label the vertices of $\overline{K}_s$ first and then the vertices of K_r. Under this labelling the skew Laplacian matrix takes the form

$$\widetilde{SL}(\overrightarrow{G}) = \begin{pmatrix} rI_{|U_1|} & 0_{|U_1|\times|U_2|} & -J_{|U_1|\times r} \\ 0_{|U_2|\times|U_1|} & -rI_{|U_2|} & J_{|U_2|\times r} \\ J_{r\times|U_1|} & -J_{r\times|U_2|} & B \end{pmatrix}, \quad \text{where} \quad B = \widetilde{SL}(\overrightarrow{K_r}) + |U_2| - |U_1|$$

Since $e_r = (1,1,\ldots,1)^t$, the all ones vector of order r is an eigenvector corresponding to eigenvalue 0 of $\widetilde{SL}(\overrightarrow{K_r})$. Let x be a vector orthogonal to e_r, satisfying $\widetilde{SL}(\overrightarrow{K_r})x = \lambda x$. Taking $X = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$ and using $-J_{|U_1|\times r}x = 0$, $J_{|U_2|\times r}x = 0$, we have

$$\widetilde{SL}(\overrightarrow{G})X = \begin{pmatrix} rI_{|U_1|} & 0_{|U_1|\times|U_2|} & -J_{|U_1|\times r} \\ 0_{|U_2|\times|U_1|} & -rI_{|U_2|} & J_{|U_2|\times r} \\ J_{r\times|U_1|} & -J_{r\times|U_2|} & B \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} = (\lambda + |U_2| - |U_1|)X.$$

This shows that $\lambda+|U_2|-|U_1|$ is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$ corresponding to the eigenvalue λ of $\widetilde{SL}(\overrightarrow{K_r})$. The equitable quotient matrix of $\widetilde{SL}(\overrightarrow{G})$ is

$$M=\begin{pmatrix} r & 0 & -r \\ 0 & -r & r \\ |U_1| & -|U_2| & |U_2|-|U_1| \end{pmatrix}.$$

Since the characteristic polynomial of M is $x(x^2-(|U_2|-|U_1|)x-r^2)$ and, by Theorem 2.2, any eigenvalue of M is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$, the result follows.

(4). Assume that the edges in K_r are oriented in such a way that its skew Laplacian matrix is diagonalizable. Let $V(\overline{K}_s)=U_1\cup U_2\cup U_3\cdots\cup U_k$. With out loss of generality, we orient the edges between K_r and $\overline{K}_s$ in such a way that $N^+(U_i)=V(K_r), N^-(U_i)=\emptyset$, for $i=1,2,\ldots,t$ and $N^+(U_i)=\emptyset, N^-(U_i)=V(K_r)$, for $i=t+1,\ldots,k$. Since U_i is an independent set for $i=1,2,\ldots,t$, from Lemma 2.1, it follows that $|N^+(U_i)|-|N^-(U_i)|=r$ is a skew Laplacian eigenvalue of $\overrightarrow{G}$ with multiplicity at least $\sum_{i=1}^{t}(|U_i|-1)$. Also, for $i=t+1,\ldots,k$, U_i is an independent set, from Lemma 2.1, it follows that $|N^+(U_i)|-|N^-(U_i)|=-r$ is a skew Laplacian eigenvalue of $\overrightarrow{G}$ with multiplicity at least $\sum_{i=t+1}^{k}(|U_i|-1)$. To find the other eigenvalues, we label the vertices of $\overline{K}_s$ first and then the vertices of K_r. Under this labelling the skew Laplacian matrix takes the form

$$\widetilde{SL}(\overrightarrow{G})=\begin{pmatrix} rI_{|U_1|} & \cdots & 0_{|U_1|\times|U_t|} & 0_{|U_1|\times|U_{t+1}|} & \cdots & 0_{|U_1|\times|U_k|} & -J_{|U_1|\times r} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0_{|U_t|\times|U_1|} & \cdots & rI_{|U_t|} & 0_{|U_t|\times|U_{t+1}|} & \cdots & 0_{|U_t|\times|U_k|} & -J_{|U_1|\times r} \\ 0_{|U_{t+1}|\times|U_1|} & \cdots & 0_{|U_{t+1}|\times|U_t|} & -rI_{|U_{t+1}|} & \cdots & 0_{|U_{t+1}|\times|U_k|} & J_{|U_1|\times r} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0_{|U_k|\times|U_1|} & \cdots & 0_{|U_k|\times|U_t|} & 0_{|U_k|\times|U_{t+1}|} & \cdots & -rI_{|U_k|} & J_{|U_1|\times r} \\ J_{r\times|U_1|} & \cdots & J_{r\times|U_t|} & -J_{r\times|U_{t+1}|} & \cdots & -J_{r\times|U_k|} & B \end{pmatrix},$$

where $B=\widetilde{SL}(\overrightarrow{K_r})+\left(\sum_{i=t+1}^{k}|U_i|-\sum_{i=1}^{t}|U_i|\right)I_r$.

Since $e_r=(1,1,\ldots,1)^t$, the all ones vector of order r is an eigenvector corresponding to eigenvalue 0 of $\widetilde{SL}(\overrightarrow{K_r})$. Let x be a vector orthogonal to e_r, satisfying $\widetilde{SL}(\overrightarrow{K_r})x=\lambda x$. Taking $X=\begin{pmatrix}0\\0\\ \vdots\\ x\end{pmatrix}$ and using $-J_{|U_i|\times r}x=0$, $J_{|U_i|\times r}x=0$, we have

$$\widetilde{SL}(\overrightarrow{G})X=(\lambda+\sum_{i=t+1}^{k}|U_i|-\sum_{i=1}^{t}|U_i|)X.$$

This shows that $\lambda+\sum_{i=t+1}^{k}|U_i|-\sum_{i=1}^{t}|U_i|$ is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$ corresponding to the eigenvalue λ of $\widetilde{SL}(\overrightarrow{K_r})$. The equitable quotient matrix of $\widetilde{SL}(\overrightarrow{G})$ is

$$M=\begin{pmatrix} r & \cdots & 0 & 0 & \cdots & 0 & -r \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -r \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & r & 0 & \cdots & 0 & -r \\ 0 & \cdots & 0 & -r & \cdots & 0 & r \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -r & r \\ |U_1| & \cdots & |U_t| & -|U_{t+1}| & \cdots & -|U_k| & \alpha \end{pmatrix}, \text{ where } \alpha=\sum_{i=t+1}^{k}|U_i|\text{-}\sum_{i=1}^{t}|U_i|.$$

Let $P(x,M)=|xI_{k+1}-M|$, be the characteristic polynomial of M. Operating $C_1\to C_1+C_2+\cdots+C_{k+1}$ in $P(x,M)$ and then $C_{k+1}\to C_{k+1}-rC_1$ in the resulting determinant, it can be seen that the characteristic polynomial of M is

$$P(x,M)=x(x-r)^{t-1}\begin{vmatrix} x+r & 0 & \cdots & 0 & -2r \\ 0 & x+r & \cdots & 0 & -2r \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & x+r & -2r \\ -|U_{t+1}| & -|U_{t+2}| & \cdots & -|U_k| & x-\alpha \end{vmatrix}_{k-t}.$$

Now, evaluating along the first row repeatedly, we obtain

$$P(x,M)=x(x-r)^{t-1}(x+r)^{k-t-1}\Big[x^2-(\alpha-r)x-r\big(\alpha+2|U_k|+2|U_{k-1}|-2\sum_{i=t+1}^{k-2}|U_i|\big)\Big].$$

Since, by Theorem 2.2, any eigenvalue of M is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$, the result follows. $\square$

Some new families of skew Laplacian integral digraphs can be obtained as under.

COROLLARY 4.8. *Let $\overrightarrow{G}$ be an orientation of the complete split graph $C(r,s)$.*

(1). Let $\overrightarrow{G}$ be obtained by orienting the edges in K_r in such a way that its skew Laplacian matrix is diagonalizable and the edges between K_r and $\overline{K}_s$, are directed from K_r to $\overline{K}_s$ or from $\overline{K}_s$ to K_r. Then $\overrightarrow{G}$ is skew Laplacian integral digraph if and only if the orientation chosen for K_r is skew Laplacian integral digraph.

(2). Let $V(\overline{K}_s)=U_1\cup U_2\cup U_3\cup\ldots\cup U_k$ with $N^+(U_i)=V(K_r), N^-(U_i)=\emptyset$, for $i=1,2,\ldots,t$ and $N^+(U_i)=\emptyset, N^-(U_i)=V(K_r)$, for $i=t+1,\ldots,k$. Then $\overrightarrow{G}$ is skew Laplacian integral digraph if and only if the orientation chosen for K_r (where edges in K_r are oriented in such a way that its skew Laplacian matrix is diagonalizable) is skew Laplacian integral digraph provided

$$\Big(\sum_{i=t+1}^{k}|U_i|-\sum_{i=1}^{t}|U_i|-r\Big)^2-4r\Big(s-4|U_{k-1}|-4|U_k|\Big)$$

is a perfect square.

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