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HARMONIC HARDY SPACE AND THEIR OPERATORS

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Abstract. Let H^2 be the Hardy space on the unit disk. For inner functions u and v, the harmonic Hardy space $H^2_{u,v}$ is defined by $H^2_{u,v} = uH^2 \oplus \overline{vzH^2}$. Assume one of u and v is a nonconstant, then $H^2_{u,v}$ is a proper closed subspace of $L^2(\partial\mathbb{D})$. We can define the Toeplitz operator on the $H^2_{u,v}$ by $\hat{T}_f x = Qfx$ for $x \in H^2_{u,v}$, where Q is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto $H^2_{u,v}$. We studied some algebraic properties of the Toeplitz operator on $H^2_{u,v}$ and obtained some interesting results that are different from the Toeplitz operators in the classical function space.

1. Harmonic Hardy spaces

Toeplitz operators on classical Hardy space H^2 on the open unit disk were widely studied such as their algebraic properties and the spectral theory. In particular on the shift-invariant subspaces in H^2 that are identified as uH^2 for some inner function u by Beurling theorem and its associated model space $K^2_u := H^2 \ominus uH^2$, it has been under investigation for more than 50 years. Many progress in this direction have been made.

In this paper, we are interested in Toeplitz operators on the newly-defined harmonic Hardy space, which is related to the model space and Hardy space. Some interesting results which is different from the classic case are obtained.

In order to state our results, we first introduce the notations and definitions. Let $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be its boundary. The Hardy space H^2 is the Hilbert space consisting of analytic functions in $\mathbb{D}$ that are also square-integrable on the boundary $\partial\mathbb{D}$.

DEFINITION 1.1. Let u and v be inner functions, that at least one of them is not a constant, we define the harmonic Hardy space $H^2_{u,v}$ by

$$H^2_{u,v} = uH^2 \oplus \bar{v}(H^2)^{\perp} = uH^2 \oplus \overline{vzH^2}. \tag{1.1}$$

$H^2_{u,v}$ is a Hilbert space of harmonic funcitons with the inner product

$$\langle F, G\rangle = \int_{\partial\mathbb{D}} F(z)\overline{G(z)}\,dm(z).$$

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It is easy to check that $H^2_{u,v}$ is a proper closed subspace of $L^2(\partial\mathbb{D})$ and $\{uz^m, \bar{v}\bar{z}^n : m,n \in \mathbb{Z}, m \geqslant 0, n \geqslant 1\}$ is an orthonormal basis of $H^2_{u,v}$. For $z \in \mathbb{D}$,

$$\mathbf{R}_z(w) = \bar{u}(z)u(w)\mathbf{K}_z(w) + v(z)z\bar{v}(w)\overline{w\mathbf{K}_z(w)}$$

is the reproducing kernel of $H^2_{u,v}$, where $\mathbf{K}_z(=\frac{1}{1-w\bar{z}})$ is Szegö kernel.

The study of $H^2_{u,v}$ is motivated by D.Sarason's result. Recall that for each nonconstant inner function u, the model space is defined as $K^2_u := H^2 \ominus uH^2$, and for $f \in L^2$, the truncated Toeplitz operator on K^2_u with symbol f is the operator A^u_f densely defined by

$$A^u_f x = P_u(fx), \text{ for } x \in K^2_u.$$

where P_u is the orthogonal projection from $L^2(\partial\mathbb{D})$ to K^2_u. Such subspaces are useful for modeling a large class of contraction operators [16, 15], which have attracted a lot of attention in recent years. D.Sarason [15, Theorem 3.1] showed that the truncated Toeplitz operator $A^u_f = 0$ if and only if $f \in H^2_{u,u} = uH^2 \oplus \bar{u}[H^2]^\perp$. Later, J. Jurasik and B.Łanucha [13] proved that asymmetric truncated Toeplitz operator $A^{v,u}_f(K^2_v \to K^2_u)$ is a zero operator if and only if $f \in H^2_{u,v} = uH^2 \oplus \bar{v}[H^2]^\perp$. We should state that

$$(H^2_{u,v})^\perp = K^2_u \oplus \overline{zK^2_v} \tag{1.2}$$

is also a Hilbert space of harmonic functions. Especially, $(H^2_{u,1})^\perp = K^2_u$ is the model space. In fact, $K^2_u \subseteq H^2 \subseteq H^2_{1,u} \subseteq L^2$.

Now we can define Toeplitz operators on $H^2_{u,v}$. Let P be the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2 and $P_- = I - P$ be the orthogonal projection from $L^2(\partial\mathbb{D})$ onto $(H^2)^\perp$. Denote M_u and $M_{\bar{u}}$ be the multiplication operators on $L^2(\partial\mathbb{D})$ induced by u and $\bar{u}$. Then direct calculation shows that

$$Q = Q_{u,v} := M_u P M_{\bar{u}} + M_{\bar{v}} P_- M_v \tag{1.3}$$

is the orthogonal projection form $L^2(\partial\mathbb{D})$ onto $H^2_{u,v}$.

DEFINITION 1.2. For f in $L^2(\partial\mathbb{D}), x \in H^2_{u,v}$, the harmonic Toeplitz operator $\widehat{T}_f$ with the symbol f is densely defined on $H^2_{u,v}$, given by

$$\widehat{T}_f x = Q(fx) = \int_{\partial\mathbb{D}} f(\xi)x(\xi)\overline{\mathbf{R}_z} dm(\xi).$$

$\widehat{T}_f$ is an integral operator.

Here we mainly study some algebraic and spectral properties of the harmonic Toeplitz operators. We show that as on a harmonic function space with an asymmetric structure, $\widehat{T}_f$ is differ in many ways from Toeplitz operators on Hardy space [9], harmonic Bergman space [4, 5] and the harmonic Dirichlet space[17], while shares some properties as same.

The paper is organised as following. Section 2 is the preliminary that will be used. In section 3 and section 4, we study fundamental properties of harmonic Toeplitz operators and dual harmonic Toeplitz operators. Boundedness and compactness of harmonic Toeplitz operators are similar to Toeplitz operator on Hardy space, but there exist the essential spectrum of a harmonic Toeplitz operators is unconnected. In section 5, we characterize zero product, semi-commutator of harmonic Toeplitz operators. In section 6, we discuss the finite rank perturbations of semi-commutator of harmonic Toeplitz operators.

2. Preliminary

For f in $L^2(\partial\mathbb{D})$, the standard Toeplitz operator with symbol f is the operator T_f on H^2 defined by

$$T_f x = P(fx), \text{ for } x \in H^2.$$

The dual Toeplitz operator S_f, on the orthogonal complement of H^2 would be defined as follow:

$$S_f y = (I-P)(fy), \text{ for } y \in [H^2]^{\perp}.$$

Define operator V on $L^2(\partial\mathbb{D})$ by

$$Vf(w) = \overline{w}\overline{f(w)}.$$

It is easy to check that V is anti-unitary, and also satisfies the following equations

$$V = V^{-1}, \;\; VP = P_{-}V, \;\; VH^2_{u,v} = H^2_{v,u} \tag{2.1}$$

T_f and $S_{\overline{f}}$ is Anti-unitary equivalent [12]. That is

$$VT_f = S_{\overline{f}}V. \tag{2.2}$$

The Hankel operator H_f with symbol f are defined by

$$H_f x = (I-P)(fx), \text{ for } x \in H^2,$$

and H_f^* are defined by

$$H_f^* y = P(\overline{f}y), \text{ for } y \in [H^2]^{\perp}.$$

Write M_f for the multiplication operator defined on L^2 by $M_f\phi = f\phi$. If M_f is expressed as an operator matrix with respect to the decomposition $L^2 = H^2 \oplus \overline{zH^2}$, the result is of the form

$$M_f = \begin{pmatrix} T_f & H^*_{\overline{f}} \\ H_f & S_f \end{pmatrix}.$$

Since $M_f M_g = M_{fg}$, we have

$$T_{fg} = T_f T_g + H^*_{\overline{f}} H_g; \tag{2.3}$$

$$H_{fg} = H_f T_g + S_f H_g; \tag{2.4}$$

$$H^*_{\overline{fg}} = T_f H^*_{\overline{g}} + H^*_{\overline{f}} S_g; \tag{2.5}$$

$$S_{fg} = S_f S_g + H_f H^*_{\overline{g}}. \tag{2.6}$$

The M_f is expressed as an operator matrix with respect to the decomposition $L^2 = H^2_{u,v} \oplus [H^2_{u,v}]^\perp$, the result is of the form

$$M_f = \begin{pmatrix} \widehat{T}_f & \widehat{H}^*_{\overline{f}} \\ \widehat{H}_f & \widehat{S}_f \end{pmatrix}.$$

where $\widehat{H}_f$ be called harmonic Hankel operator is defined by

$$\widehat{H}_f x = (I-Q)(fx), \text{ for } x \in H^2_{u,v}.$$

Moreover, $\widehat{H}^*_{f} y = Q\overline{f}y$ for $y \in [H^2_{u,v}]^\perp$ and $\widehat{S}_f$ define dual harmonic Toeplitz operator by

$$\widehat{S}_f x = (I-Q)(fx), \text{ for } x \in [H^2_{u,v}]^\perp.$$

In particular, the dual harmonic Toeplitz operator $\widehat{S}_f$ on the $[H^2_{u,1}]^\perp$ is the truncated Toeplitz operator A_f on the K^2_u.

Since $M_f M_g = M_{fg}$, we have

$$\widehat{T}_{fg} - \widehat{T}_f \widehat{T}_g = \widehat{H}^*_{\overline{f}} \widehat{H}_g, \tag{2.7}$$

$$\widehat{S}_{fg} - \widehat{S}_f \widehat{S}_g = \widehat{H}_f \widehat{H}^*_{\overline{g}}, \tag{2.8}$$

$$\widehat{H}_{fg} - \widehat{S}_f \widehat{H}_g = \widehat{H}_f \widehat{T}_g. \tag{2.9}$$

3. Fundamental properties of harmonic Toeplitz operators

It is known that T_f is bounded if and only if f is in $L^\infty(\partial\mathbb{D})$, in which case $\|T_f\| = \|f\|_\infty$. The only compact Toeplitz operator is zero. These still hold for harmonic Toeplitz operators.

THEOREM 3.1. *Let $f \in L^\infty(\partial\mathbb{D})$.*

1. *Let $\phi \in L^2(\partial\mathbb{D})$, then $\widehat{T}_\phi$ is bounded on $H^2_{u,v}$ if and only if $\phi \in L^\infty(\partial\mathbb{D})$. If $\widehat{T}_\phi$ is bounded, then $\|\widehat{T}_\phi\| = \|\phi\|_\infty$;*

2. *$\widehat{T}_f$ is compact on $H^2_{u,v}$ if and only if $f = 0$ a.e. on $\partial\mathbb{D}$;*

3. $\mathscr{R}(f) \subseteq \sigma(\widehat{T}_f)$. *Where* $\sigma(T)$ *denotes the spectrum of a bounded linear operator* T *on a Hilbert space* H, $\mathscr{R}(f)$ *is the essential range of* f.

Proof. (1) $\mathbf{k}_z = \frac{\sqrt{1-|z|^2}}{(1-w\bar{z})}$ is the normalized reproducing kernel for H^2. We have

$$\begin{aligned}\|\widehat{T}_f\| &\geqslant |\langle \widehat{T}_f u\mathbf{k}_z, u\mathbf{k}_z\rangle| \\ &= |\langle f u\mathbf{k}_z, u\mathbf{k}_z\rangle| \\ &= |\langle f\mathbf{k}_z, \mathbf{k}_z\rangle| = |f(z)|.\end{aligned}$$

Hence,

$$\|\widehat{T}_f\| \geqslant \|f\|_\infty.$$

Furthermore,

$$\|\widehat{T}_f x\| \leqslant \|Qfx\| \leqslant \|f\|_\infty \|x\|.$$

(2) Assume $\widehat{T}_f$ is compact, then for any $\{x_n\} \subseteq H^2_{u,v}$ and $\{x_n\}$ is weakly convergent to zero, then $\|\widehat{T}_f x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. If u is a non-constant inner function, let $x_n = uy_n$ and $y_n \in H^2$. It is easy to see that $\{x_n\}$ weakly convergent to zero on uH^2 if and only if $\{y_n\}$ weakly converges to zero on H^2. Note that $\|T_f y_n\| = \|M_u P M_{\bar{u}} f x_n\| \leqslant \|\widehat{T}_f x_n\|$. Thus $\widehat{T}_f$ is compact imply that T_f is compact, by [9, 7.15], we have $f = 0$.

(3) According to [9, corollary 4.24], where $\sigma(M_f) = \mathscr{R}(f)$, which that f is invertible in L^∞ equivalent to M_f is an invertible operator on L^2. If $\widehat{T}_f$ is invertible, then there exists $\varepsilon > 0$ such that $\|\widehat{T}_f x\| \geqslant \varepsilon\|x\|$ for $x \in H^2_{u,v}$. Thus for each integer $k, y \in H^2$ and $z \in \partial\mathbb{D}$, we have

$$\|M_f z^k y\| = \|f z^k y\| = \|fuy\| \geqslant \|Qfuy\| = \|\widehat{T}_f uy\| \geqslant \varepsilon\|uy\| = \varepsilon\|z^k y\|.$$

Since $\{z^k y : y \in H^2, \text{k takes all integers}\}$ is a dense subset of L^2, it follows that $\|M_f x\| \geqslant \varepsilon\|x\|$ for x in L^2. Similarly, $\|M_{\bar{f}} x\| \geqslant \varepsilon\|x\|$, since $\widehat{T}_{\bar{f}} = \widehat{T}_f^*$ is also invertible and thus M_f is invertible by [9, Corollary 4.9]. Since $\widehat{T}_f - \lambda = \widehat{T}_{f-\lambda}$ for $\lambda \in \mathbb{C}$. The proof is completed. $\square$

The following theorem is unlike classical Toeplitz operator T_f.

THEOREM 3.2. *Assume that* u *and* v *are both finite Blaschke products, for* $f \in L^\infty$, *then*

$$\sigma_e(\widehat{T}_f) = \sigma_e(M_f) = \mathscr{R}(f).$$

where $\sigma_e(T)$ *denotes the essential spectrum of a bounded linear operator* T *on a Hilbert space* H.

Proof. Since u and v are finite Blaschke products, the dimension of $[H^2_{u,v}]^\perp = K^2_u \oplus \overline{zK^2_v}$ is finite. On $L^2 = H^2_{u,v} \oplus [H^2_{u,v}]^\perp$,

$$M_f = \begin{pmatrix} \widehat{T}_f & \widehat{H}^*_{\bar{f}} \\ \widehat{H}_f & \widehat{S}_f \end{pmatrix},$$

$\widehat{H}^*_{\bar{f}}, \widehat{H}_f$ and $\widehat{S}_f$ are finite rank operators. $\sigma_e(M_f) = \sigma_e(\widehat{T}_f)$.

By [1, Theorem 2.1.4], $\sigma(M_f) = \mathscr{R}(f)$. Since $\sigma_e(M_f) \subset \sigma(M_f)$. We only need to prove that if M_f is a Fredholm operator then M_f is invertible on $L^2(\partial\mathbb{D})$. In fact, assume M_f is a Fredholm operator, then $dimKer\{M_f\} < \infty$. Easy to see that if $g|_E = 0$ with $mE > 0$, then $dimKer\{M_g\} = \infty$. Thus $f \neq 0$ a.e. on $\partial\mathbb{D}$ and $Ker\{M_f\} = \{0\}$. Also M_f has closed range, follows M_f is bounded below. Similarly, $M_f^* = M_{\bar{f}}$ is also bounded below. So M_f is invertible. $\square$

In fact,

$$\begin{aligned} I - Q &= P - M_u P M_{\bar{u}} + P_- - M_{\bar{v}} P_- M_v \\ &= P - M_u P M_{\bar{u}} + M_{\bar{v}}(P - M_v P M_{\bar{v}}) M_v \end{aligned}$$

Since $(H^2_{u,v})^\perp = K_u^2 \oplus \overline{zK_v^2}$, $P - M_u P M_{\bar{u}}$ and $M_{\bar{v}}(P - M_v P M_{\bar{v}})M_v$ is the orthogonal projection form $L^2(\partial\mathbb{D})$ onto K_u^2 and $\overline{zK_v^2}$ respectively. In addition, $K_v^2 = H^2 \cap v\overline{zH^2}$, hence $\overline{zK_v^2} = \bar{v}K_v^2$.

THEOREM 3.3. *Let $f \in L^\infty(\partial\mathbb{D})$. $\widehat{S}_f = 0$ if and only if $f \in H^2_{uv,uv}$.*

Proof. For $x \in K_u^2$, we have

$$\begin{aligned} \widehat{S}_f x &= \{P - M_u P M_{\bar{u}} + M_{\bar{v}}(P - M_v P M_{\bar{v}}) M_v\} f x \\ &= (P - M_u P M_{\bar{u}}) f x + M_{\bar{v}}(P - M_v P M_{\bar{v}}) M_v f x \\ &= A_f^u x + M_{\bar{v}} A_{vf}^{u,v} x. \end{aligned}$$

For $y \in K_v^2$, we have

$$\begin{aligned} \widehat{S}_f \bar{v} y &= \{P - M_u P M_{\bar{u}} + M_{\bar{v}}(P - M_v P M_{\bar{v}}) M_v\} f \bar{v} y \\ &= (P - M_u P M_{\bar{u}}) f \bar{v} y + M_{\bar{v}}(P - M_v P M_{\bar{v}}) M_v f \bar{v} y \\ &= A_{\bar{v}f}^{v,u} y + M_{\bar{v}} A_f^v y. \end{aligned}$$

Hence,

$$\begin{aligned} \widehat{S}_f \begin{pmatrix} x \\ \bar{v}y \end{pmatrix} &= \widehat{S}_f \begin{pmatrix} I & 0 \\ 0 & M_{\bar{v}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & M_{\bar{v}} \end{pmatrix} \begin{pmatrix} A_f^u & A_{\bar{v}f}^{v,u} \\ A_{vf}^{u,v} & A_f^v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

and

$$\begin{pmatrix} I & 0 \\ 0 & M_v \end{pmatrix} \widehat{S}_f \begin{pmatrix} I & 0 \\ 0 & M_{\bar{v}} \end{pmatrix} = \begin{pmatrix} A_f^u & A_{\bar{v}f}^{v,u} \\ A_{vf}^{u,v} & A_f^v \end{pmatrix},$$

where $M_{\bar{v}}$ is a unitary operator maps K_v^2 to $\bar{v}K_v^2$. Thus $\widehat{S}_f = 0$ if and only if $A_f^u, A_{\bar{v}f}^{v,u}, A_{vf}^{u,v}$ and A_f^v are all zero operator.

By [15, Theorem 3.1] and [13, Theorem 2.1], we have

1. $A_f^u = 0$ if and only if $f \in uH^2 \oplus \overline{uzH^2}$;

2. $A_f^v = 0$ if and only if $f \in vH^2 \oplus \overline{vzH^2}$;

3. $A_{\bar{v}f}^{v,u} = 0$ if and only if $f \in uvH^2 \oplus \overline{zH^2}$;

4. $A_{vf}^{u,v} = 0$ if and only if $f \in H^2 \oplus \overline{uvzH^2}$.

Since $uvH^2 \subset uH^2 \subset H^2$ and $uvH^2 \subset vH^2 \subset H^2$, $\widehat{S}_f = 0$ if and only if $f \in uvH^2 \oplus \overline{uvzH^2} = H^2_{uv,uv}$. □

4. Examples and Questions

The harmonic Toeplitz operator is a new type of Toeplitz operator with many properties that are completely different from the classical Toeplitz operator.

EXAMPLE 4.1. Let u and v be finite Blaschke products, f is a characteristic function of set $E \subseteq \partial\mathbb{D}$ $(0 < mE < 1)$, so $\sigma_e(\widehat{T}_f) = \mathscr{R}(f) = \{0,1\}$ by Theorem 3.2, and $\sigma_e(\widehat{T}_f)$ disconnected. For classic Toeplitz operators, if $f \in L^\infty$, then $\sigma_e(T_f)$ is connected (see [9, Theorem.7.45]).

EXAMPLE 4.2. On the classical Hardy space, by Coburn's Theorem [9, Proposition.7.24], if f is a function in $L^\infty(\partial\mathbb{D})$ not almost everywhere zero, then either $\ker T_f = \{0\}$ or $\ker T_f^* = \{0\}$. On $H^2_{u,v}$, there exists a nonzero function f in L^∞, such that $\ker \widehat{T}_f \neq \{0\}$ and $\ker \widehat{T}_f^* \neq \{0\}$. Let $f = u$, and take $h \in K_u^2 (= \ker T_{\bar{u}})$, we have

$$\begin{aligned}
\widehat{T}_u \bar{v}\,\bar{z}\bar{h} &= [M_u P M_{\bar{u}} + M_{\bar{v}} P_- M_v] u \bar{v}\,\bar{z}\bar{h} \\
&= M_u P \bar{v}\,\bar{z}\bar{h} + M_{\bar{v}} P_- u\,\bar{z}\bar{h} \\
&= 0 + M_{\bar{v}} V P V u\,\bar{z}\bar{h} \\
&= M_{\bar{v}} V P \bar{u} h \\
&= M_{\bar{v}} V T_{\bar{u}} h = 0.
\end{aligned}$$

and

$$\begin{aligned}
\widehat{T}_u^* uh = \widehat{T}_{\bar{u}} uh &= [M_u P M_{\bar{u}} + M_{\bar{v}} P_- M_v] \bar{u} uh \\
&= M_u T_{\bar{u}} h + M_{\bar{v}} P_- vh = 0.
\end{aligned}$$

EXAMPLE 4.3. If u and v are finite Blaschke products, $(H^2_{u,v})^\perp = K_u^2 \oplus \bar{z}\overline{K_v^2}$ is a finite dimensional space, $\widehat{H}_{\bar{f}}^*$ has finite rank. By (2.7), $\widehat{T}_{fg} - \widehat{T}_f \widehat{T}_g$ has finite rank. Since Theorem 3.1(2), $\widehat{T}_f \widehat{T}_g$ has finite rank if and only if $fg = 0$. Thus $\widehat{T}_f \widehat{T}_g$ is a finite rank operator, can't implies that either f or g is a zero function.

On classical Hardy space, zero product question for Toeplitz operators is very interesting. Naturally, we have following question.

QUESTION 4.4. Assume $f,g \in L^\infty$ and $\widehat{T}_f\widehat{T}_g = 0$, whether f or g is a zero function?

EXAMPLE 4.5. By famous Brown-Halmos Theorem [3, Theorem 8], if $\overline{f}$ or g is analytic, then $T_fT_g = T_{fg}$. But the result does not hold for harmonic Toeplitz operators. Assume $u = v\theta$, θ is an inner function and $\theta(0) = 0$. $\psi = uv, \varphi \in zH^2 \cap H^\infty$, ψ and φ both are analytic, but

$$\begin{aligned}\widehat{T}_\varphi\widehat{T}_\psi \overline{z}\,\overline{v} &= \widehat{T}_\varphi Q\psi\overline{z}\,\overline{v} = \widehat{T}_\varphi Q\overline{z}u\\ &= \widehat{T}_\varphi[M_uPM_{\overline{u}} + M_{\overline{v}}P_-M_v]\overline{z}u\\ &= \widehat{T}_\varphi[uP\overline{z} + \overline{v}P_-v\overline{z}u]\\ &= \widehat{T}_\varphi[0+0] = 0\end{aligned}$$

and

$$\begin{aligned}\widehat{T}_{\psi\varphi}\overline{z}\,\overline{v} &= Q\psi\varphi\overline{z}\,\overline{v} = Quv\varphi\overline{z}\,\overline{v} = Qu\varphi\overline{z} = Qv\varphi\theta\overline{z}\\ &= M_uPM_{\overline{u}}v\varphi\theta\overline{z} + M_{\overline{v}}P_-M_vv\varphi\theta\overline{z}.\end{aligned}$$

Note that $\theta\overline{z} \in H^2, M_{\overline{v}}P_-M_vv\varphi\theta\overline{z} = 0$,

$$\widehat{T}_{\psi\varphi}\overline{z}\,\overline{v} = M_uPM_{\overline{u}}v\varphi\theta\overline{z} = u\overline{z}\varphi \neq 0.$$

Hence

$$\widehat{T}_\varphi\widehat{T}_\psi \neq \widehat{T}_{\psi\varphi}.$$

Naturally, we can ask the following question.

QUESTION 4.6. For which functions f and g, we have $\widehat{T}_f\widehat{T}_g = \widehat{T}_{fg}$?

EXAMPLE 4.7. It is known that $I - T_zT_{\overline{z}} = 1 \otimes 1$ on Hardy space. On harmonic Hardy space, an easy computation gives

$$\begin{aligned}I - \widehat{T}_{\overline{z}}\widehat{T}_z &= (1 - |u(0)v(0)|^2)(\overline{z}\,\overline{v} \otimes \overline{z}\,\overline{v}),\\ I - \widehat{T}_z\widehat{T}_{\overline{z}} &= (1 - |u(0)v(0)|^2)(u \otimes u).\end{aligned}$$

In addition, we are concerned with the following question.

QUESTION 4.8. For which functions f and g, $\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg}$ is a finite rank operator?

5. The product of harmonic Toeplitz operators

LEMMA 5.1. *Let $f_1, f_2, \cdots, f_n$ belong to $L^\infty(\partial\mathbb{D})$, such that $\prod_{i=1}^n \widehat{T}_{f_i} - \widehat{T}_h$ is compact, then $\prod_{i=1}^n f_i = h$ a.e on $\partial\mathbb{D}$.*

Proof. First, we will prove the following formula by induction.

$$\lim_{r\to 1^-} \left\langle \prod_{i=1}^n \widehat{T}_{f_i} u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \right\rangle = \prod_{i=1}^n f_i(\xi) \tag{5.1}$$

for almost all $\xi \in \partial\mathbb{D}$, where $\mathbf{k}_z(w) = \frac{(1-|z|^2)^{\frac{1}{2}}}{(1-w\bar{z})}$ is the normalized Hardy reproducing kernel. For $k = 1$, we have

$$\begin{aligned}
\lim_{r\to 1^-} \langle \widehat{T}_{f_1} u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle &= \lim_{r\to 1^-} \langle f_1 u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle \\
&= \lim_{r\to 1^-} \langle f_1 \mathbf{k}_{r\xi}, \mathbf{k}_{r\xi} \rangle \\
&= \lim_{r\to 1^-} \int_{\partial\mathbb{D}} f_1(\zeta) |\mathbf{k}_{r\xi}|^2 dm(\zeta) \\
&= f_1(\xi)
\end{aligned}$$

for almost all $\xi \in \partial\mathbb{D}$. Assume the result true up to $n-1$. Observe that

$$\begin{aligned}
\langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} \widehat{T}_{f_n} u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle =& \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} [M_u P M_{\bar{u}} + M_{\bar{v}} P_- M_v] f_n u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle \\
=& \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} M_{\bar{v}} P_- M_v f_n u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle \\
&+ \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} M_u P M_{\bar{u}} f_n u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle \\
=& \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} M_{\bar{v}} H_{f_n vu} \mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle \\
&+ \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} u P f_n \mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle.
\end{aligned}$$

Also

$$|\langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} M_{\bar{v}} H_{f_n vu} \mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle| \leqslant \|\widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}}\| \|H_{f_n vu} \mathbf{k}_{r\xi}\|.$$

By [10, (B5)], we have $\|H_{f_n vu} \mathbf{k}_{r\xi}\| \to 0$ radially. Hence

$$\lim_{r\to 1^-} |\langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} M_{\bar{v}} H_{f_n vu} \mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle| = 0.$$

On the other hand,

$$\begin{aligned}
\langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} u P f_n \mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle =& \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} u P f_{n+} \mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle \\
&+ \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} u P f_{n-} \mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle \\
=& \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1} f_{n+}} u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle \\
&+ f_{n-}(r\xi) \langle \widehat{T}_{f_1} \cdots \widehat{T}_{f_{n-1}} u P \mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi} \rangle
\end{aligned}$$

where $f_{n+} = Pf_n$, $f_{n-} = P_- f_n$, by induction hypothesis, (5.1) holds. For every $f = uh + \bar{v}\bar{g} \in H^2_{u,v}, h \in H^2, g \in zH^2$, we have

$$\begin{aligned}\langle u\mathbf{k}_{r\xi}, uh + \bar{v}\bar{g}\rangle &= \langle u\mathbf{k}_{r\xi}, uh\rangle \\ &= \langle \mathbf{k}_{r\xi}, h\rangle.\end{aligned}$$

Since $\mathbf{k}_{r\xi}$ converges weakly to zero as $r \to 1^-$, $u\mathbf{k}_{r\xi}$ converges weakly to zero as $r \to 1^-$, Recall that compact operators map weakly convergent sequences to norm-convergent sequences. Note that the Cauchy-Schwarz inequality yields

$$|\langle (\prod_{i=1}^{n} \widehat{T}_{f_i} - \widehat{T}_h) u\mathbf{k}_{r\xi}, u\mathbf{k}_{r\xi}\rangle| \leqslant \|(\prod_{i=1}^{n} \widehat{T}_{f_i} - \widehat{T}_h) u\mathbf{k}_{r\xi}\|.$$

Hence $\prod_{i=1}^{n} f_i = h$ a.e on $\partial\mathbb{D}$. $\square$

LEMMA 5.2. [8, Lemma 3.5] *Let θ be a noncostant inner function. On the Hardy space H^2, for $f, g \in L^\infty(\partial\mathbb{D})$, if $T_f T_g = T_{\overline{\theta} f} T_{\theta g}$, then either $\overline{f}$ or g is analytic on $\mathbb{D}$.*

THEOREM 5.3. *Let f and g are non-constant functions in $L^\infty(\partial\mathbb{D})$. Assume $\widehat{T}_f \widehat{T}_g = \widehat{T}_{fg}$, then one and only one of the following possibilities occurs:*

1. *both f and g are analytic;*
2. *both f and g are co-analytic.*

Proof. Assume $\widehat{T}_f \widehat{T}_g = \widehat{T}_{fg}$, for every $x \in H^2_{u,v}$, we have $\widehat{T}_f \widehat{T}_g x = \widehat{T}_{fg} x$. Take $x = uh, h \in H^2$,

$$\begin{aligned} M_u P M_{\overline{u}} f \widehat{T}_g uh &= M_u P M_{\overline{u}} f [M_u P M_{\overline{u}} + M_{\overline{v}} P_- M_v] guh \\ &= M_u P f P g h + M_u P M_{\overline{u}} f M_{\overline{v}} P_- M_v guh \\ &= M_u P f P g h + M_u P \overline{u} f \overline{v} (I - P) gvuh \\ &= M_u P f P g h + M_u P \overline{u} f \overline{v} gvuh - M_u P \overline{u} f \overline{v} P gvuh \\ &= M_u T_f T_g h + M_u P f g h - M_u T_{\overline{u}\overline{v} f} T_{vug} h, \end{aligned} \tag{5.2}$$

and

$$M_u P M_{\overline{u}} f \widehat{T}_g uh = M_u P f g h. \tag{5.3}$$

Thus,

$$T_f T_g = T_{f\overline{v}\overline{u}} T_{gvu}. \tag{5.4}$$

If $x=\overline{v}\overline{h}, h\in zH^2$, we have

$$\begin{aligned} M_{\overline{v}}P_{-}M_{v}f\widehat{T}_{g}\overline{v}\overline{h} &= M_{\overline{v}}P_{-}M_{v}f[M_{u}PM_{\overline{u}}+M_{\overline{v}}P_{-}M_{v}]g\overline{v}\overline{h} \\ &= M_{\overline{v}}P_{-}M_{v}fM_{u}PM_{\overline{u}}g\overline{v}\overline{h}+M_{\overline{v}}P_{-}M_{v}fM_{\overline{v}}P_{-}M_{v}g\overline{v}\overline{h} \\ &= M_{\overline{v}}P_{-}vufP\overline{v}\overline{u}g\overline{h}+M_{\overline{v}}P_{-}fP_{-}g\overline{h} \\ &= M_{\overline{v}}H_{vuf}H^{*}_{vu\overline{g}}\overline{h}+M_{\overline{v}}S_{f}S_{g}\overline{h}, \end{aligned} \tag{5.5}$$

and

$$M_{\overline{v}}P_{-}M_{v}fg\overline{v}\overline{h}=M_{\overline{v}}P_{-}fg\overline{h}=M_{\overline{v}}S_{fg}\overline{h}. \tag{5.6}$$

That means,

$$\begin{aligned} H_{vuf}H^{*}_{vu\overline{g}}+S_{f}S_{g} &= S_{fg}; \\ V(H_{vuf}H^{*}_{vu\overline{g}}+S_{f}S_{g})V &= VS_{fg}V; \\ H^{*}_{vuf}H_{vu\overline{g}}+T_{\overline{f}}T_{\overline{g}} &= T_{\overline{f}\overline{g}}; \\ T_{\overline{v}\overline{u}\overline{f}vu\overline{g}}-T_{\overline{v}\overline{u}\overline{f}}T_{vu\overline{g}}+T_{\overline{f}}T_{\overline{g}} &= T_{\overline{f}\overline{g}}. \end{aligned}$$

Hence

$$T_{\overline{v}\overline{u}\overline{f}}T_{vu\overline{g}}=T_{\overline{f}}T_{\overline{g}}. \tag{5.7}$$

By using the Lemma 5.2 to $T_{f}T_{g}=T_{f\overline{v}\overline{u}}T_{gvu}$ and $T_{\overline{v}\overline{u}\overline{f}}T_{vu\overline{g}}=T_{\overline{f}}T_{\overline{g}}$, we have eithor both f and g are analytic or both f and g are co-analytic. $\square$

Now, we can affirmatively answer Question 4.4.

COROLLARY 5.4. *Assume $f,g\in L^{\infty}$, if $\widehat{T}_{f}\widehat{T}_{g}=0$, then either f or g is a zero function.*

Proof. If $\widehat{T}_{f}\widehat{T}_{g}=0$, then $fg=0$ a.e.on $\partial\mathbb{D}$ by Lemma 5.1. If one of f and g is constant, obviously either f or g is zero. If f and g aren't constant, then f and g are both analytic, or f and g are both co-analytic, this implies that either f or g is zero function. $\square$

COROLLARY 5.5. *For $f\in L^{\infty}$, then $\widehat{T}_{f}$ is a isometry if and only if f is unimodular constant.*

Proof. Assume $\widehat{T}_{f}$ is a isometry, then $\widehat{T}^{*}_{f}\widehat{T}_{f}=\widehat{T}_{\overline{f}}\widehat{T}_{f}=I=\widehat{T}_{1}$. Thus $\overline{f}$ and f b are both analytic by Theorem 5.3. Hence f is a constant and $|f|^2=\overline{f}f=1$. $\square$

COROLLARY 5.6. *For $f\in L^{\infty}$, if $\widehat{T}_{f}$ is a multiplication operators on $H^{2}_{u,v}$, then f is a constant.*

Proof. Assume $\widehat{T}_f$ is a multiplication operators on $H^2_{u,v}$, then $\widehat{T}_{\bar{f}}\widehat{T}_f = \widehat{T}_{\bar{f}f}$. It follows that $\bar{f}$ and f are both analytic by Theorem 5.3. Hence f is a constant. □

In order to answer Question 4.6, assume u and v are nonzero functions. We only need to consider the case that f and g both are analytic.

LEMMA 5.7. *If $f,g \in H^\infty$, the following statements are equivalent.*

1. $\widehat{T}_{\bar{g}}\widehat{T}_{\bar{f}} = \widehat{T}_{\overline{fg}}$;
2. $\widehat{T}_f\widehat{T}_g = \widehat{T}_{fg}$;
3. $H^*_{\bar{f}}H_{\bar{v}\bar{u}}H^*_{\bar{g}} = 0: \ \ [H^2]^\perp \to H^2$.

Proof. Since $\widehat{T}^*_f = \widehat{T}_{\bar{f}}$, it is clear that (1) is equivalent to (2). Assume $f,g \in H^\infty$, then

$$(\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg})|_{\overline{vzH^2}} = 0.$$

Thus

$$[M_uPM_{\bar{u}}f\widehat{T}_g - M_uPM_{\bar{u}}fg]\bar{v}\bar{y} = 0$$

for every $\bar{y} \in (H^2)^\perp$. Multiplying both sides of the above equation by $M_{\bar{u}}$, we have,

$$P\bar{u}f[M_uPM_{\bar{u}}g + M_{\bar{v}}P_-M_vg]\bar{v}\bar{y} - P\overline{uv}fg\bar{y}$$

$$= PfP\overline{uv}g\bar{y} + P\overline{uv}fP_-g\bar{y} - P\overline{uv}fg\bar{y} = 0.$$

This implies that

$$T_fH^*_{uv\bar{g}} + H^*_{uv\bar{f}}S_g = H^*_{uv\overline{fg}}$$

on $[H^2]^\perp$. By equation (2.5), $H^*_{uv\bar{f}}S_g = H^*_{uv\overline{fg}} - T_{\overline{uv}f}H^*_{\bar{g}}$. Thus we obtain that

$$T_fH^*_{uv\bar{g}} - T_{\overline{uv}f}H^*_{\bar{g}} = [T_fT_{\overline{uv}} - T_{\overline{uv}f}]H^*_{\bar{g}} = -H^*_{\bar{f}}H_{\overline{uv}}H^*_{\bar{g}} = 0,$$

The second equality follows from that $H_{fg} = H_fT_g$ and $T_{\bar{g}}H^*_f = H^*_{fg}$ when $g \in H^\infty$. On the other hand, if $H^*_{\bar{f}}H_{\overline{uv}}H^*_{\bar{g}} = -[T_fH^*_{uv\bar{g}} - T_{\overline{uv}f}H^*_{\bar{g}}] = 0$, then

$$M_uPM_{\bar{u}}(\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg})|_{\overline{vzH^2}} = 0.$$

Since f and g are both analytic, we have

$$[M_{\bar{v}}P_-M_vf\widehat{T}_g - M_{\bar{v}}P_-M_vfg]|_{\overline{vzH^2}} = M_{\bar{v}}[S_fS_g - S_{fg}]|_{\overline{zH^2}} = 0$$

where S_f and S_g are dual Toeplitz operators on $[H^2]^\perp$. Thus

$$[\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg}]|_{\overline{vzH^2}} = 0.$$

Similarly,

$$[\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg}]|_{uH^2} = 0.$$

Hence $\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg} = 0$. Thus (2) equivalent to (3). □

LEMMA 5.8. [8, Lemma 4.6] *Let* φ *and* ψ *be nonconstant functions in* H^∞, *and* θ *is a nonconstant inner function. Then* $H^*_{\bar\psi}H_{\bar\theta}H^*_{\bar\varphi}$ *is zero if and only if* $\bar\varphi(\theta-\lambda),\bar\psi(\theta-\lambda)$ *and* $\bar\varphi\bar\psi(\theta-\lambda)$ *are in* H^2 *for some constant* λ.

Sum up Theorem 5.3, Lemma 5.7 and Lemma 5.8, we give an answer to Question 4.6.

THEOREM 5.9. *Let* $f,g\in L^\infty(\partial\mathbb{D})$. *Then* $\widehat{T}_f\widehat{T}_g=\widehat{T}_{fg}$ *if and only if one of the following cases holds:*

1. $f,g,\bar f(vu-\lambda),\bar g(vu-\lambda)$ *and* $\overline{fg}(vu-\lambda)$ *all belong to* H^2 *for some constant* λ.
2. $\bar f,\bar g,f(vu-\lambda),g(vu-\lambda)$ *and* $fg(vu-\lambda)$ *all belong to* H^2 *for some constant* λ.
3. *either* f *or* g *is constant.*

This result is different from classical Hardy Toeplitz operator theory.

EXAMPLE 5.10. Assume u and v are inner functions and u isn't constant. Let $f=u$ and $g=v$, take $\lambda=0$, then $f,g,\bar f(uv-\lambda),\bar g(uv-\lambda)$ and $\overline{fg}(uv-\lambda)$ all are belong to H^2, hence $\widehat{T}_f\widehat{T}_g=\widehat{T}_{fg}$ by Theorem 5.9. This result is different from harmonic Bergman Toeplitz operator theory[4].

COROLLARY 5.11. *Assume* $f,g\in L^\infty$. *If* $\hat T_f\hat T_g=\hat T_{fg}$, *then* $\hat T_f\hat T_g=\hat T_g\hat T_f$.

Proof. If $\hat T_f\hat T_g=\hat T_{fg}$, Then $\hat T_g\hat T_f=\hat T_{fg}$ by Theorem 5.9. Hence $\hat T_f\hat T_g=\hat T_g\hat T_f$. □

6. The finite rank perturbutions

For convenience, we use

$$A=B \;\; mod \;\; (F)$$

to denote that the operator $A-B$ has finite rank.

THEOREM 6.1. *Let* $f,g\in L^\infty(\partial\mathbb{D})$. *Then* $\widehat{T}_f\widehat{T}_g=\widehat{T}_{fg}$ *mod (F) if and only if the following conditions all holds*

1. $T_fT_g=T_{\bar v\bar u f}T_{gvu}$ *mod (F);*
2. $H_{vuf}T_g=H_fT_{vug}$ *mod (F);*
3. $T_fH^*_{uv\bar g}=T_{\overline{uv}f}H^*_{\bar g}$ *mod (F);*
4. $H_{vuf}H^*_{uv\bar g}=H_fH^*_{\bar g}$ *mod (F).*

Proof. $\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg}$ has finite rank if and only if

$$(\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg})|_{uH^2} = 0 \bmod (F)$$

and

$$(\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg})|_{\overline{v}[H^2]^\perp} = 0 \bmod (F).$$

Note that $\{\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg}\}|_{uH^2} = 0 \bmod (F)$ if and only if set $\{P\bar{u}f\widehat{T}_g x - P\bar{u}fgx : x \in uH^2\}$ and $\{P_-vf\widehat{T}_g x - P_-vfgx : x \in uH^2\}$ all have finite dimension. An easy calculation gives

$$\begin{aligned}
&\{P\bar{u}f\widehat{T}_g x - P\bar{u}fgx : x \in uH^2\}\\
=&\{P\bar{u}f[uP\bar{u}gx + \overline{v}(I-P)vgx] - P\bar{u}fgx : x \in uH^2\}\\
=&\{PfP\bar{u}gx - P\overline{uv}fPvgx : x \in uH^2\}\\
=&\{PfPgy - P\overline{uv}fPuvgy : y \in H^2\}\\
=&range\{T_fT_g - T_{\overline{uv}f}T_{uvg}\}
\end{aligned}$$

and

$$\begin{aligned}
&\{P_-vf\widehat{T}_g x - P_-vfgx : x \in uH^2\}\\
=&\{P_-vf[uP\bar{u}gx + \overline{v}(I-P)vgx] - P_-vfgx : x \in uH^2\}\\
=&\{P_-uvfP\bar{u}gx - P_-fPvgx : x \in uH^2\}\\
=&\{P_-uvfPgy - P_-fPuvgy : y \in H^2\}\\
=&range\{H_{uvf}T_g - H_fT_{uvg}\}.
\end{aligned}$$

Thus $\{\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg}\}|_{uH^2} = 0 \bmod (F)$ if and only if both $(T_fT_g - T_{\overline{uv}f}T_{uvg})$ and $(H_{uvf}T_g - H_fT_{uvg})$ are finite rank operators. Thus (1) and (2) hold.

Similarly, we have $\{\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg}\}|_{\overline{v}[H^2]^\perp} = 0 \bmod (F)$ if and only if both $\{P\bar{u}f\widehat{T}_g x - P\bar{u}fgx : x \in \overline{v}[H^2]^\perp\}$ and $\{P_-vf\widehat{T}_g x - P_-vfgx : x \in \overline{v}[H^2]^\perp\}$ are finite dimension. Easy calculations give

$$\begin{aligned}
&\{P\bar{u}f\widehat{T}_g x - P\bar{u}fgx : x \in \overline{v}[H^2]^\perp\}\\
=&\{P\bar{u}f[uP\bar{u}gx + \overline{v}(I-P)vgx] - P\bar{u}fgx : x \in \overline{v}[H^2]^\perp\}\\
=&\{PfP\bar{u}gx - P\overline{uv}fPvgx : x \in \overline{v}[H^2]^\perp\}\\
=&\{PfP\overline{uv}gy - P\overline{uv}fPgy : y \in [H^2]^\perp\}\\
=&range\{T_fH^*_{uv\bar{g}} - T_{\overline{uv}f}H^*_{\bar{g}}\}
\end{aligned}$$

and

$$\begin{aligned}&\{P_{-}vf\widehat{T}_{g}x - P_{-}vfgx : x \in \bar{v}[H^2]^{\perp}\}\\ =&\{P_{-}vf[uP\bar{u}gx + \bar{v}(I-P)vgx] - P_{-}vfgx : x \in \bar{v}[H^2]^{\perp}\}\\ =&\{P_{-}uvfP\bar{u}gx - P_{-}fPvgx : x \in \bar{v}[H^2]^{\perp}\}\\ =&\{P_{-}uvfP\overline{uv}gy - P_{-}fPgx : y \in [H^2]^{\perp}\}\\ =&range\{H_{uvf}H^{*}_{uv\bar{g}} - H_{f}H^{*}_{\bar{g}}\}.\end{aligned}$$

Hence $(\widehat{T}_f\widehat{T}_g - \widehat{T}_{fg})|_{\bar{v}[H^2]^{\perp}} = 0 \bmod (F)$ is equivalent to $(T_f H^*_{uv\bar{g}} - T_{\overline{uv}f}H^*_{\bar{g}})$ and $(H_{uvf}H^*_{uv\bar{g}} - H_f H^*_{\bar{g}})$ are both finite rank operators. That is, (3) and (4) hold, which completes the proof. □

LEMMA 6.2. *Let $f, g \in L^\infty(\partial\mathbb{D})$. Then $T_f T_g = T_{\bar{v}\bar{u}f}T_{gvu}$ mod (F) if and only if one of the following conditions holds:*

1. *There exist nonzero analytic polynomials $A(z), B(z)$ such that $A(z)\bar{f}(z) \in H^\infty$ or $B(z)g(z) \in H^\infty$;*
2. *there exist nonzero analytic polynomials $A_1(z), A_2(z), B_1(z)$ and $B_2(z)$ with $A_1\bar{B}_1 = A_2\bar{B}_2$ such that*

$$\{A_1 + A_2 vu\}\bar{f} \in H^\infty, \{B_1 + B_2 vu\}g \in H^\infty.$$

Proof. By Kronecker's theorem [14, Corollary 3.3.], for $\phi \in L^\infty$, the Hankel operator H_ϕ has finite rank if and only if there is a analytic polynomials $A(z)$ such that $A\phi \in H^\infty$. Since Axler-Chang-Sarason theorem [2], for $\phi, \psi \in L^\infty$, $H^*_\phi H_\psi$ has finite rank if and only if the operators H_ϕ or H_ψ does. Since $H_{uvg} = H_g T_{uv}$, H_g is a finite rank operator implies that H_{uvg} is also finite rank operator. The condtions (1) and (2) follow from [7, Theorem 3.4]. □

LEMMA 6.3. *Let $f, g \in L^\infty(\partial\mathbb{D})$. Then $H_{vuf}H^*_{uv\bar{g}} = H_f H^*_{\bar{g}}$ mod (F) if and only if one of the following conditions holds:*

1. *There exist nonzero analytic polynomials $A(z)$ and $B(z)$ such that $A(z)f(z) \in H^\infty$ or $B(z)\bar{g}(z) \in H^\infty$;*
2. *there exist nonzero analytic polynomials $A_1(z), A_2(z), B_1(z)$ and $B_2(z)$ with $A_1\bar{B}_1 = A_2\bar{B}_2$ such that*

$$\{A_1 + A_2 vu\}f \in H^\infty, \{B_1 + B_2 vu\}\bar{g} \in H^\infty.$$

Proof. Since

$$H_{vuf}H^*_{uv\bar{g}} - H_f H^*_{\bar{g}} = V(H^*_{vuf}H_{uv\bar{g}} - H^*_f H_{\bar{g}})V$$

and

$$H^*_{vuf}H_{uv\bar{g}} - H^*_f H_{\bar{g}} = T_{\bar{f}}T_{\bar{g}} - T_{\overline{uv}\bar{f}}T_{uv\bar{g}},$$

$$H_{vuf}H^*_{uv\bar{g}} - H_f H^*_{\bar{g}} = 0 \ \ mod \ \ (F)$$

if and only if

$$T_{\bar{f}}T_{\bar{g}} - T_{\overline{uv}\bar{f}}T_{uv\bar{g}} = 0 \ \ mod \ \ (F).$$

The result follows from Lemma 6.2. □

LEMMA 6.4. *Let $f,g \in L^\infty(\partial\mathbb{D})$. Then $H_{vuf}T_g = H_f T_{vug}$ mod (F) if and only if one of the following conditions holds:*

1. *At least one of P_-f, P_-uvf, P_-g and P_-uvg is a rational function all of whose poles are in $\mathbb{D}$.*
2. *There exist nonzero analytic polynomials $A_1(z), A_2(z), B_1(z)$, $B_2(z)$ and $q(z)$ such that*
$$(A_1 + A_2uv)f \in H^\infty, (B_1 + B_2vu)g \in H^\infty.$$
and $q(B_1 + B_2uv)fg \in H^\infty$ with $A_1B_1 + A_2B_2 = 0$ on $\partial\mathbb{D}$.

Proof. By Kronecker's theorem [14, Corollary 3.3.], at least one of H_f, H_{uvf}, H_g and H_{uvg} has finite rank if and only if at least one of P_-f, P_-uvf, P_-g and P_-uvg is a rational function all of whose poles are in $\mathbb{D}$. By [6, Theorem 4.2], if none of H_f, H_{uvf}, H_g and H_{uvg} has finite rank, then

$$H_{vuf}T_g = H_f T_{vug} \ \ mod \ \ (F)$$

if and only if there exist nonzero analytic polynomials $A_1(z), A_2(z), B_1(z)$, $B_2(z)$ such that

$$(A_1 + A_2uv)f \in H^\infty, (B_1 + B_2vu)g \in H^\infty.$$

with $A_1B_1 + A_2B_2 = 0$ on $\partial\mathbb{D}$ and $H_{A_2(B_1+B_2uv)fg}$ has finite rank.

$$H_{A_2(B_1+B_2uv)fg} = H_{(B_1+B_2uv)fg}T_{A_2}$$

has finite rank if and only if

$$H_{(B_1+B_2uv)fg}$$

has finite rank if and only if there is none zero analytic polynomial $q(z)$ such that $q(B_1 + B_2uv)fg \in H^\infty$. □

LEMMA 6.5. *Let $f,g \in L^\infty(\partial\mathbb{D})$. Then $T_f H^*_{uv\bar{g}} - T_{\overline{uv}f}H^*_{\bar{g}} = 0$ mod (F) if and only if one of the following conditions holds:*

1. *At least one of $P_-\overline{f}, P_-uv\overline{f}, P_-\overline{g}$ and $P_-uv\overline{g}$ is a rational function all of whose poles are in $\mathbb{D}$.*

2. *there exist nonzero analytic polynomials $A_1(z), A_2(z), B_1(z)$, $B_2(z)$ and $q(z)$ such that*
$$(A_1 + A_2uv)\overline{g} \in H^\infty, (B_1 + B_2vu)\overline{f} \in H^\infty.$$
and $q(B_1 + B_2uv)\overline{fg} \in H^\infty$ with $A_1B_1 + A_2B_2 = 0$ on ∂D.

Proof. Since $T_fH^*_{uv\overline{g}} - T_{\overline{uv}f}H^*_{\overline{g}} = 0$ mod (F) if and only if $(T_fH^*_{uv\overline{g}} - T_{\overline{uv}f}H^*_{\overline{g}})^* = H_{uv\overline{g}}T_{\overline{f}} - H_{\overline{g}}T_{uv\overline{f}} = 0$ mod (F), the result follows from Lemma 6.4. $\square$

Combining Theorem 6.1, Lemma 6.2, Lemma 6.3, Lemma 6.4 and Lemma 6.5. we obtain the following theorem.

THEOREM 6.6. *Let $f, g \in L^\infty(\partial\mathbb{D})$. Then $\widehat{T}_f\widehat{T}_g = \widehat{T}_{fg}$ mod (F) if and only if the following conditions all holds*

1. *The condition (1) or (2) holds in Lemma 6.2;*

2. *The condition (1) or (2) holds in Lemma 6.3;*

3. *The condition (1) or (2) holds in Lemma 6.4;*

4. *The condition (1) or (2) holds in Lemma 6.5.*

COROLLARY 6.7. *Let u, v be inner functions, and $f, g \in L^\infty(\partial\mathbb{D})$. If one of u and v is not a finite Blaschke product, then $\widehat{T}_f\widehat{T}_g$ has finite rank if and only if one of f and g is zero function.*

Proof. Assume one of u and v is not a finite Blaschke product, then uv is not a finite Blaschke product. If $\widehat{T}_f\widehat{T}_g$ has finite rank, then $fg = 0$ (Lemma 5.1) and $T_fT_g = T_{\overline{vu}f}T_{gvu}$ mod (F)(Theorem 6.1).

By Lemma 5.2, if $T_fT_g - T_{\overline{vu}f}T_{gvu} = 0$, thus either $\overline{f}$ or g is analytic. Since $fg = 0$, one of f and g is zero function.

If $T_fT_g - T_{\overline{vu}f}T_{gvu}$ is a nonzero finite rank operator, Since
$$T_fT_g - T_{\overline{vu}f}T_{gvu} = H^*_{uv\overline{f}}H_{uvg} - H^*_{\overline{f}}H_g,$$
we need to consider two cases: the case $H^*_{uv\overline{f}}H_{uvg}$ and $H^*_{\overline{f}}H_g$ both are finite rank operators, the case $H^*_{uv\overline{f}}H_{uvg}$ and $H^*_{\overline{f}}H_g$ both are not finite rank operators.

In the previous case, $H_{\overline{f}}$ or H_g is finite rank operator by Axler-Chang-Sarason theorem in [2]. By Kronecker's theorem [14, Corollary 3.3.], there is a nonzero analytic polynomial $A(z)$ such that $A(z)\overline{f}(z) \in H^\infty$ or $A(z)g(z) \in H^\infty$. Since $fg = 0$, $(A(z)\overline{f}(z))\overline{g}(z) = 0$ or $(A(z)g(z))f(z) = 0$ a.e on $\partial\mathbb{D}$, one of f and g is zero function.

In the latter case, By [7, Theorem 3.4], there exist nonzero analytic polynomials $A_i(z), B_i(z), i = 1,2$ with $A_1(z)\overline{B_1(z)} = A_2(z)\overline{B_2(z)}$, such that $A_1uv\overline{f} + A_2\overline{f} \in H^\infty$

and $B_1uvg+B_2g \in H^\infty$. If there is a set $E \subseteq \partial\mathbb{D}$ such that $0 < mE < 1$, $f|_E = 0$ and $f|_{\partial\mathbb{D}-E} \neq 0$, then $A_1(z)u(z)v(z)+A_2(z) = 0$. Thus $uv = -\frac{A_2}{A_1}$ is a rational inner function.

Note that rational inner function must be finite Blaschke product. In fact, if rational function $\frac{b(z)}{a(z)}$ is a inner function, then $\frac{b(z)}{a(z)}$ have the form

$$\frac{b(z)}{a(z)} = c_0B(z)Q(z),$$

where c_0 is a constant with $|c_0| = 1$, $B(z)$ is finite Blaschke Product, and polynomial $Q(z) = c_1(z-\lambda_1)\cdots(z-\lambda_k)$ with $|\lambda_j| \geqslant 1$. By Theory[11, page.72], $Q(z)$ is a singular inner function. Let $Q(z)$ be the singular function determined by measure μ on $\partial\mathbb{D}$, and let $E \subset \partial\mathbb{D}$ be the closed support of μ, then $|Q(z)|$ dose not extend continuously from $\mathbb{D}$ to any point of E. But $|Q(z)|$ is continuously on $\mathbb{C}$. Hence polynomial is impossible singular inner function. Thus $Q(z)$ is a constant. It follows that uv is a finite Blaschke product. This leads to a contradiction. □

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REFERENCES

[1] W. ARVESON, *A short course on spectral theory*, vol. 209, Springer Science & Business Media, 2006.
[2] S. AXLER, S. Y. A. CHANG, D. SARASON, *Products of Toeplitz operators*, Integral Equations and Operator Theory 1 (1978) 285–309.
[3] A. BROWN, P. HALMOS, *Algebraic Properties of Toeplitz operators*, Journal für die reine und angewandte Mathematik 213 (1964) 89–102.
[4] B. R. CHOE, Y. J. LEE, *Commuting Toeplitz operators on the harmonic Bergman space*, Michigan Math. J 46 (1999) 163–174.
[5] B. R. CHOE, Y. J. LEE, *Commutants of analytic Toeplitz operators on the harmonic Bergman space*, Integral Equations and Operator Theory 50 (2004) 559–564.
[6] X. DING, *The finite rank perturbations of the product of Hankel and Toeplitz operators*, Journal of Mathematical Analysis and Applications 337 (2008) 726–738.
[7] X. DING, *The finite sum of the products of two Toeplitz operators*, Journal of the Australian Mathematical Society 86 (2009) 45–60.
[8] X. DING, Y. SANG, *Dual truncated Toeplitz operators*, Journal of Mathematical Analysis and Applications 461 (2018) 929–946.
[9] R. G. DOUGLAS, *Banach algebra techniques in operator theory*, vol. 179, Springer Science & Business Media, 2012.
[10] M. ENGLIŠ, *Toeplltz operators and the Berezin transform on H^2*, Linear algebra and its applications 223 (1995) 171–204.
[11] J. GARNETT, *Bounded analytic functions*, vol. 236, Springer Science & Business Media, 2007.
[12] H. GUEDIRI, *Dual Toeplitz operators on the sphere*, Acta Mathematica Sinica, English Series 29 (2013) 1791–1808.
[13] J. JURASIK, B. ŁANUCHA, *Asymmetric truncated Toeplitz operators equal to the zero operator*, Annales Universitatis Mariae Curie-Sklodowska, sectio A-Mathematica, 2016, 70 (2): 51.

[14] V. V. PELLER, *Hankel operators and their applications*, Springer-Verlag, 2003.
[15] D. SARASON, *Algebraic properties of truncated Toeplitz operators*, Operators and Matrices 4 (2007).
[16] B. SZ.-NAGY, C. FOIAS, H. BERCOVICI, L. KRCHY, *Harmonic Analysis of Operators on Hilbert Space*, Springer New York, 2010.
[17] L. ZHAO, *Commutativity of Toeplitz operators on the harmonic Dirichlet space*, Journal of Mathematical Analysis and Applications 339 (2008) 1148–1160.

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