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COMMUTATIVITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACES OF THE UNIT DISK

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Abstract. In this paper we give some necessary and sufficient conditions that Toeplitz operators with special symbols commute with Toeplitz operators with harmonic symbols on the Bergman space.

1. Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, let $\partial\mathbb{D}$ be the unit cicle and let dA be the normalized area measure on $\mathbb{D}$. The Bergman space $A^2(\mathbb{D})$ is the space of functions analytic on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty.$$

It is well known that $A^2(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$. Let P be the orthogonal projection form $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$. Let $L^\infty(\mathbb{D})$ be the space of essentially bounded area measurable functions on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator with the symbol φ is the operator $T_\varphi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ defined by

$$T_\varphi f = P(\varphi f), \qquad f \in A^2(\mathbb{D}).$$

This paper is motivated by the following problem: find the necessary and sufficient conditions that two Toeplitz operators acting on the Bergman space commute. Much work has been done in studying the commutativity of Toeplitz operators.

In the case of the classical Hardy space $H^2(\partial\mathbb{D})$, Brown and Halmos [1] completely answered the problem and obtained the following result:

THEOREM A. *Bounded Toeplitz operators T_φ and T_ψ commute if and only if either both symbols are analytic or both symbols are conjugate analytic or $\alpha\varphi + \beta\psi$ is constant for some constants α, β not both 0.*

On the Bergman space of the unit disk, the first complete result was obtained by Axler and Čučković in [2] who described the commutativity of Toeplitz operators with harmonic symbols, which was similar to the above result proved by Brown and Halmos in [1]:

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THEOREM B. *Suppose that φ and ψ are bounded harmonic functions on $\mathbb{D}$, then T_φ and T_ψ commute if and only if either both symbols are analytic or both symbols are conjugate analytic or $\alpha\varphi+\beta\psi$ is constant for some constants α, β not both 0.*

Stroethoff [3] completely characterized the essential commutativity of Toeplitz operators with harmonic symbols on the Bergman space of the unit disk. In [4] Axler, Čučković and Rao studied the problem for Toeplitz operators with analytic symbols and obtained the following result:

THEOREM C. *Suppose that Ω is a bounded open domain in $\mathbb{C}$. If φ is a nonconstant bounded analytic function on Ω and ψ is a bounded measurable function on Ω such that T_φ and T_ψ commute, then ψ is analytic.*

Čučković and Rao [5] studied Toeplitz operators that commute with Toeplitz operators with monomial and radial symbols on the Bergman space of the unit disk and obtained the result as bellow:

THEOREM D. *Let $\varphi,\psi \in L^\infty(\mathbb{D})$ be nontrivial functions and φ also be a radial function. If T_φ and T_ψ commute, then ψ is a radial function.*

In [6] Yufeng Lu and Chaomei Liu extended some results obtained by Čučković and Rao in [5] to the weighted Bergman space of the unit disk.

Encouraged by the above results, it is natural to come up with the following problem:

If $\varphi,\psi \in L^\infty(\mathbb{D})$ and φ is a nonconstant bounded harmonic function, what are the necessary and sufficient conditions such that T_φ and T_ψ commute?

In this paper we will find some necessary and sufficient conditions for a symbol that produces a Toeplitz operator on the Bergman space that commutes with Toeplitz operator with harmonic symbol.

2. Commutativity of Topelitz operators

In this section, we start with a decomposition of the space $L^2(\mathbb{D})$. Let

$$\mathscr{R} = \{a : \mathbb{D} \to \mathbb{C} \text{ radial} \;\Big|\; \int_0^1 r|a(r)|^2 dr < \infty\},$$

and let $\mathscr{R}_k = e^{ik\theta}\mathscr{R}$ for $k \in \mathbb{Z}$. By using the fact that the trigonometric polynomials are dense in $L^2(\mathbb{D})$ and that $\mathscr{R}_{k_1}$ is orthogonal to $\mathscr{R}_{k_2}$ for $k_1 \neq k_2$, we can see that

$$L^2(\mathbb{D}) = \bigoplus_{k\in\mathbb{Z}} \mathscr{R}_k,$$

i.e., every function $u \in L^2(\mathbb{D})$ has the form

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} u_k(r), \qquad u_k \in \mathscr{R}.$$

One of our most useful tools in the following calculations is the Mellin transform (closely related, using the change of variables $x=e^{-u}$, to the Laplace transform). The Mellin transform $\widehat{\varphi}$ of a function φ is defined by the equation

$$\widehat{\varphi}(z)=\int_0^\infty \varphi(x)x^{z-1}dx.$$

By [8] we know that $\widehat{\varphi}(z)$ is well defined on $\{z\in\mathbb{C}:\mathrm{Re}\,z\geqslant 2\}$ and analytic on $\{z\in\mathbb{C}:\mathrm{Re}\,z>2\}$ if $\varphi\in L^1([0,1],rdr)$.

The following lemmas will be frequently used in the following calculations.

LEMMA 2.1. *Let p be a nonnegative integer, let m_0 and n_0 be positive integers with $n_0\geqslant m_0$, and let $\varphi\in L^\infty(\mathbb{D})$ be a radial function. If for any positive integer n with $n\geqslant n_0$,*

$$\widehat{\varphi}(2n+p+2)=\frac{n-m_0+1}{n+1}\widehat{\varphi}(2n-2m_0+p+2), \tag{1}$$

where $\widehat{\varphi}(z)$ is the Mellin transform of the function φ, then

(1) *if* $p>0$, $\varphi\equiv 0$;

(2) *if* $p=0$, φ *is constant on* $\mathbb{D}$.

Proof. Let $[\frac{n_0}{m_0}]+1=s_0$, where $[x]$ denotes the integer part of x, then we have $s_0\geqslant 1$ and $s_0m_0\geqslant n_0$. From the equality (1) we get that for any integer s with $s\geqslant s_0$,

$$\widehat{\varphi r^p}(2m_0s+2)=(2(s_0-1)m_0+2)\widehat{\varphi}(2m_0(s_0-1)+2+p)\widehat{1}(2m_0s+2).$$

Let $\psi(r)=\varphi r^p-(2(s_0-1)m_0+2)\widehat{\varphi}(2m_0(s_0-1)+2+p)$, then the above equality implies that for any integer s with $s\geqslant s_0$,

$$\widehat{\psi}(2m_0s+2)=0.$$

Hence by Lemma 3.1 in [8] we get that $\psi\equiv 0$, that is, $\varphi(re^{i\theta})=\frac{1}{r^p}(2(s_0-1)m_0+2)\widehat{\varphi}(2m_0(s_0-1)+2+p)$.

When $p=0$, then $\varphi(re^{i\theta})=(2(s_0-1)m_0+2)\widehat{\varphi}(2m_0(s_0-1)+2)$ is a constant.

When $p>0$, we claim that $\widehat{\varphi}(2m_0(s_0-1)+2+p)=0$. Otherwise, suppose that $\widehat{\varphi}(2m_0(s_0-1)+2+p)\neq 0$. Since $p>0$ is an integer, we have that

$$\begin{aligned}\int_D|\varphi(z)|^2dA(z)&=2\Big((2(s_0-1)m_0+2)\widehat{\varphi}(2m_0(s_0-1)+2+p)\Big)^2\int_0^1\frac{1}{r^{2p}}rdr\\&\geqslant 2\Big((2(s_0-1)m_0+2)\widehat{\varphi}(2m_0(s_0-1)+2+p)\Big)^2\int_0^1\frac{1}{r}dr\\&=+\infty.\end{aligned}$$

But $\varphi\in L^\infty(\mathbb{D})\subset L^2(\mathbb{D})$, then we have that

$$\int_D|\varphi(z)|^2dA(z)<+\infty,$$

which is contradictory to the above inequality, so $\widehat{\varphi}(2m_0(s_0-1)+2+p)=0$. Hence $\varphi\equiv 0$. $\square$

LEMMA 2.2. *Let p , m_0 and n_0 be positive integers and let $\varphi\in L^\infty(\mathbb{D})$ be a radial function. If for any nonnegative integer n with $n\geqslant n_0$,*

$$\widehat{\varphi}(2n+2m_0+p+2)=\frac{n+p+1}{n+m_0+p+1}\widehat{\varphi}(2n+p+2), \tag{2}$$

where $\widehat{\varphi}(z)$ is the Mellin transform of the function φ, then $\varphi=Cr^p$, where $C\in\mathbb{C}$ is a constant.

Proof. Let $[\frac{n_0}{m_0}]+1=s_0$, where $[x]$ denotes the integer part of x, then we have $s_0\geqslant 1$ and $s_0m_0\geqslant n_0$. Then from equality (2) we obtain that

$$\begin{aligned}
&\widehat{\varphi}(2mm_0+2m_0+p+2)\\
&=\frac{mm_0+p+1}{mm_0+m_0+p+1}\widehat{\varphi}(2mm_0+p+2)\\
&=\frac{mm_0+p+1}{mm_0+m_0+p+1}\,\frac{(m-1)m_0+p+1}{mm_0+p+1}\widehat{\varphi}(2(m-1)m_0+p+2)\\
&=\frac{(m-1)m_0+p+1}{(m+1)m_0+p+1}\widehat{\varphi}(2(m-1)m_0+p+2)\\
&=\frac{(m-1)m_0+p+1}{(m+1)m_0+p+1}\widehat{\varphi}(2(m-1)m_0+p+2)=\cdots\\
&=\frac{s_0m_0+p+1}{(m+1)m_0+p+1}\widehat{\varphi}(2s_0m_0+p+2)\\
&=(2s_0m_0+2p+2)\widehat{\varphi}(2s_0m_0+p+2)\widehat{r^p}(2mm_0+2m_0+p+2)
\end{aligned}$$

for any integers m with $m\geqslant m_0$. Let $\psi=\varphi-(2s_0m_0+2p+2)\widehat{\varphi}(2s_0m_0+p+2)r^p$, then we have that

$$\widehat{\psi}(2mm_0+2m_0+p+2)=0$$

for any integers m with $m\geqslant m_0$. So by Lemma 3.1 in [8] we get that $\psi\equiv 0$, that is, $\varphi=(2s_0m_0+2p+2)\widehat{\varphi}(2s_0m_0+p+2)r^p\doteq Cr^p$.

Now we start to discuss the commuting problem of two Toeplitz operators on the Bergman space. First suppose that $f\in L^\infty(\mathbb{D})$ is a nonconstant harmonic function on $\mathbb{D}$ in the following discussion except for some special case.

PROPOSITION 2.1. *Let $g(re^{i\theta})=\varphi(r)e^{ip\theta}\in L^\infty(\mathbb{D})$, where p is an integer, then $T_fT_g=T_gT_f$ if and only if one of the following conditions is satisfied:*

(1) *$p>0$, f and g are both analytic;*

(2) *$p<0$, f and g are both conjugate analytic;*

(3) *$p=0$, g is constant on $\mathbb{D}$.*

Proof. Suppose that $T_f T_g = T_g T_f$. Since p is an integer, there exist three cases: (1) $p > 0$; (2) $p < 0$; (3) $p = 0$.

When $p > 0$, then by Theorem 11 in [7] we have that f is analytic and so we get that g is analytic by Theorem C.

When $p < 0$, then by the properties of Toeplitz operator we get that $T_{\bar{f}} T_{\bar{g}} = T_{\bar{g}} T_{\bar{f}}$ and $\bar{g}(re^{i\theta}) = \varphi(r)e^{i(-p)\theta}$, so by Theorem 11 in [7] we obtain that $\bar{f}$ is analytic and that $\bar{g}$ is analytic by Theorem C, which means that f and g are conjugate analytic.

When $p = 0$, then g is radial and we claim that g is constant on $\mathbb{D}$. Otherwise, suppose that g is not constant on $\mathbb{D}$, then by Theorem D we get that f is radial. Since f is harmonic, then f is constant on $\mathbb{D}$, which is contradictory to the supposition that f is not constant. Thus g is constant on $\mathbb{D}$.

It is clear that the converse is also true. □

REMARK 2.1. Similar to the analysis of the above proposition, we could get the following result: Let f be a bounded harmonic function on $\mathbb{D}$ and $g(re^{i\theta}) = \varphi(r)e^{ip\theta} \in L^\infty(\mathbb{D})$, where p is an integer. If g is not constant on $\mathbb{D}$, then $T_f T_g = T_g T_f$ if and only if one of the following conditions is satisfied: (1) $p > 0$, f and g are both analytic; (2) $p < 0$, f and g are both conjugate analytic; (3) $p = 0$, f is constant on $\mathbb{D}$.

According to the above proposition and remark we have the following corollaries.

COROLLARY 2.1. *Let f be a bounded harmonic function on $\mathbb{D}$ and $g(re^{i\theta}) = \varphi(r)e^{ip\theta} \in (b^2(\mathbb{D}))^\perp$, where $b^2(\mathbb{D})$ is the harmonic Bergman space on $\mathbb{D}$, then T_f and T_g commute if and only if $g \equiv 0$.*

COROLLARY 2.2. *Suppose that f is a bounded harmonic function on $\mathbb{D}$ and $g(z) = z^n \bar{z}^m - \frac{n-m+1}{n+1} z^{n-m}$, where n and m are both positive integers with $m < n$, then T_f and T_g commute if and only if f is constant on $\mathbb{D}$.*

THEOREM 2.1. *Let $g(re^{i\theta}) = Cr^q e^{iq\theta} + \varphi(r)e^{ip\theta} \in L^\infty(\mathbb{D})$, where p and q are both integers with $0 \leqslant q < p$ and C is a constant with $C \neq 0$, then T_f and T_g commute if and only if one of the following conditions is satisfied:*

(1) *f and g are both analytic;*

(2) *f is conjugate analytic and $g = C$.*

Proof. Since f is bounded and harmonic on $\mathbb{D}$, there exist functions f_1 and f_2 analytic on $\mathbb{D}$ such that $f = f_1 + \overline{f_2}$, where $f_1(z) = \sum_{s=0}^{\infty} a_s z^s$ and $f_2(z) = \sum_{s=1}^{\infty} \overline{b_s} z^s$. Let $g_1(z) = Cz^q$ and $g_2(re^{i\theta}) = \varphi(r)e^{ip\theta}$, then $g = g_1 + g_2$.

Suppose that T_f and T_g commute, then for any nonnegative integers n we have

$$(T_f T_{g_1} - T_{g_1} T_f) z^n = (T_{g_2} T_f - T_f T_{g_2}) z^n, \tag{3}$$

and

$$\begin{aligned} &(T_f T_{g_1} - T_{g_1} T_f) z^n \\ &= C \sum_{s=q}^{n+q-1} b_{n+q-s} \left[\frac{s+1}{n+q+1} - \frac{s-q+1}{n+1} \right] z^s + C \sum_{s=0}^{q-1} b_{n+q-s} \frac{s+1}{n+q+1} z^s, \end{aligned} \tag{4}$$

$$
\begin{aligned}
&(T_{g_2}T_f - T_fT_{g_2})z^n \\
&= \sum_{s=0}^{\infty} a_s[2(n+s+p+1)\widehat{\varphi}(2n+2s+p+2) - 2(n+p+1)\widehat{\varphi}(2n+p+2)]z^{n+p+s} \\
&\quad + \sum_{s=p}^{n+p-1} b_{n+p-s}2(s+1)\left[\frac{s-p+1}{n+1}\widehat{\varphi}(2s-p+2) - \widehat{\varphi}(2n+p+2)\right]z^s \\
&\quad - \sum_{s=0}^{p-1} b_{n+p-s}2(s+1)\widehat{\varphi}(2n+p+2)z^s, \qquad (5)
\end{aligned}
$$

where $\widehat{\varphi}(z)$ is the Mellin transform of the function φ.

Since $q < p$, the equalities (3), (4) and (5) give that

$$
\textstyle\sum_{s=0}^{\infty} a_s[2(n+s+p+1)\widehat{\varphi}(2n+2s+p+2) - 2(n+p+1)\widehat{\varphi}(2n+p+2)]z^{n+p+s} = 0
$$

for any integer n with $n \geqslant p-q+1$. The above equality implies that for any integer n with $n \geqslant p-q+1$ and for any positive integer s,

$$
a_s[2(n+s+p+1)\widehat{\varphi}(2n+2s+p+2) - 2(n+p+1)\widehat{\varphi}(2n+p+2)] = 0. \qquad (6)
$$

Now we will continue the discuss in two cases.

First case. If $a_s = 0$ for any positive integer s, then f is co-analytic. Since $T_fT_g = T_gT_f$, it is obvious that $T_{\bar{f}}T_{\bar{g}} = T_{\bar{g}}T_{\bar{f}}$, so by Theorem C we get that $\bar{g}$ must be analytic. Since $\overline{g(re^{i\theta})} = \bar{C}r^qe^{-iq\theta} + \varphi(r)e^{-ip\theta}$ and $0 \leqslant q < p$, we obtain that $q = 0$ and $\varphi(r) = 0$, that is, $g = C$.

Second case. If there exists a positive integer s_0 such that $a_{s_0} \neq 0$, then from the equality (6) we get that for any integer n with $n \geqslant p-q+1$,

$$
\widehat{\varphi}(2n+2s_0+p+2) = \frac{n+p+1}{n+s_0+p+1}\widehat{\varphi}(2n+p+2),
$$

and by Lemma 2.2 we obtain that $\varphi(r) = C_1r^p$, where $C_1 \in \mathbb{C}$, which means that $g(z) = Cz^q + C_1z^p$ is analytic. Since $T_fT_g = T_gT_f$, by Theorem C we get that f must be analytic.

Conversely, if f and g satisfy the condition (1) or (2) , then it is obvious that T_f and T_g commute. $\square$

By the properties of Toeplitz operators and the above theorem it is easy to obtain the following corollary.

COROLLARY 2.3. *Suppose that $g(re^{i\theta}) = Cr^qe^{-iq\theta} + \varphi(r)e^{-ip\theta} \in L^\infty(\mathbb{D})$, where p and q are both positive integers with $0 \leqslant q < p$ and $C \neq 0$, then T_f and T_g commute if and only if one of the following conditions is satisfied:*

(1) *f and g are both conjugate analytic;*

(2) *f is analytic and $g = C$.*

THEOREM 2.2. *Let* $g(re^{i\theta}) = \varphi(r)e^{ip\theta} + Cr^q e^{iq\theta} \in L^\infty(\mathbb{D})$, *where* p *and* q *are both integers with* $1 \leqslant p < q$ *and* $C \neq 0$, *then* T_f *and* T_g *commute if and only if* f *and* g *are both analytic.*

Proof. Since f is a bounded harmonic function on $\mathbb{D}$, there exist functions f_1 and f_2 analytic on $\mathbb{D}$ such that $f = f_1 + \overline{f_2}$, where $f_1(z) = \sum_{s=0}^\infty a_s z^s$ and $f_2(z) = \sum_{s=1}^\infty \overline{b_s} z^s$. Let $g_1(z) = Cz^q$ and $g_2(re^{i\theta}) = \varphi(r)e^{ip\theta}$, then $g = g_1 + g_2$.

Suppose that T_f and T_g commute, then for any nonnegative integers n we have

$$(T_f T_{g_1} - T_{g_1} T_f)z^n = (T_{g_2} T_f - T_f T_{g_2})z^n,$$

and

$$(T_f T_{g_1} - T_{g_1} T_f)z^n = C\sum_{s=q}^{n+q-1} b_{n+q-s}\left[\frac{s+1}{n+q+1} - \frac{s-q+1}{n+1}\right]z^s + C\sum_{s=0}^{q-1} b_{n+q-s}\frac{s+1}{n+q+1}z^s,$$

$$\begin{aligned}(T_{g_2} T_f - T_f T_{g_2})z^n = &\sum_{s=n+p}^{\infty} a_{s-n-p}[2(s+1)\widehat{\varphi}(2s-p+2) - 2(n+p+1)\widehat{\varphi}(2n+p+2)]z^s \\ &+ \sum_{s=p}^{n+p-1} b_{n+p-s}2(s+1)\left[\frac{s-p+1}{n+1}\widehat{\varphi}(2s-p+2) - \widehat{\varphi}(2n+p+2)\right]z^s \\ &- \sum_{s=0}^{p-1} b_{n+p-s}2(s+1)\widehat{\varphi}(2n+p+2)z^s,\end{aligned}$$

where $\widehat{\varphi}(z)$ is the Mellin transform of the function φ.

The above equalities give that for any integer n with $n \geqslant q-p+1$,

$$\begin{aligned}&C\sum_{s=q}^{n+q-1} b_{n+q-s}\left[\frac{s+1}{n+q+1} - \frac{s-q+1}{n+1}\right]z^s + C\sum_{s=0}^{q-1} b_{n+q-s}\frac{s+1}{n+q+1}z^s \\ = &\sum_{s=n+p}^{\infty} a_{s-n-p}[2(s+1)\widehat{\varphi}(2s-p+2) - 2(n+p+1)\widehat{\varphi}(2n+p+2)]z^s \\ &+ \sum_{s=p}^{n+p-1} b_{n+p-s}2(s+1)\left[\frac{s-p+1}{n+1}\widehat{\varphi}(2s-p+2) - \widehat{\varphi}(2n+p+2)\right]z^s \\ &- \sum_{s=0}^{p-1} b_{n+p-s}2(s+1)\widehat{\varphi}(2n+p+2)z^s.\end{aligned}$$

Since $1 \leqslant p < q$, the above equality implies that for any integer n with $n \geqslant q-p+1$,

$$\begin{aligned}&a_{q-p-s}\left[2(n+q-s+1)\widehat{\varphi}(2n+2q-2s-p+2) - 2(n+p+1)\widehat{\varphi}(2n+p+2)\right] \\ =&Cb_s\left[\frac{s}{n+1} - \frac{s}{n+q+1}\right];\end{aligned} \tag{7}$$

for any integer s with $1 \leqslant s \leqslant q-p$ and

$$\begin{aligned}
&Cb_s\left[\frac{s}{n+1}-\frac{s}{n+q+1}\right] \\
&= b_{s+p-q}2(n+q-s+1)\left[\frac{n+q-p-s+1}{n+1}\widehat{\varphi}(2n+2q-2s-p+2)-\widehat{\varphi}(2n+p+2)\right]
\end{aligned} \tag{8}$$

for any integer s with $q-p+1 \leqslant s \leqslant n$.

Now we start to show that $b_s = 0$ for any integer s with $1 \leqslant s \leqslant q-p$.

Since $C \neq 0$ and $\frac{s}{n+1}-\frac{s}{n+q+1} \neq 0$ for any integer s with $1 \leqslant s \leqslant q-p$ and any fixed integer n, it is evident that $b_{q-p} = 0$ from the equality (8).

If $q-p > 1$, we claim that $b_s = 0$ for any integer s with $1 \leqslant s \leqslant q-p-1$. Otherwise, suppose that there exists an integer s_0 with $1 \leqslant s_0 \leqslant q-p-1$ such that $b_{s_0} \neq 0$, then from the equality (8) we obtain that $a_{q-p-s_0} \neq 0$ and

$$\begin{aligned}
&(2n+2q-2s_0+2)\widehat{\varphi}(2n+2q-2s_0-p+2)-(2n+2p+2)\widehat{\varphi}(2n+p+2) \\
&= M\left[\frac{s_0}{n+1}-\frac{s_0}{n+q+1}\right]
\end{aligned}$$

for any integer n with $n \geqslant q-p+1$, where $M = C\frac{b_{s_0}}{a_{q-p-s_0}}$. Let $q_0 = q-s_0$, then by the Mellin convolution theorem we obtain that

$$\begin{aligned}
&\widehat{\varphi r^{2q_0-p}}(2n+2)-\widehat{\varphi r^{p}}(2n+2)+(2q_0-2p)(\widehat{\varphi r^{p}*r^{2q_0}})(2n+2) \\
=\ &\frac{Ms_0}{q_0}\widehat{1}(2n+2)-\left[M+\frac{Ms_0}{q_0}\right]\widehat{r^{2q_0}}(2n+2)+M\widehat{r^{2q}}(2n+2)
\end{aligned}$$

for any integer n with $n \geqslant q-p+1$, that is,

$$\psi(2n+2) = 0, \quad \text{for any integer } n \text{ with } n \geqslant q-p+1,$$

where $\psi = \varphi r^{2q_0-p} - \varphi r^p + (2q_0-2p)(\varphi r^p)*r^{2q_0} - \frac{Ms_0}{q_0} + \left[M+\frac{Ms_0}{q_0}\right]r^{2q_0} - Mr^{2q}$. So by Lemma 3.1 in [8] we get that $\psi \equiv 0$, which is equivalent to that

$$\varphi r^{2q_0-p} - \varphi r^p + (2q_0-2p)(\varphi r^p)*r^{2q_0} = \frac{Ms_0}{q_0} - \left[M+\frac{Ms_0}{q_0}\right]r^{2q_0} + Mr^{2q}.$$

Applying the Mellin convolution we obtain that

$$((\varphi r^p)*r^{2q_0})(r) = \int_r^1 \varphi\left(\frac{r}{t}\right)\left(\frac{r}{t}\right)^p t^{2q_0}\frac{1}{t}dt = r^p\int_r^1 \varphi\left(\frac{r}{t}\right)t^{2q_0-p-1}dt \doteq r^p h(r)$$

and $h(r) \in L^\infty(\mathbb{D})$, since $\varphi \in L^\infty(\mathbb{D})$ and $p+1 \leqslant q_0 \leqslant q+1$. Thus we have that

$$\varphi r^{2q_0-2p} - \varphi + (2q_0-2p)h(r) + \left[M+\frac{Ms_0}{q_0}\right]r^{2q_0-p} - Mr^{2q-p} = \frac{Ms_0}{q_0}\frac{1}{r^p}.$$

and $\varphi r^{2q_0-2p}-\varphi+(2q_0-2p)h(r)+\left[M+\frac{Ms_0}{q_0}\right]r^{2q_0-p}-Mr^{2q-p}\in L^\infty(\mathbb{D})$. But

$$\int_D\left|\frac{Ms_0}{q_0}\frac{1}{r^p}\right|^2dA(z)=\left|\frac{Ms_0}{q_0}\right|^2\pi\int_0^1\frac{1}{r^{2p-1}}dr=+\infty,$$

which is contradictory to the fact that $\varphi r^{2q_0-2p}-\varphi+(2q_0-2p)h(r)+\left[M+\frac{Ms_0}{q_0}\right]r^{2q_0-p}$ $-Mr^{2q-p}$ is in $L^\infty(\mathbb{D})$, so $b_{s_0}=0$. Hence we obtain that $b_s=0$ for any integer s with $1\leqslant s\leqslant q-p$.

From the equality (9) and the preceding analysis we get that $b_s=0$ for any integer s with $q-p+1\leqslant s\leqslant n$ and for any any integer n with $n\geqslant q-p+1$, which means that $b_s=0$ for any positive integer s. Thus we obtain that f is analytic, and by Theorem C we obtain that g must be analytic, since T_f and T_g commute.

Conversely, if f and g are both analytic, then it is obvious that T_f and T_g commute. □

By the properties of Toeplitz operators and the above theorem it is easy to obtain the following corollary.

COROLLARY 2.4. *Suppose that* $g(re^{i\theta})=Cr^qe^{-iq\theta}+\varphi(r)e^{-ip\theta}\in L^\infty(\mathbb{D})$, *where* p *and* q *are both positive integers with* $1\leqslant p<q$ *and* $C\neq 0$, *then* T_f *and* T_g *commute if and only if* f *and* g *are both conjugate analytic.*

THEOREM 2.3. *Let* $g(re^{i\theta})=Cr^qe^{iq\theta}+\varphi(r)\in L^\infty(\mathbb{D})$, *where* q *is a positive integer and* $C\neq 0$, *then* T_f *and* T_g *commute if and only if* f *and* g *are both analytic.*

Proof. Since f is a bounded harmonic function on $\mathbb{D}$, there exist functions f_1 and f_2 analytic on $\mathbb{D}$ such that $f=f_1+\overline{f_2}$, where $f_1(z)=\sum_{s=0}^\infty a_sz^s$ and $f_2(z)=\sum_{s=1}^\infty \overline{b_s}z^s$. Let $g_1(re^{i\theta})=\varphi(r)$ and $g_2(re^{i\theta})=Cr^qe^{iq\theta}$, then $g=g_1+g_2$.

Suppose that T_f and T_g commute, then we have

$$[T_{f_1}T_{g_1}-T_{g_1}T_{f_1}](z^n)=[T_{g_2}T_{\overline{f_2}}-T_{\overline{f_2}}T_{g_2}+T_{g_1}T_{\overline{f_2}}-T_{\overline{f_2}}T_{g_1}](z^n) \tag{9}$$

for any positive integers n and obtain that

$$[T_{f_1}T_{g_1}-T_{g_1}T_{f_1}](z^n)=\sum_{s=n}^\infty a_{s-n}\left[2(n+1)\widehat{\varphi}(2n+2)-2(s+1)\widehat{\varphi}(2s+2)\right]z^s, \tag{10}$$

$$
\begin{aligned}
&[T_{g_2}T_{\overline{f_2}} - T_{\overline{f_2}}T_{g_2} + T_{g_1}T_{\overline{f_2}} - T_{\overline{f_2}}T_{g_1}](z^n)\\
&= \sum_{s=0}^{q-1}\left\{ b_{n-s}2(s+1)\left[\frac{s+1}{n+1}\widehat{\varphi}(2s+2) - \widehat{\varphi}(2n+2)\right] - Cb_{n+q-s}\frac{s+1}{n+q+1}\right\} z^s\\
&\quad + \sum_{s=q}^{n-1}\left\{ Cb_{n+q-s}\left[\frac{s-q+1}{n+1} - \frac{s+1}{n+q+1}\right] + b_{n-s}2(s+1)\left[\frac{s+1}{n+1}\widehat{\varphi}(2s+2) - \widehat{\varphi}(2n+2)\right]\right\} z^s\\
&\quad + \sum_{s=n}^{n+q-1} Cb_{n+q-s}\left[\frac{s-q+1}{n+1} - \frac{s+1}{n+q+1}\right] z^s
\end{aligned}
\tag{11}
$$

for any positive integers n with $n \geqslant q+1$.

From the above equalities we obtain that

$$
a_s[2(n+1)\widehat{\varphi}(2n+1) - 2(n+s+1)\widehat{\varphi}(2n+2s+2)] = 0
$$

for any integers s with $s \geqslant q$ and for any integers n with $n \geqslant q+1$. Now we continue the discuss in two cases.

First case. If there exists an integer s_0 with $s_0 \geqslant q$ such that $a_{s_0} \neq 0$, then we have that

$$
2(n+1)\widehat{\varphi}(2n+1) - 2(n+s_0+1)\widehat{\varphi}(2n+2s_0+2) = 0
$$

for any integers n with $n \geqslant q+1$. By Lemma 2.1 we get that φ is constant on $\mathbb{D}$, which means that g is analytic on $\mathbb{D}$, so by Theorem C we obtain that f must be analytic on $\mathbb{D}$, since f is not constant.

Second case. If $a_s = 0$ for any integers s with $s \geqslant q$, then we claim that there exists an integer s_1 with $1 \leqslant s_1 \leqslant q-1$ such that $a_{s_1} \neq 0$. Otherwise, if $a_s = 0$ for any integers s with $1 \leqslant s \leqslant q-1$, then we have that f is co-analytic on $\mathbb{D}$, so by Theorem C and the properties of Toeplitz operators we obtain that g must be co-analytic on $\mathbb{D}$, which is contradictory to the assumption of g.

From equalities (10), (11) and (12) we obtain that

$$
\begin{aligned}
&Cb_s\left[\frac{s}{n+1} - \frac{s}{n+q+1}\right]\\
&= a_{q-s}\left[(2n+2q-2s+2)\widehat{\varphi}(2n+2q-2s+2) - (2n+2)\widehat{\varphi}(2n+2)\right]
\end{aligned}
\tag{12}
$$

for all integers s with $1 \leqslant s \leqslant q$ and for any integers n with $n \geqslant q+1$.

Now we claim that $b_s = 0$ for all integers s with $1 \leqslant s \leqslant q$. From the equality (13) it is evident that $b_q = 0$, so we only need to prove that $b_s = 0$ for all integers s with $1 \leqslant s \leqslant q-1$.

Suppose that there exists an integer s_0 with $1 \leqslant s_0 \leqslant q-1$ such that $b_{s_0} \neq 0$. Then by the equality (13) we have that $a_{q-s_0} \neq 0$ and

$$
(2n+2q-2s_0+2)\widehat{\varphi}(2n+2q-2s_0+2) - (2n+2)\widehat{\varphi}(2n+2) = M\left[\frac{s_0}{n+1} - \frac{s_0}{n+q+1}\right]
$$

for any integers n with $n \geqslant q+1$, where $M = \frac{Cb_{s_0}}{a_{q-s_0}} \neq 0$.

Let $m_0 = q - s_0$, then $1 \leqslant m_0 \leqslant q-1$. Then by the above equality we get that

$$\begin{aligned}&\widehat{\varphi}(2n+2m_0+2) + \frac{2m_0}{2n+2}\widehat{\varphi}(2n+2m_0+2) - \widehat{\varphi}(2n+2)\\ &= M2s_0\left[\frac{1}{2n+2}\cdot\frac{1}{2n+2} - \frac{1}{2q}\frac{1}{2n+2} + \frac{1}{2q}\frac{1}{2n+2q+2}\right]\end{aligned}$$

for any integers n with $n \geqslant q+1$, that is,

$$\begin{aligned}&\widehat{\varphi r^{2m_0}}(2n+2) + 2m_0\widehat{1}(2n+2)\cdot\widehat{\varphi r^{2m_0}}(2n+2) - \widehat{\varphi}(2n+2)\\ &= M2s_0\left[\widehat{1}(2n+2)\cdot\widehat{1}(2n+2) - \frac{1}{2q}\widehat{1}(2n+2) + \frac{1}{2q}\widehat{r^{2q}}(2n+2)\right]\end{aligned}$$

for any integers n with $n \geqslant q+1$. So by the properties of Mellin transform and Mellin convolution we obtain that

$$\varphi r^{2m_0} + 2m_0(\varphi r^{2m_0}) * 1 - \varphi = M2s_0\left[1*1 - \frac{1}{2q} + \frac{1}{2q}r^{2q}\right],$$

$(1*1)(r) = \int_r^1 \frac{1}{t}dt = -\ln r$ and $1*(\varphi r^{2m_0})(r) = \int_r^1 \varphi(t)t^{2m_0-1}dt$. Since $g \in L^\infty(\mathbb{D})$, we have that $\varphi \in L^\infty(\mathbb{D})$ and $\left|1*(\varphi r^{2m_0})(r)\right| \leqslant \int_r^1 |\varphi(t)|t^{2m_0-1}dt \leqslant \int_0^1 |\varphi(t)|t^{2m_0-1}dt \leqslant \frac{\|\varphi\|_\infty}{2m_0} < \infty$ for any real numbers r with $0 \leqslant r < 1$, which means that $1*(\varphi r^{2m_0}) \in L^\infty(\mathbb{D})$. Thus we get that

$$\ln r = \frac{1}{M2s_0}\left[\varphi - \varphi r^{2m_0} - 2m_0(\varphi r^{2m_0})*1\right] - \frac{1}{2q} + \frac{1}{2q}r^{2q} \in L^\infty(\mathbb{D}),$$

which is contradictory to the fact that $\ln r$ doesn't belong to $L^\infty(\mathbb{D})$. Hence the assumption doesn't hold, that is, $b_s = 0$ for all integers s with $1 \leqslant s \leqslant q-1$.

Then by the equality (13) we obtain that

$$a_{q-s}\left[(2n+2q-2s+2)\widehat{\varphi}(2n+2q-2s+2) - (2n+2)\widehat{\varphi}(2n+2)\right] = 0$$

for all integers s with $1 \leqslant s \leqslant q$ and for any integers n with $n \geqslant q+1$.

Since $a_{s_1} \neq 0$, where $1 \leqslant s_1 \leqslant q-1$, then from the above equality we get that

$$(2n+2q-2s_1+2)\widehat{\varphi}(2n+2q-2s_1+2) - (2n+2)\widehat{\varphi}(2n+2) = 0$$

for any integers n with $n \geqslant q+1$. So by Lemma 2.1 we have that φ is constant on $\mathbb{D}$, which means that g is analytic on $\mathbb{D}$. Thus by Theorem C we obtain that f must be analytic on $\mathbb{D}$, since f is not constant.

Conversely, if f and g are both analytic, then it is obvious that T_f and T_g commute. □

By the properties of Toeplitz operators and the above theorem we could easily obtain the following corollary.

COROLLARY 2.5. *Suppose that* $g(re^{i\theta}) = Cr^q e^{-iq\theta} + \varphi(r) \in L^\infty(\mathbb{D})$, *where* q *is a positive integer and* $C \neq 0$, *then* T_f *and* T_g *commute if and only if* f *and* g *are both conjugate analytic.*

THEOREM 2.4. *Let* $g(re^{i\theta}) = Cr^p e^{-ip\theta} + \varphi(r)e^{iq\theta} \in L^\infty(\mathbb{D})$, *where* p *and* q *are both positive integers and* $C \neq 0$, *then* T_f *and* T_g *commute if and only if one of the following conditions is satisfied:*

(1) f *and* g *are both conjugate analytic;*

(2) *there exist scalars* α *and* β, *not both zero, such that* $\alpha f + \beta g$ *is constant on* $\mathbb{D}$.

Proof. Since f is a bounded harmonic function on $\mathbb{D}$, there exist functions f_1 and f_2 analytic on $\mathbb{D}$ such that $f = f_1 + \overline{f_2}$, where $f_1(z) = \sum_{s=0}^\infty a_s z^s$ and $f_2(z) = \sum_{s=1}^\infty \overline{b_s} z^s$. Let $g_1(re^{i\theta}) = Cr^p e^{-ip\theta}$ and $g_2(re^{i\theta}) = \varphi(r)e^{iq\theta}$, then $g = g_1 + g_2$.

Suppose that T_f and T_g commute, then we have

$$[T_{f_1}T_{g_1} - T_{g_1}T_{f_1} + T_{f_1}T_{g_2} - T_{g_2}T_{f_1}](z^n) = [T_{g_2}T_{\overline{f_2}} - T_{\overline{f_2}}T_{g_2}](z^n)$$

for any positive integers n and obtain that

$$[T_{f_1}T_{g_1} - T_{g_1}T_{f_1} + T_{f_1}T_{g_2} - T_{g_2}T_{f_1}](z^n) = \sum_{s=0}^{\infty} Ca_s \left[\frac{p}{n+1+s} - \frac{p}{n+1}\right] z^{s+n-p}$$
$$+ \sum_{s=0}^{\infty} a_s \Big[(2n+2q+2)\widehat{\varphi}(2n+q+2) - (2n+2s+2q+2)\widehat{\varphi}(2n+2s+q+2)\Big] z^{s+n+q},$$

$$[T_{g_2}T_{\overline{f_2}} - T_{\overline{f_2}}T_{g_2}](z^n) = -\sum_{s=n+1}^{n+q} b_s 2(n+q-s+1)\widehat{\varphi}(2n+q+2) z^{n+q-s}$$
$$+ \sum_{s=1}^{n} b_s 2(n+q-s+1)\left[\frac{n-s+1}{n+1}\widehat{\varphi}(2n-2s+q+2) - \widehat{\varphi}(2n+q+2)\right] z^{n+q-s}$$

for any positive integers n with $n \geqslant p$.

From the above equalities we obtain that

$$a_s[(2n+2s+2q+2)\widehat{\varphi}(2n+2s+q+2) - (2n+2q+2)\widehat{\varphi}(2n+q+2)]$$
$$= Ca_{s+p+q}\left[\frac{2p}{2n+2p+2q+2s+2} - \frac{2p}{2n+2}\right]$$

for any nonnegative integers s and for any integers n with $n \geqslant p$. Now we continue the discuss in two cases.

First case. If $a_s = 0$ for all positive integers s, we have that f is co-analytic on $\mathbb{D}$, since f is not constant on $\mathbb{D}$. So by Theorem C we obtain that g must also be co-analytic on $\mathbb{D}$.

Second case. If there exists an positive integer s_0 such that $a_{s_0} \neq 0$, then we have

$$(2n+2s_0+2q+2)\widehat{\varphi}(2n+2s_0+q+2) - (2n+2q+2)\widehat{\varphi}(2n+q+2)$$
$$= C_1 a_{s_0+p+q}\left[\frac{2p}{2n+2p+2q+2s_0+2} - \frac{2p}{2n+2}\right]$$

for any integers n with $n \geqslant p$, where $C_1 = \frac{C}{a_{s_0}}$. Now we claim that $a_{s_0+p+q} = 0$.

Otherwise, suppose that $a_{s_0+p+q} \neq 0$, then we have

$$\begin{aligned}&\widehat{\varphi r^{2s_0+q}}(2n+2) - \widehat{\varphi r^q}(2n+2) + 2s_0 \widehat{r^{2s_0+2q}}(2n+2) \cdot \widehat{\varphi r^q}(2n+2) \\ &= M\left[M_1 \widehat{r^{2s_0+2q}}(2n+2) - \frac{1}{2p}\widehat{r^{2s_0+2q+2p}}(2n+2) - \frac{1}{2p+2s_0}\widehat{1}(2n+2)\right],\end{aligned}$$

for any integers n with $n \geqslant p$, where $M = C_1 a_{s_0+p+q} \neq 0$ and $M_1 = \frac{1}{2p} + \frac{1}{2p+2s_0} \neq 0$. So by the properties of Mellin transform we obtain that

$$\varphi r^{2s_0+q} - \varphi r^q + 2s_0 r^{2s_0+2q} * (\varphi r^q) = M\left[M_1 r^{2s_0+2q} - \frac{1}{2p} r^{2s_0+2q+2p} - \frac{1}{2p+2s_0}\right],$$

and

$$[r^{2s_0+2q} * (\varphi r^q)](r) = \int_r^1 \varphi\left(\frac{r}{t}\right)\left(\frac{r}{t}\right)^q t^{2s_0+2q}\frac{1}{t}dt = r^q \int_r^1 \varphi\left(\frac{r}{t}\right) t^{2s_0+q-1}dt \doteq r^q h(r)$$

for any nonnegative scalars r with $0 \leqslant r < 1$. Since $g \in L^\infty(\mathbb{D})$, then we have $\varphi \in L^\infty(\mathbb{D})$ and

$$|h(r)| \leqslant \int_r^1 \left|\varphi\left(\frac{r}{t}\right)\right| t^{2s_0+q-1}dt \leqslant \|\varphi\|_\infty \int_0^1 t^{2s_0+q-1}dt = \frac{\|\varphi\|_\infty}{2s_0+q} < \infty,$$

that is, $h \in L^\infty(\mathbb{D})$. Hence we have that

$$\frac{1}{r^q} = (2p+2s_0)\left[M_1 r^{2s_0+q} - \frac{1}{2p} r^{2s_0+q+2p} - \frac{1}{p}\right] - \frac{2p+2s_0}{M}\left[\varphi r^{2s_0} - \varphi + 2s_0 h(r)\right] \in L^\infty(\mathbb{D}),$$

which is contradictory to the fact that $\frac{1}{r^q}$ doesn't belong to $L^\infty(\mathbb{D})$, since $q > 0$. Thus the assumption doesn't hold, which means that $a_{s_0+p+q} = 0$.

Then we have

$$(2n+2s_0+2q+2)\widehat{\varphi}(2n+2s_0+q+2) - (2n+2q+2)\widehat{\varphi}(2n+q+2) = 0$$

for any integers n with $n \geqslant p$. So by Lemma 2.2 we get that $\varphi(r) = C_2 r^q$ and $g(z) = C\overline{z}^p + C_2 z^q$. Since $C \neq 0$ and f is a bounded harmonic function on $\mathbb{D}$ and not constant, by Theorem B we obtain that f and g are both co-analytic or there exist scalars α and β, not both zero, such that $\alpha f + \beta g$ is constant on $\mathbb{D}$.

Conversely, if f and g satisfy the condition (1) or (2) , then it is obvious that T_f and T_g commute. □

By the properties of Toeplitz operators and the above theorem it is easy to obtain the following corollary.

COROLLARY 2.6. *Let* $g(re^{i\theta}) = Cr^p e^{ip\theta} + \varphi(r)e^{-iq\theta} \in L^\infty(\mathbb{D})$, *where* p *and* q *are both positive integers and* $C \neq 0$, *then* T_f *and* T_g *commute if and only if one of the following conditions is satisfied:*

(1) f *and* g *are both analytic;*

(2) *there exist scalars* α *and* β, *not both zero, such that* $\alpha f + \beta g$ *is constant on* $\mathbb{D}$.

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