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$\mathscr{C}$–SYMMETRIC SECOND ORDER DIFFERENTIAL OPERATORS

HORST BEHNCKE AND DON HINTON

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Abstract. We consider a $\mathscr{C}$-Symmetric second order linear differential operator on a half interval or the real line. We determine the spectrum and construct the resolvent and m-function. In addition we analyze the resolvent and m-function near their poles. Under the conditions of Theorem 2.2 we prove the essential spectrum is empty, and the operator has a compact resolvent. Integral conditions on the operator coefficients are given in Theorem 3.4 for the operator to be Hilbert-Schmidt. These conditions are new even in the selfadjoint case. This analysis is based on asymptotic integration. A central role is played by the Titchmarsh-Weyl m-function which is defined by square integrable functions and not by a nesting circle analysis.

1. Introduction

Second order linear differential equations with complex valued potential term, and more generally complex symmetric operators have many applications. There has been considerable research in recent years in this area including complex symmetric matrices which are the most basic of complex symmetric operators. We refer the reader to the survey article by Garcia, Prodan, and Putinar [17] for applications and recent results.

Here we consider the singular second order operator

$$L[y] = \frac{1}{w}\left[-(py')' + qy\right], \quad a \leqslant x < \infty \text{ or} - \infty < x < \infty, \tag{1.1}$$

where $w > 0$ is real and $p = p_1 + ip_2 \neq 0$, $q = q_1 + iq_2 \neq 0$ are complex valued. Further in section 2 we will assume sectorial conditions on the function q/p.

The classical Sturm-Liouville equation with real coefficients is certainly one of the most studied differential equation. Thousands of scientific papers contrast with fewer than a hundred or so for the complex version. This is not surprising as the spectral theorem and Sturm's results are no more available, and in fact new phenomena arise in the complex setting like higher order and non-simple poles or strange spectral behavior. For this reason we restrict ourselves to $\mathscr{C}$-symmetric operators, which for Sturm-Liouville operators is hardly a restriction.

In 1957 Sims [36] extended part of the Titchmarsh-Weyl program to complex Sturm-Liouville operators by constructing the m-function. This was further developed by Brown, McCormack, Evans, and Plum [10], and later extended to non-selfadjoint

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Hamiltonian systems by Brown, Evans, and Plum [11] also by Muzzolini [31]. Their construction is based on the Weyl disc method, which not only requires numerical range conditions but also an auxiliary matrix $\mathscr{U}_{2n}$. The role of this auxiliary matrix remains obscure, even in the Sturm-Liouville case. For selfadjoint operators there are essentially two ways to construct the m-matrix. The Weyl disc method and the square integrability technique. This latter method was used by these authors [8, 9] in conjunction with asymptotic integration. For higher order operators this approach has to be supplemented by numerical range conditions. For second order operators even these are not needed in all cases, though a dichotomy condition is necessary for asymptotic integration. With the knowledge of the asymptotics of the eigenfunctions the resolvent can be constructed with the aid of the m-function. Otherwise the role of the m-function remains obscure. It is somewhat related to the resolvent, but spectral properties cannot be inferred from the imaginary part of m anymore.

In this paper primarily compact or even Hilbert-Schmidt resolvents arise, and we give criteria on the coefficients of L so that the resolvent is Hilbert-Schmidt or even in the Schatten class $\mathscr{C}_p$. In the non-selfadjoint case the resolvent may have poles of order greater than one, which correspond to poles of the m-function of the same order and also to the algebraic multiplicity of the eigenvalue. However in section 8 we consider a complex version of the classical Wigner-von Neumann potential [37] which fails to have an eigenvalue embedded in the essential spectrum.

Our objective is to develop a spectral theory for (1.1) in the case where the spectrum is discrete, but that the numerical range may cover the entire complex plane. The results here extend some of those of Behncke and Hinton [8] where the numerical range was contained in a half plane. But missing under the hypotheses here will be the Dirichlet condition for members of the domain of the maximal operator. The method of analysis in this paper is by asymptotic integration which was not used in [8]. While covering in some sense a larger class of operators the method does require greater smoothness of the coefficients. A Green's function will be constructed with the aid of the Titchmarsh-Weyl function. This leads to a representation of the resolvent operator. Under further conditions it will be proved that the resolvent operator is Hilbert-Schmidt. The Hilbert-Schmidt condition gives a new criterion even in the self-adjoint case. The analysis here includes some self-adjoint operators as well as the $\mathscr{C}$-symmetric ones. We will define an m-function without the aid of a nesting circle analysis.

Even if the notation is largely standard, a few remarks on this are needed. We study the operator (1.1). It will act in the weighted Hilbert space $\mathscr{L}^2_w[a,\infty)$ or in $\mathscr{L}^2_w(-\infty,\infty)$. The norm and inner product will be denoted by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, respectively.

We now give some definitions and quote some basic results. With the conditions w is continuous and $1/p, q$ are locally Lebesgue integrable, the differential expression L determines a maximal operator T and an unclosed minimal operator T_0' defined by the action of L on the domains, respectively, for the case $[a,\infty)$, with similar definitions for $(-\infty,\infty)$,

$$D(T) = \{y \in \mathscr{L}^2_w[a,\infty) : y, py' \in AC_{\mathrm{loc}} \text{ and } L[y] \in \mathscr{L}^2_w[a,\infty)\}$$

and

$$D(T_0') = \{y \in D(T) : y \text{ has compact support in } (a,\infty)\}$$

where AC_{loc} means locally absolutely continuous. The operator T is closed and T_0' has a closure T_0 and both are densely defined. For a discussion of these properties we refer to the paper of Knowles [27].

The formal adjoint of L is given by

$$L^+[y] = \frac{1}{w}\left[(-\bar{p}y')' + \bar{q}y\right], \tag{1.2}$$

and we define the maximal operator T^+ and minimal operator T_0^+ for L^+ analogous to those for L. We have the adjoint relations Goldberg [20, p. 130] or Kauffman, Read, and Zettl [25, p. 14].

$$T_0^* = T^+, \quad T = T_0^{+*}, \quad T_0 = T^{+*}, \quad T^* = T_0^+.$$

Recall that the numerical range $N(K)$ of a linear operator K acting in a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ is defined by

$$N(K) = \{\langle Kf, f\rangle : f \in D(K), \|f\| = 1\}.$$

In certain cases studied here, the numerical range of T_0 is not all of $\mathbb{C}$, and in this case one can say more about the structure of $\mathscr{C}$-symmetric extensions of T_0. However in the general case studied here the numerical range of T_0 may be $\mathbb{C}$.

For $z \notin N(T_0)$, we have from Kato [24, p. 268] that $T_0 - z$ has a closed range, nullity $T_0 - z = 0$, and the defect of $T_0 - z$ is constant on each connected component of $\overline{N(T_0)}^C$. Under the hypotheses of Theorem 2.2 we will be able to show that $T_0 - z$ has a closed range for all $z \in \mathbb{C}$. From the fact that $T_0 - z$ has a closed range, we also have that $T - z$, $T_0^+ - \bar{z}$, $T^+ - \bar{z}$ all have a closed range Goldberg [20, p. 130] or Kauffman, Read, and Zettl [25, p. 15].

Until section 7, we now only consider the case $[a, \infty)$. Define

$$s = \dim\,(D(T)/D(T_0)). \tag{1.3}$$

Then $s \geqslant 2$ since one can construct compactly supported independent functions y_1, y_2 in $D(T)/D(T_0)$ with initial values $y_1(a) = 1$, $(py_1)'(a) = 0$, $y_2(a) = 0$, $(py_2)'(a) = 1$. Further it follows that when $T_0 - z$ has a closed range, Kauffman, Read, and Zettl [25, p. 16], that

$$s = \operatorname{nul}\,(T - z) + \operatorname{nul}\,(T^+ - \bar{z}). \tag{1.4}$$

In section 2 we will prove in Theorem 2.2 that for all $z \in \mathbb{C}$,

$$\operatorname{nul}\,(T - z) = 1, \quad \operatorname{nul}\,(T^+ - \bar{z}) = 1, \tag{1.5}$$

and in Theorem 3.2 that $T_0 - z$ has a closed range for all $z \in \mathbb{C}$. Under these conditions one has $s = 2$.

For a closed, densely defined operator S on a Hilbert space, the regularity field, $\Pi(S)$, is defined by

$$\Pi(S) = \{z \in \mathbb{C} : \|(S - z)(x)\| \geqslant k_z\|x\|, x \in D(S), \text{ for some } k_z > 0.\}$$

The resolvent set $\rho(S)$ of S is the set of all z in $\Pi(S)$ such that the range of $S-z$ is H. The spectrum $\sigma(S)$ of S is the complement of $\rho(S)$. The set $\sigma(S)$ is the union three sets: the eigenvalues of S, $\sigma_p(S)$, the residual spectrum $\sigma_r(S)$ which is the set of values of $z \notin \sigma_p(S)$ for which the range of $S-z$ is closed but $\neq H$ (a $\mathscr{C}$-selfadjoint operator has no residual spectrum), and finally, the essential spectrum of S, $\sigma_{ess}(S)$ which is the set of z such that the range of $S-z$ is not closed. Glazman [18, p. 9] proves that this is equivalent (when there are no eigenvalues of infinite geometric multiplicity) to there being a singular sequence for z, i.e., a bounded noncompact sequence $\{f_n\}$ such that $(S-z)(f_n) \to 0$ as $n \to \infty$. In general then, $\sigma(S) = \sigma_p(S) \cup \sigma_r(S) \cup \sigma_{ess}(S)$ and $\sigma(S) = \sigma_p(S) \cup \sigma_{ess}(S)$ if S is a $\mathscr{C}$-selfadjoint operator. Let $\mathscr{N}(S)$, respectively, $\mathscr{R}(S)$, denote the nullspace and the range of S. Then we have the well known relations, Kato [24, p. 267], $T_0^* - \bar{z} = T^+ - \bar{z}$, $T_0^{+*} - z = T - z$ and

$$\mathscr{N}(T_0^* - \bar{z}) = (\mathscr{R}(T_0 - z))^{\perp}, \mathscr{N}(T_0^{+*} - \bar{z}) = (\mathscr{R}(T_0^+ - z))^{\perp}.$$

From these we get in Theorem 2.2, for all $z \in \mathbb{C}$,

$$\operatorname{def}(T_0 - z) := \dim(\mathscr{R}(T_0 - z))^{\perp} = 1, \quad \text{and also } \operatorname{def}(T_0^+ - \bar{z}) = 1. \tag{1.6}$$

Knowles [27] has shown that for operators T with $\Pi(T_0) \neq \emptyset$ and $\operatorname{def} T_0 = \dim \mathscr{R}(T)^{\perp} = 1$, then all maximal $\mathscr{C}$-symmetric, i.e., $\mathscr{C}$-selfadjoint, extensions T_1 of T_0 are given by $T_1(y) = T(y)$ on

$$D(T_1) = \{y \in D(T) : \gamma_1 y(a) + \gamma_2 (py')(a) = 0, |\gamma_1| + |\gamma_2| \neq 0\}.$$

The form of $\mathscr{C}$-symmetry in [27] is conjugation and that is the form we use. More general types can be found in [17]. If p, q, γ_1, γ_2 are real, then these maximal $\mathscr{C}$-symmetric extensions T_1 of T_0 are selfadjoint.

The following description, which is adapted from the selfadjoint case, is more convenient. Let $\alpha \in \mathbb{C}$, and restrict T to

$$D(T_\alpha) = \{y \in D(T) : (\cos\alpha) y(a) + (\sin\alpha)(py')(a) = 0\}. \tag{1.7}$$

Then the domain of $D(T_\alpha)$ is a one dimensional extension of $D(T_0)$. In Theorem 3.2 below conditions are given for $z \in \mathbb{C}, T_\alpha - z$ to have a closed range. In Theorem 3.3 conditions are given for $z \in \mathbb{C}, T_\alpha - z$ to have a compact resolvent which ensures that the resolvent set of T_α is nonempty.

The domain $D(T_\alpha^*)$ is given by

$$D(T_\alpha^*) = \{y \in D(T^+) : (\cos\bar{\alpha}) y(a) + (\sin\bar{\alpha})(py')(a) = 0\}.$$

since $\overline{\cos\alpha} = \cos\bar{\alpha}, \overline{\sin\alpha} = \sin\bar{\alpha}$. If we define the conjugation operator $\mathscr{J}$ by $\mathscr{J}(y) = \bar{y}$ for $y \in \mathscr{L}_w^2[a,\infty)$, then we see that $\mathscr{J}$ is one to one from $D(T_\alpha)$ onto $D(T_\alpha^*)$. It also follows from the results of Knowles [27] that T_α is $\mathscr{J}$ selfadjoint, i.e., $T_\alpha = \mathscr{J} T_\alpha^* \mathscr{J}$, and that $\operatorname{nul}(T - z) = \operatorname{nul}(T^+ - \bar{z})$.

We will sometimes need an additional hypothesis to avoid a degenerate case. It is:

$H:$ For no $\alpha \in \mathbb{C}$ does there exist an eigenvalue of T_α of infinite algebraic multiplicity.

We prove in section 6 that the set of such endpoints a for which H fails for some α is a set of isolated points, and further that if an eigenvalue of infinite algebraic multiplicity occurs for one boundary condition then it occurs for no other at the same endpoint. We know of no example of a second order Sturm-Liouville $\mathscr{C}$-symmetric operator with an eigenvalue of infinite algebraic multiplicity. It is obvious that H is satisfied for all operators with a compact resolvent, as they are studied in this paper. Likewise it holds for all operators T, for which $N(T) \neq \mathbb{C}$ as they have been studied in [8, 10, 11]. Finally H holds for all operators $T[y] = -y'' + qy$ with Im q semibounded. This follows from Sim's theory or $N(T) \neq \mathbb{C}$. So it is not surprising that we know of no nontrivial example for this case. We conjecture that such a degenerate case does not occur for operators on the half line.

REMARK 1.1. Under this hypothesis, and since we prove $T_\alpha - z$ is a Fredholm operator for all $z \in \mathbb{C}$, it follows that all $z \in \mathbb{C}$, $z \in \rho(T_\alpha)$ or z is an isolated eigenvalue of T_α, see Locker [28, p. 56]. Eigenvalues of infinite algebraic multiplicity do occur in boundary value problems, one such given by Locker [28, p. 85], is

$$L[y] = -y'', \quad y'(-1) + y'(1) = 0, \quad y(-1) - y(1) = 0,$$

in which case every $z \in \mathbb{C}$ is an eigenvalue. See also the example in Coddington and Levinson [13, p. 300].

By the criterion in Knowles [27], the boundary value problem on $[a,b]$,

$$L[y] = -y'', \quad A \begin{bmatrix} y(a) \\ y'(a) \end{bmatrix} + B \begin{bmatrix} y(b) \\ y'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{rank } [A,B] = 2, \tag{1.8}$$

with complex matrices

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \quad B = \begin{pmatrix} \alpha_{13} & \alpha_{14} \\ \alpha_{23} & \alpha_{24} \end{pmatrix},$$

is $\mathscr{C}$-symmetric iff

$$\text{det A} = \text{det B} \Leftrightarrow AJA^T = BJB^T, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The problem (1.8) is selfadjoint iff $AJA^* = BJB^*$, see problem 1 [13, p. 297]. The problem of Locker above is not $\mathscr{C}$-symmetric.

The spectral theory of $\mathscr{C}$-symmetric operators is far more complicated than that of their real brethren. There are operators where the m-function has a higher order pole,e.g., see Example 5.2, Also there are examples for which the operator has an empty spectrum, e.g., see the example in [13, p. 300]. We prove in Example 1.1 this can also occur with a $\mathscr{C}$-symmetric case of (1.8) (but in only two cases), and we also prove no $\mathscr{C}$-symmetric problem of (1.8) can have an eigenvalue of infinite algebraic multiplicity. By translation and scaling we can take $[a,b] = [0,1]$.

EXAMPLE 1.1. By the criterion of [13, p. 302], the BVP (1.8) has eigenvalues the zeros of an entire function so that eigenvalues of infinite algebraic multiplicity do not occur unless

$$A_{24} = 0, \quad A_{13} = 0, \quad A_{23} = A_{14} \tag{1.9}$$

where

$$A_{jk} = \det \begin{pmatrix} \alpha_{1j} & \alpha_{1k} \\ \alpha_{2j} & \alpha_{2k} \end{pmatrix}.$$

Suppose now (1.8) is $\mathscr{C}$-symmetric and (1.9) holds. First consider the case $\det A \neq 0$. Since we may multiply the boundary conditions by A^{-1} to obtain equivalent boundary conditions, we may assume $A = I$. With $A = I$, we have

$$A_{24} = -\alpha_{14} = 0, \quad A_{13} = \alpha_{23} = 0, \quad A_{23} = A_{14} \Leftrightarrow \alpha_{13} = -\alpha_{24}.$$

Let $c = \alpha_{13}$. Then the boundary conditions of (1.8) are

$$y(0) = -cy(1), \quad y'(0) = cy'(1) \tag{1.10}$$

which is $\mathscr{C}$-symmetric iff $c^2 = -1$ or $c = \pm i$. We now show these boundary conditions have no eigenvalues. For $z = 0$, the general solution of $-y'' = 0$ is $y(x) = c_0 + c_1 x$ and substitution into (1.10) implies $c_0 = c_1 = 0$. For $z = s^2 \neq 0$, the general solution of $-y'' = zy$ is $y(x) = c_1 \exp(isx) + c_2 \exp(-isx)$. One may set $c_1 = 1$ and a simple computation shows that (1.10) is incompatible with $c = \pm i$. Thus with boundary conditions (1.10) there are no eigenvalues. If T_1 is the $\mathscr{C}$-symmetric operator defined by $L[y] = -y''$ and (1.10), then $(T_1 - z)^{-1}$ is compact for all $z \in \mathbb{C}$ as is seen from computing the Green's function. Hence $\sigma(T_1) = \emptyset$, and $\sigma((T_1 - z)^{-1}) = \sigma_{ess}((T_1 - z)^{-1}) = \{-z\}$ as the range of $(T_1 - z)^{-1} = D(T_1)$ is not closed.

Secondly, let $\det A = 0$. Then rank $A =$rank $B = 1$ for otherwise rank $[A, B] < 2$. Now an elementary study of all cases shows that in this case there are no $\mathscr{C}$-symmetric boundary conditions satisfying (1.9).

An example of a singular half line operator has been given by McLeod [30] on $\mathscr{L}^2[0,\infty)$ which has $\sigma_p(T_\alpha) = \emptyset$ for all α. The equation is

$$-y'' - 2ie^{2(1+i)x}y = zy.$$

McLeod proves the solutions of this equation can be expressed in terms of Bessel functions and no nontrivial solution is in $\mathscr{L}^2[0,\infty)$. A less artificial example is the Airy equation of example 7.1. Both Sims and McLeod have shown that $\mathscr{L}^2$ solutions exist if Im q is semibounded. So the Airy equation is about the simplest operator, that shows the limits of Sim's theorem, one can think of.

2. Asymptotic solutions

In this section we derive conditions for the asymptotic solutions of (1.1). There are quite a number of results on the asymptotics of Sturm-Liouville equations. For us the most convenient is Corollary 2.2.1 of Eastham below [15, p. 58] for the equation

$$-(py')' + qy = 0 \tag{2.1}$$

which has to be adapted to spectral problems.

THEOREM 2.1. *Suppose the complex valued functions p and q are nowhere zero and have absolutely continuous first derivatives on $[a,\infty)$. Assume the following hold:*

$$(pq)'/pq = o((q/p)^{1/2}) \quad (x \to \infty), \tag{2.2}$$

$$[p^{-1/2}q^{-3/2}(pq)']',\, p^{-3/2}q^{-5/2}((pq)')^2 \in \mathscr{L}[a,\infty), \tag{2.3}$$

$$re\,(q/p)^{1/2} \text{ has one sign in } [a,\infty). \tag{2.4}$$

Then (2.1) *has solutions y_1 and y_2 such that as $x \to \infty$,*

$$y_1(x) = (pq)^{-1/4}(x)[1+o(1)]\exp\left(-\int_a^x (q/p)^{1/2}\,dt\right), \tag{2.5}$$

$$(py_1')(x) = -(pq)^{1/4}(x)[1+o(1)]\exp\left(-\int_a^x (q/p)^{1/2}\,dt\right), \tag{2.6}$$

with similar formulae for y_2 deleting the minus signs.

We want to apply Theorem 2.1 to the equation

$$L[y] = \frac{1}{w}[-(py')' + qy] = zy, \tag{2.7}$$

where the hypotheses are independent of z.

THEOREM 2.2. *Suppose $w(x) > 0$ and the complex valued functions p and q are nowhere zero and w, p, q have absolutely continuous first derivatives on $[a,\infty)$. Assume the following hold:*

$$p'/p,\, q'/q,\, w'/w = o((q/p)^{1/2}) \quad (x \to \infty), \tag{2.8}$$

$$p^{-1/2}q^{-5/2}(pq)'q',\, p^{-3/2}q^{-3/2}(pq)'p' \in \mathscr{L}[a,\infty), \tag{2.9}$$

$$w/q = o(1),\, (w/q)' = O((pq)'/pq) \quad (x \to \infty), \tag{2.10}$$

$$(pq)''p^{-1/2}q^{-3/2},\, p^{1/2}q^{-1/2}(w/q)'' \in \mathscr{L}[a,\infty), \tag{2.11}$$

$$\text{for some } \delta > 0,\, -\pi+\delta \leqslant \arg\,(q/p) \leqslant \pi-\delta. \tag{2.12}$$

Then (2.7) *has solutions y_1 and y_2 such that as $x \to \infty$, with $\tilde{q} = q - zw$,*

$$y_1(x) = (pq)^{-1/4}(x)[1+o(1)]\exp\left(-\int_a^x (\tilde{q}/p)^{1/2}\,dt\right), \tag{2.13}$$

$$(py_1')(x) = -(pq)^{1/4}(x)[1+o(1)]\exp\left(-\int_a^x (\tilde{q}/p)^{1/2}\,dt\right), \tag{2.14}$$

with similar formulae for y_2 deleting the minus signs. Further $y_1 \in \mathscr{L}_w^2[a,\infty)$ and $y_2 \notin \mathscr{L}_w^2[a,\infty)$.

REMARK 2.3. Note that $\bar{p},\bar{q}$ will also satisfy the hypotheses of Theorem 2.2 when p,q do. Hence under these conditions for all $z\in\mathbb{C}$, $\dim \mathcal{N}(T-z)=\dim \mathcal{N}(T^+-\bar{z})=1$. We will sometimes use $(\tilde{q}/p)^{1/2}=(q/p)^{1/2}[1+o(1)]$ in the above asymptotic formulae.

Proof. We will first show the conditions (2.8), (2.9), (2.11) (first part), and (2.12), of Theorem 2.2 imply the conditions of Theorem 2.1. First note that

$$\begin{aligned}(2.8)&\Rightarrow (pq)'/(pq)=o((q/p)^{1/2}),\\ (2.9)&\Rightarrow p^{-3/2}q^{-5/2}((pq)')^2\in\mathcal{L}[a,\infty),\\ (2.9),\ (2.11)&\Rightarrow [p^{-1/2}q^{-3/2}(pq)']'\in\mathcal{L}[a,\infty).\end{aligned}$$

Clearly, (2.4) follows from (2.12).

Let $\tilde{q}=q-zw=q[1+o(1)]$. We now proceed to prove that conditions (2.8), (2.9), (2.11) (first part), and (2.12), of Theorem 2.2 hold with q replaced by $\tilde{q}$. This is where we need additional conditions on w. Adding w to q will imply that all necessary inequalities hold asymptotically. So it may be necessary to restrict the interval $[a,\infty)$. The results of (2.5), (2.6), however remain valid. This means that (2.8), (2.12) hold for some $0<\delta'<\delta$ on some $[a',\infty)$. Without loss of generality we take this to be δ on $[a,\infty)$ and assume also $|w/q|\leqslant 1/2\sin(\delta/2)$. A calculation shows that the second condition of (2.10) implies (2.8) holds for $\tilde{q}$.

We now consider (2.9).

$$\begin{aligned}p^{-1/2}\tilde{q}^{-5/2}(p\tilde{q})'\tilde{q}'&=p^{-1/2}q^{-5/2}\left(1-\frac{zw}{q}\right)^{-5/2}\\ &\quad\times\left[(pq)'\left(1-\frac{zw}{q}\right)-zpq\left(\frac{w}{q}\right)'\right]\left[q'\left(1-\frac{zw}{q}\right)-zq\left(\frac{w}{q}\right)'\right].\end{aligned}\tag{2.15}$$

Using (2.10), we have

$$p^{-1/2}\tilde{q}^{-5/2}(p\tilde{q})'\tilde{q}'=p^{-1/2}q^{-5/2}O((pq)')[O(q')+O((pq)'/p)]\in\mathcal{L}[a,\infty).$$

The expression for $p^{-1/2}\tilde{q}^{-5/2}(p\tilde{q})'p'$ is similar, but simpler. Thus (2.9) holds for $\tilde{q}$. Finally we must prove the first part of (2.11) holds for $\tilde{q}$.

A calculation gives that

$$\begin{aligned}(p\tilde{q})''&=(pq)''p^{-1/2}q^{-3/2}\left(1-\frac{zw}{q}\right)^{-1/2}\\ &\quad+2z(pq)'p^{-1/2}q^{-3/2}\left(1-\frac{zw}{q}\right)^{-3/2}(w/q)'+zp^{1/2}q^{-1/2}\left(1-\frac{zw}{q}\right)^{-1/2}(w/q)''.\end{aligned}\tag{2.16}$$

The first term of (2.16) $\in\mathcal{L}[a,\infty)$ by (2.11), the second term of (2.16) $\in\mathcal{L}[a,\infty)$ by (2.9), and the last term of (2.16) $\in\mathcal{L}[a,\infty)$ by (2.11).

We now show $y_1 \in \mathscr{L}_w^2[a,\infty)$. We will use below and later that (2.8) gives

$$\frac{p(x)}{p(s)} = \exp\left(\int_s^x \frac{p'(t)}{p(t)}\,dt\right) = \exp\left(\int_s^x o\left(\frac{q(t)}{p(t)}\right)^{1/2} dt\right),\ s \leqslant x, \tag{2.17}$$

with similar expressions for q, w.

If we write $q/p = \rho e^{i\phi}$, $\rho = |q/p|$, $-\pi+\delta \leqslant \phi \leqslant \pi-\delta$, then $(q/p)^{1/2} = \rho^{1/2}e^{i\phi/2}$. Thus

$$y_1(x) = q(x)^{-1/2}\left[\frac{q(x)}{p(x)}\right]^{1/4}[1+o(1)]\exp\left(-\int_a^x (q/p)^{1/2}[1+o(1)]dt\right)$$

and using

$$\left|\exp\left(-2i\int_a^x \rho(t)^{1/2}\sin(\phi(t)/2)\,dt\right)\right| = 1,$$

we have, with $\cos(\phi(t)/2) \geqslant \cos((\pi-\delta)/2) = \sin(\delta/2)$,

$$\begin{aligned} w(x)|y_1(x)|^2 &= \left|\frac{w(x)}{q(x)}\right|\rho(x)^{1/2}[1+o(1)]\exp\left(-2\int_a^x \rho(t)^{1/2}[\cos(\phi(t)/2)+o(1)]\right)dt \\ &\leqslant \left|\frac{w(x)}{q(x)}\right|\rho(x)^{1/2}[1+o(1)]\exp\left(-2\int_a^x \rho(t)^{1/2}[\sin(\delta/2)+o(1)]\right)dt. \end{aligned} \tag{2.18}$$

Since $w/q \to 0$ as $x \to \infty$ and

$$\int_a^\infty \rho(x)^{1/2}\exp\left(-2\int_a^x \rho(t)^{1/2}\sin(\delta/2)\,dt\right)dx = \frac{1}{2\sin(\delta/2)} < \infty, \tag{2.19}$$

we have $y_1 \in \mathscr{L}_w^2[a,\infty)$.

A similar analysis shows that $|q|^{1/2}y_1 \in \mathscr{L}_2([a,\infty))$ and using asymptotically $py' = (pq)^{1/2}y_1$, we also obtain $|p|^{1/2}y_1' \in \mathscr{L}_2([a,\infty))$. Thus y_1 satisfies a Dirichlet condition. Multiplying (2.7) by $w\bar{y}_1$ and integrating by parts gives

$$(py_1'\bar{y}_1)(a) + \int_a^\infty [p|y_1'|^2 + q|y_1|^2]\,dt = z\int_a^\infty w|y_1|^2\,dt. \tag{2.20}$$

To prove $y_2 \notin \mathscr{L}_w^2[a,\infty)$, we first show $\int_a^\infty \rho^{1/2}\,dt = \infty$. From $\rho^2 = q\bar{q}/p\bar{p}$, differentation, and using (2.8), it follows that $\rho'/\rho^{3/2} = o(1)$. Thus $\rho'(t)/\rho^{3/2}(t) \geqslant -1$ on some $[t_0,\infty)$, and an integration yields, for $t \geqslant t_0$,

$$2\rho^{-1/2}(t_0) + t - t_0 \geqslant 2\rho^{-1/2}(t)$$

from which $\int_a^\infty \rho^{1/2}\,dt = \infty$ follows.

Following a similar argument, and using (2.17) for q, w,

$$\frac{w(x)}{w(a)}, \frac{q(x)}{q(a)} = \exp\left(\int_a^x o\left(\frac{q(t)}{p(t)}\right)^{1/2} dt\right) \Rightarrow \frac{w(x)}{q(x)} = \exp\left(\int_a^x o\left(\frac{q(t)}{p(t)}\right)^{1/2} dt\right),$$

as in (2.18), we arrive at

$$w(x)|y_2(x)|^2 = \rho(x)^{1/2}[1+o(1)]\exp\left(2\int_a^x \rho(t)^{1/2}[\cos(\phi(t)/2)+o(1)]\right)dt$$
$$\geqslant \rho(x)^{1/2}[1+o(1)]$$

which gives $y_2 \notin \mathscr{L}_w^2[a,\infty)$ since $\int_a^\infty \rho^{1/2}\,dt = \infty$. □

EXAMPLE 2.1. Let $p(x) = x^\alpha$, $w(x) = x^\beta$, $q(x) = c_1x^{\delta_1} + \ldots + c_kx^{\delta_k}$ on $[a,\infty)$, $a > 0$, where α, β, δ_i are real, $c_i \in \mathbb{C}$ with $c_1 \notin (-\infty,0]$, and $\delta_1 > \delta_2 > \ldots > \delta_k$. Further suppose $\delta_1 > \beta$ and $\alpha - \delta_1 < 2$. Then all conditions of Theorem 2.2 are satisfied.

REMARK 2.4. Equation (2.20) can be used for partial location of eigenvalues. Note an eigenfunction must be a multiple of y_1 as $y_2 \notin \mathscr{L}_w^2[a,\infty)$. For example, if p is real and the boundary condition is $y(a) = h(py')(a)$, h real , and $z = z_1 + iz_2$ is an eigenvalue, then taking imaginary parts of (2.20) with $q = q_1 + iq_2$ gives

$$\int_a^\infty q_2|y_1|^2\,dx = z_2\int_a^\infty w|y_1|^2\,dx. \tag{2.21}$$

Thus if $q_2 \geqslant d$, ($q_2 \leqslant d$), it then follows that $z_2 \geqslant d$ ($z_2 \leqslant d$).

The standard form of (1.2) for $p = w = 1$ can achieved by the Kummer Liouville transformation. In this case the conditions of Theorem 2.1 and Theorem 2.2 are somewhat simpler. The Kummer Liouville transformation is based on the transformation

$$y(x) = \mu(x)u(t), \quad \mu(x) > 0, \quad t = f(x), \quad \gamma := f'(x) > 0.$$

Details may be found in Ahlbrandt, Hinton, and Lewis [2] or in Behncke and Hinton [3]. In this case we have, with

$$' = d/dx, \quad \cdot = d/dt, \quad t = f(x) = \int_a^x (w/p)^{1/2}\,ds, \quad \mu = 1/(pw)^{1/4},$$

that

$$-(py')' + qy = zwy, \quad \Leftrightarrow -\ddot{u} + Q(t)u = zu,$$

where

$$Q(t) = \frac{\mu}{\gamma}\left[-\left(p\mu'\right)' + q\mu\right] = -\frac{\mu}{\gamma}(p\mu')' + \frac{q}{w},$$

since $\mu^2 w = \gamma = (w/p)^{1/2}$. Note that the term $(\mu/\gamma)(p\mu')'$ will under general conditions be small with respect to q/w. However imposing the conditions of Theorem 2.2 on the transformed equations makes for rather complicated hypotheses. In Example 2.1, $\gamma(x) = x^{(\beta-\alpha)/2}$ and $\mu(x) = x^{-(\alpha+\beta)/4}$.

Asymptotic tools can also be used to cover the case where essential spectrum arises. As an example consider a special case [9],

$$L[y] = -(py')' + qy, \quad 1/p = p_0 + p_1 + p_2, \quad q = q_0 + q_1 + q_2, \tag{2.22}$$

where $p_0 \neq 0, q_0$ are constants, $p_1, q_1 \to 0$ as $x \to \infty$, and p_2, q_2 are integrable on $[a,\infty)$, then the operator T has an essential spectrum contained in the algebraic curve

$$\Sigma := \{z \in \mathbb{C} : z = p_0\lambda^2 + q_0 \text{ for some real } \lambda\}.$$

Further, by using singular sequences, it can be shown that all $z \in \Sigma$ belong to the essential spectrum of T. The conditions on p, q above in (2.22) are too weak to obtain asymptotic solutions of $L[y] = -(py')' + qy = zy$ as one generally must require the Levinson dichotomy condition. However they are sufficient to obtain an exponential dichotomy, c.f., Ju and Wiggins [23].

These results on asymptotic integration can be extended to a situation where the operators satisfy less smoothness but rather combination of smoothness and decay. For this one decomposes the coefficient q as $q = q_1 + q_2 + \ldots + q_m$ so that q_k becomes integrable at the k-th diagonalization. For the simple equation $-y'' + qy = zy$, this is particularly easy to follow [6, 9].

3. The operator R

In this section we assume the conditions of Theorem 2.2. General results for Sturm-Liouville operators imply the Wronskian $W(y_1,y_2)$ to be independent of x. Hence evaluating $W(y_1,y_2)$, with y_1, y_2 given by Theorem 2.2, shows that

$$W(y_1,y_2) = y_1(x)(py_2')(x) - y_2(x)(py_1')(x) = 2.$$

With this we can define an operator $R = R(z)$ on $\mathscr{L}^2_w[a,\infty)$ by

$$(Rf)(x) = \int_a^x \frac{1}{2} y_1(x)y_2(s)w(s)f(s)\,ds + \int_x^\infty \frac{1}{2} y_2(x)y_1(s)w(s)f(s)\,ds. \tag{3.1}$$

As is to be expected from the selfadjoint case R will turn out to be the resolvent of $L[y] = (1/w)[-(py')' - qy]$. For this reason denote the kernel of the first summand of (3.1) by K_+ and that of the second by K_-. Since $y_1 \in \mathscr{L}^2_w[a,\infty)$, it is clear that Rf is defined. Let $y(x) = (Rf)(x)$. Then a calculation shows that $-(py')' + qy = zwy + wf$. The following special case of a theorem of Okikiolu [32, p. 190] will prove the boundedness of R.

THEOREM 3.1. *Let the measures on $X, Y \subseteq [a,\infty)$ be defined by $m_X(x) = w(x)dx$, $m_Y(y) = w(y)dy$, and let $K(x,y)$ be a measurable function on $[a,\infty)\times[a,\infty)$ such that*

$$\int_X |K(x,y)|\,dm_X(x) \leqslant M_1^2, a.e., y; \quad \int_Y |K(x,y)|\,dm_Y(y) \leqslant M_2^2, a.e., x$$

for some constants M_1, M_2. Let T be the integral operator defined on $\mathscr{L}^2_w(X)$ by

$$T(f)(y) = \int_X K(x,y)f(x)\,dm_X(x).$$

Then T is a bounded operator from $\mathscr{L}^2_w(X)$ to $\mathscr{L}^2_w(Y)$ with $\|T\| \leqslant M_1M_2$.

To apply Theorem 3.1, we must obtain bounds on the functions $K_{\pm}(x,s)$. From Theorem 2.2 we have

$$
\begin{aligned}
&2K_{+}(x,s)\\
&= y_1(x)y_2(s)\\
&= (pq)^{-1/4}(x)(pq)^{-1/4}(s)[1+o(1)]\exp\left(-\int_s^x (\tilde{q}/p)^{1/2}\,dt\right)\\
&= \frac{1}{q(x)}\left[\frac{p(x)}{p(s)}\right]^{\frac{1}{4}}\left[\frac{q(x)}{q(s)}\right]^{\frac{3}{4}}\left[\frac{q(s)}{p(s)}\right]^{\frac{1}{2}}[1+o(1)]\exp\left(-\int_s^x (q/p)^{1/2}[1+o(1)]\,dt\right)\\
&= \frac{1}{q(x)}\left[\frac{q(s)}{p(s)}\right]^{\frac{1}{2}}[1+o(1)]\exp\left(-\int_s^x (q/p)^{1/2}[1+o(1)]\,dt\right)
\end{aligned}
\tag{3.2}
$$

where we have used (2.17) in the last equation. Thus with $q/p = \rho e^{i\phi}$,

$$
\begin{aligned}
&|2K_{+}(x,s)|\\
&= \frac{1}{|q(x)|}\rho(s)^{\frac{1}{2}}[1+o(1)]\exp\left(-\int_s^x \rho(t)^{1/2}[\cos(\phi(t)/2)+o(1)]\,dt\right)\\
&\leqslant \frac{1}{|q(x)|}\rho(s)^{\frac{1}{2}}[1+o(1)]\exp\left(-\int_s^x \rho(t)^{1/2}[\sin(\delta/2)+o(1)]\,dt\right).
\end{aligned}
\tag{3.3}
$$

For later purposes, we have that a similar analysis shows, using for $r \geqslant 1$,

$$
\begin{aligned}
|2K_{+}(x,s)|^r =& |q(x)|^{(-r-1)/2}|p(x)|^{(-r+1)/2}\left|\frac{p(x)}{p(s)}\right|^{\frac{r-2}{4}}\left|\frac{q(x)}{q(s)}\right|^{\frac{r+2}{4}}\\
&\times\left|\frac{q(s)}{p(s)}\right|^{\frac{1}{2}}[1+o(1)]\exp\left(-r\int_s^x (q/p)^{1/2}[1+o(1)]\,dt\right)
\end{aligned}
$$

so that as in (3.3),

$$
\begin{aligned}
&|2K_{+}(x,s)|^r\\
&\leqslant |p(x)|^{\frac{1-r}{2}}|q(x)|^{\frac{-r-1}{2}}\rho(s)^{\frac{1}{2}}[1+o(1)]\exp\left(-r\int_s^x \rho(t)^{1/2}[\sin(\delta/2)+o(1)]\,dt\right).
\end{aligned}
\tag{3.4}
$$

Since the properties we are proving are independent of the endpoint a, it is sufficient to assume that the $o(1)$ in the integral of (3.3) satisfies $o(1) \leqslant (1/2)\sin(\delta/2)$. Hence, using (2.17) for w, there is a constant C so that

$$
\begin{aligned}
\int_a^x |K_{+}(x,s)|w(s)ds &\leqslant \frac{w(x)}{2|q(x)|}\int_a^x \rho(s)^{1/2}[1+o(1)]\exp\left(-\int_s^x \frac{1}{2}\rho(t)^{1/2}\sin(\delta/2)dt\right)ds\\
&\leqslant C\frac{w(x)}{|q(x)|}.
\end{aligned}
\tag{3.5}
$$

Here we have also used (2.19). Analogously, for $r \geqslant 1$,

$$\int_a^x |K_+(x,s)|^r w(s)\,ds \leqslant Cw(x)|p(x)|^{\frac{1-r}{2}}|q(x)|^{\frac{-r-1}{2}}. \tag{3.6}$$

A similar argument proves that

$$\int_x^\infty |K_-(x,s)|^r w(s)\,ds \leqslant Cw(x)|p(x)|^{\frac{1-r}{2}}|q(x)|^{\frac{-r-1}{2}}; \tag{3.7}$$

hence

$$\int_a^\infty |K(x,s)|^r w(s)\,ds \leqslant Cw(x)|p(x)|^{\frac{1-r}{2}}|q(x)|^{\frac{-r-1}{2}}. \tag{3.8}$$

For use in Theorem 3.4, we note that a similar argument proves that

$$\int_a^\infty |K(x,s)|^r w(s)^{r/2}\,ds \leqslant Cw(x)^{r/2}|p(x)|^{\frac{1-r}{2}}|q(x)|^{\frac{-r-1}{2}}. \tag{3.9}$$

We may now apply Theorem 3.1 to (3.8) using $K(x,s) = K(s,x)$, $X = Y = [a,\infty)$, $r = 1$, with

$$M_1 = C\sup_{y\in Y}\frac{w(y)}{|q(y)|}, \quad M_2 = C\sup_{x\in X}\frac{w(x)}{|q(x)|}. \tag{3.10}$$

The suprema above are finite since $w(x)/q(x) \to 0$ as $x \to \infty$.

Thus $R(f) \in \mathscr{L}_w^2[a,\infty)$ and R is bounded by Theorem 3.1; further $y = R(f) \in D(T)$.

Since for the maximal operator T, $(T-z)R(f) = f$ for all $f \in \mathscr{L}_w^2[a,\infty)$, we have that $T-z$ has the closed range $\mathscr{L}_w^2[a,\infty)$. This proves

THEOREM 3.2. *Under the hypotheses of Theorem 2.2 $T_0 - z$ has a closed range for all $z \in \mathbb{C}$, and the maximal operator T, T_α, and minimal operator T_0 satisfy*

$$\sigma_{ess}(T) = \sigma_{ess}(T_0) = \sigma_{ess}(T_\alpha) = \emptyset, \quad \Pi(T_0) = \mathbb{C}, \quad \sigma(T_\alpha) = \sigma_p(T_\alpha).$$

The second condition $\Pi(T_0) = \mathbb{C}$ follows by the closed graph theorem since $T_0 - z$, $z \in \mathbb{C}$, has a closed range, is closed, and is one-to-one. The condition $\sigma(T_\alpha) = \sigma_p(T_\alpha)$ follows since T_α is $\mathscr{C}$-selfadjoint and $\sigma_{ess}(T_\alpha) = \emptyset$.

THEOREM 3.3. *Under the hypotheses of Theorem 2.2 the operator R has the properties:*

(i) R is compact.

(ii) $R^{-1} + z$ is an extension of T_0.

(iii) $R^{-1} + z$ is $\mathscr{C}$-symmetric and its domain is given by:

$$\{y \in D(T) : y(a)y_2'(a,z) - y'(a)y_2(a,z) = 0\}.$$

Proof.

(i) To show R is compact, let the integral operators R_b and $\tilde{R}_b$ be defined by the kernels $\chi_{[a,b]}(x)K(x,s)$ and $\chi_{[b,\infty)}(x)K(x,s)$, i. e.,

$$(R_b f)(x) = \int_a^\infty \chi_{[a,b]}(x)K(x,s)w(s)f(s)\,ds, \tag{3.11}$$

and similarly for $\tilde{R}_b$. Applying Theorem 3.1 to $\tilde{R}_b$ and using (3.9) and (3.10) with $r=1, Y=[b,\infty)$, and $X=[a,\infty)$, it follows that $\tilde{R}_b$ has an arbitrary small norm if b is sufficiently large. Thus R is compact if R_b is compact as it is the limit in operator norm of compact operators. To see that R_b is compact we employ a similar decomposition writing $R_b = R_{b1} + R_{b2}$ where

$$(R_{b1} f)(x) = \int_a^{b'} \chi_{[a,b]}(x)K(x,s)w(s)f(s)\,ds,$$
$$(R_{b2} f)(x) = \int_{b'}^\infty \chi_{[a,b]}(x)K(x,s)w(s)f(s)\,ds.$$

Now R_{b1} is compact since its kernel is continuous on the compact set $[a,b]\times[a,b']$. Repeating the argument above shows that R_{b2} has arbitrary small norm if b' is sufficiently large; thus R_b is compact.

(ii) Let $y\in D(T_0)$ and $f=(T_0-z)y, \tilde{y}=R(f)$. Then $y\in D(T_0)\Rightarrow y(a)=y'(a)=0$. Further (3.1) and using $\int_a^\infty wy_1 f = \langle f,\bar{y}_1\rangle$ gives

$$\tilde{y}(a) = \frac{1}{2}y_2(a,z)\langle f,\bar{y}_1\rangle, \quad (p\tilde{y}')(a) = \frac{1}{2}(py_2')(a,z)\langle f,\bar{y}_1\rangle. \tag{3.12}$$

From (1.6) $\mathscr{R}(T_0-z)^\perp = \mathscr{N}(T_0^*-\bar{z}) = \mathscr{N}(T^+-\bar{z})$ and $\mathscr{N}(T^+-\bar{z})$ is spanned by $\bar{y}_1$, we have $\langle f,\bar{y}_1\rangle = 0$, and that $\tilde{y}(a)=\tilde{y}'(a)=0$. Thus $\hat{y}:=y-\tilde{y}$ satisfies $\hat{y}(a)=\hat{y}'(a)=0$, and $(T-z)(\hat{y})=f-f=0$, and hence $\hat{y}$ is the zero function by uniqueness of solutions of initial value problems. This implies $R^{-1}(y)+zy=f+zy=T_0(y)$.

Finally, for part (iii), let $y\in$ domain R^{-1}. Then from (3.12), with $y=\tilde{y}$,

$$y(a)(py_2')(a,z)-(py')(a)y_2(a,z)=0. \tag{3.13}$$

Now let $y\in D(T)$ so that (3.13) holds. Let $f=(T-z)(y), \tilde{y}=R(f), \hat{y}=\tilde{y}-y$. Then $(T-z)(\hat{y})=f-f=0$; hence $\hat{y}=c_1y_1$ since $\hat{y}\in\mathscr{L}_w^2[a,\infty)$. Thus

$$0=\hat{y}(a)(py_2')(a,z)-(p\hat{y}')(a)y_2(a,z)=c_1[y_1(a,z)(py_2')(a,z)-(py_1')(a,z)y_2(a,z)]$$

which implies $c_1=0$ since the Wronskian $[y_1(a,z)(py_2')(a,z)-(py_1')(a,z)y_2(a,z)]=2$ is constant. Hence $\hat{y}=0$ and $\tilde{y}=y$. $\square$

For the next theorem recall that a compact kernel operator such as R is a Hilbert-Schmidt operator if its kernel K satisfies

$$\int_a^\infty\int_a^\infty w(s)w(x)|K(x,s)|^2\,dsdx<\infty. \tag{3.14}$$

in which case (3.14) is an upper bound for $\|R\|^2$.

Also a compact operator A defined on a Hilbert space belongs to the Schatten class $\mathscr{C}_t$ for $1 \leqslant t < \infty$ provided that $\sum_1^\infty \mu(A)^t < \infty$ where $\mu(A)$ are the t-numbers of A, i.e., eigenvalues of the compact operator $(AA^*)^{1/2}$. $\mathscr{C}_\infty$ is the class of compact operators, and $\mathscr{C}_1$ is the trace class operators. Note that $\mathscr{C}_k \subseteq \mathscr{C}_t$ if $k \leqslant t$. The norm of a class $\mathscr{C}_t$ operator A is given by

$$\|A\|_t = \left(\sum_1^\infty \mu(A)^t\right)^{1/t}.$$

THEOREM 3.4. *Under the hypotheses of Theorem 2.2 the operator R is a Schatten class $\mathscr{C}_t$ operator, $t \geqslant 2$, if*

$$\int_a^\infty w(x)^2|p(x)|^{-1/2}|q(x)|^{-3/2}\,dx = \int_a^\infty \left|\frac{w(x)}{q(x)}\right|^2 \left|\frac{q(x)}{p(x)}\right|^{1/2} dx < \infty. \tag{3.15}$$

in which case for some constant C_t, then the Schatten norm of R satisfies,

$$\|R\|_t^t \leqslant C_t \int_a^\infty w(x)^t|p(x)|^{-1/2}|q(x)|^{-t+1/2}\,dx = C_t \int_a^\infty \left|\frac{w(x)}{q(x)}\right|^t \left|\frac{q(x)}{p(x)}\right|^{1/2} dx. \tag{3.16}$$

Proof. From (3.8) with $r = 2$, we have

$$\int_a^\infty w(s)|K(x,s)|^2\,ds \leqslant Cw(x)|p(x)|^{-1/2}|q(x)|^{-3/2},$$

and hence

$$\int_a^\infty\int_a^\infty w(s)w(x)|K(x,s)|^2\,ds\,dx \leqslant C\int_a^\infty w(x)^2|p(x)|^{-1/2}|q(x)|^{-3/2}\,dx < \infty$$

which proves R is Hilbert-Schmidt from (3.14).

To establish (3.16) for $t > 2$, we employ Theorem 1 of Russo [35]. In his theorem we use the fact that $K(x,s) = K(s,x)$. Define the kernel $k(x,s)$ by

$$k(x,s) = w(x)^{1/2}K(x,s)w(s)^{1/2},$$

and the operator $\tilde{R} : \mathscr{L}^2([a,\infty)) \to \mathscr{L}^2([a,\infty))$ by

$$(\tilde{R}g)(x) = \int_a^\infty k(x,s)g(s)\,ds.$$

Note that (3.15) implies $k \in \mathscr{L}^2([a,\infty)) \times \mathscr{L}^2([a,\infty))$ so that Russo's theorem applies.

Russo's theorem gives that the Schatten class $\mathscr{C}_t$ norm of $\tilde{R}$ satisfies (using $k(x,s) = k(s,x)$)

$$\|R\|_t \leqslant \|k\|_{v,t} := \left(\int_a^\infty\left(\int_a^\infty |k(x,s)|^v\,ds\right)^{\frac{t}{v}} dx\right)^{\frac{1}{t}}, \quad \frac{1}{t}+\frac{1}{v} = 1. \tag{3.17}$$

From (3.9) with $r=v$, we have

$$\left(\int_a^\infty |K(x,s)|^v w(s)^{v/2}\,ds\right)^{\frac{t}{v}} \leqslant \left(Cw(x)^{v/2}|p(x)|^{(1-v)/2}|q(x)|^{-(1+t)/2}\right)^{\frac{t}{v}}$$
$$= C^{t/v}w(x)^{t/2}|p(x)|^{-1/2}|q(x)|^{-t+1/2}.$$

Thus

$$\begin{aligned}\int_a^\infty \left(\int_a^\infty |k(x,s)|^v ds\right)^{\frac{t}{v}} dx &= \int_a^\infty \left(\int_a^\infty w(x)^{v/2}|K(x,s)|^v w(s)^{v/2}\,ds\right)^{\frac{t}{v}} dx \\ &\leqslant \int_a^\infty C^{t/v}w(x)^t|p(x)|^{-1/2}|q(x)|^{-t+1/2}dx\end{aligned} \tag{3.18}$$

which will yield (3.16) by (3.17) after we verify $\|R\|_t = \|\tilde{R}\|_t$. To see this let $M : \mathscr{L}^2_w([a,\infty)) \to \mathscr{L}^2([a,\infty))$ be defined by $M(y) = w^{1/2}y$. Then $M^{-1}(g) = g/w^{1/2}$, and $\|M(y)\|_{\mathscr{L}^2([a,\infty))} = \|y\|_{\mathscr{L}^2_w([a,\infty))}$. Thus M is isomorphic from $\mathscr{L}^2_w([a,\infty))$ onto $\mathscr{L}^2([a,\infty))$. Since $\tilde{R} = MRM^{-1}$, it follows that $\tilde{R}$ and R are unitarily equivalent and $\|R\|_t = \|\tilde{R}\|_t$. □

COROLLARY 3.1. *Under the hypotheses of Theorem 2.2 and hypothesis (H), the operator T_α has a compact resolvent. Further T_α has a Hilbert-Schmidt resolvent if (3.15) holds.*

Proof. First we note by the first resolvent formula that if an operator has a compact (Hilbert-Schmidt) resolvent for one z_0 in its resolvent set, then it has a compact (Hilbert-Schmidt) resolvent for all z_0 in its resolvent set. By Remark 1.1 we know T_α and $R^{-1}+z$ have nonempty resolvent sets; further $\rho(T_\alpha)\cap\rho(R^{-1}+z) \neq \emptyset$ since the eigenvalues are isolated and both have empty essential spectrum. Let $z_0 \in \rho(T_\alpha)\cap\rho(R^{-1}+z)$. By a theorem of Kato [24, p. 188], the difference $(T_\alpha - z_0)^{-1} - (R^{-1}+z-z_0)^{-1}$ is a finite rank operator. Since a finite rank operator is both compact and Hilbert-Schmidt, the conclusion follows. □

EXAMPLE 3.1. Let p, w, q be as in Example 2.1. Then condition (3.15) is equivalent to

$$\int_a^\infty x^{2\beta-\alpha/2-3\delta_1/2}\,dx < \infty \Leftrightarrow 2\beta - \frac{\alpha}{2} - 3\frac{\delta_1}{2} < -1 \Leftrightarrow \frac{2}{3} < \frac{\alpha}{3} + \delta_1 - \frac{4\beta}{3}.$$

This agrees with the pointwise criterion of Example 2 of [8, p. 21] when $q = c_1 x^{\delta_1}$. (Note $\tilde{q}(x)$ there should be $\tilde{q}(x)/w(x)$ in agreement with [8, p. 20].) Theorem 3.4 also applies to self-adjoint problems and gives a new criterion for the self-adjoint boundary value problem to be Hilbert-Schmidt.

4. The Titchmarsh-Weyl function and resolvent operator for T_α

Here we define the Titchmarsh-Weyl function $m_\alpha(z)$ and construct the resolvent formula for T_α as in [8] By section 3 we know $\rho(T_\alpha) \neq \emptyset$, and T_α has a compact resolvent. Define for all $z \in \mathbb{C}$, the basis of solutions of $L[y] = zy$ by the initial conditions

$$\begin{bmatrix} \theta_\alpha & \phi_\alpha \\ p\theta_\alpha' & p\phi_\alpha' \end{bmatrix}(a,z) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}. \tag{4.1}$$

Then $\phi_\alpha(\cdot,z)$ satisfies the boundary condition for T_α,

$$\cos\alpha\, y(a) + \sin\alpha\,(py')(a) = 0, \tag{4.2}$$

while θ_α does not. For $z \in \rho(T_\alpha)$, let $\psi_\alpha(\cdot,z) = c_1\theta_\alpha(\cdot,z) + c_2\phi_\alpha(\cdot,z)$ be the $\mathscr{L}_w^2[a,\infty)$ solution of $L[y] = zy$ normalized as follows. We have $c_1 \neq 0$ since $\phi_\alpha(\cdot,z) \notin \mathscr{L}_w^2[a,\infty)$ as $z \in \rho(T_\alpha)$. Thus take $c_1 = 1$. Then c_2 is uniquely determined and we define it as m_α.

Thus m_α is defined on $\rho(T_\alpha)$, and

$$\psi_\alpha(\cdot,z) = \theta_\alpha(\cdot,z) + m_\alpha(z)\phi_\alpha(\cdot,z) \in \mathscr{L}_w^2[a,\infty).$$

For $\beta \neq \alpha$, there is a simple relation between m_α and m_β. We have $\psi_\alpha = \text{constant} \times \psi_\beta$ since there is only one independent $\mathscr{L}_w^2[a,\infty)$ solution of $L[y] = zy$. Using the initial values of the solutions θ_α, ϕ_α, for α, β, the following equation can be derived.

$$m_\beta = \frac{\sin(\beta-\alpha) + m_\alpha\cos(\beta-\alpha)}{\cos(\beta-\alpha) - m_\alpha\sin(\beta-\alpha)}. \tag{4.3}$$

Define a Green's function $H_\alpha(x,y,z)$, for $z \in \rho(T_\alpha)$, by:

$$H_\alpha(x,y,z) = \begin{cases} -\phi_\alpha(x,z)\psi_\alpha(y,z), & a \leqslant x \leqslant y, \\ -\psi_\alpha(x,z)\phi_\alpha(y,z), & a \leqslant y < x. \end{cases} \tag{4.4}$$

Define $(R_\alpha(z)f)(x)$ on $\mathscr{L}_w^2[a,\infty)$ by

$$\begin{aligned} (R_\alpha(z)f)(x) &= \int_a^\infty H_\alpha(x,y,z)w(y)f(y)\,dy \\ &= \int_a^x -\psi_\alpha(x,z)\phi_\alpha(y,z)w(y)f(y)\,dy \\ &\quad + \int_x^\infty -\phi_\alpha(x,z)\psi_\alpha(y,z)w(y)f(y)\,dy. \end{aligned} \tag{4.5}$$

We will prove that $R_\alpha(z)$ is the resolvent of T_α, i.e., $R_\alpha(z) = (T_\alpha - z)^{-1}$. It is clear that $R_\alpha(z)$ is defined since $\psi_\alpha(\cdot,z), f \in \mathscr{L}_w^2[a,\infty)$. Straightforward calculations show that $R_\alpha(z)f$ satisfies the boundary condition (4.2) and the equation

$$L[R_\alpha(z)f] = z[R_\alpha(z)f] + f.$$

To complete the proof we need to show $R_\alpha(z)f \in \mathscr{L}^2_w[a,\infty)$. Let

$$f_N = f\big|_{[a,N]}, \quad y_N = R_\alpha(z)f_N, \quad y = R_\alpha(z)f.$$

For $x \leqslant N$,

$$y(x) - y_N(x) = -\phi_\alpha(x,z)\int_N^\infty \psi_\alpha(y,z)w(y)f(y)\,dy,$$

and using $\psi_\alpha(\cdot,z), f \in \mathscr{L}^2_w[a,\infty)$, we see that $y_N \to y$ uniformly on compact intervals.

For $x \geqslant N$,

$$y_N(x) = -\psi_\alpha(x,z)\int_a^N \phi_\alpha(y,z)w(y)f(y)\,dy \in \mathscr{L}^2_w[a,\infty).$$

Thus $y_N \in D(T_\alpha)$ and $y_N = (T_\alpha - z)^{-1}f_N$. Now $f_N \to f$ in $\mathscr{L}^2_w[a,\infty)$ as $N \to \infty$, and since $(T_\alpha - z)^{-1}$ is continuous as $z \in \rho(T_\alpha)$, we conclude that $y_N = (T_\alpha - z)^{-1}f_N \to (T_\alpha - z)^{-1}f$ in $\mathscr{L}^2_w[a,\infty)$ as $N \to \infty$. But also $y_N \to y$ uniformly on compact sets; hence $y = (T_\alpha - z)^{-1}f$ and $y \in \mathscr{L}^2_w[a,\infty)$. This proves

THEOREM 4.1. *Assume* (4.1) *and the hypotheses of Theorem* 2.2*. Then the resolvent operator for T_α is given by* (4.5) *where H_α is given by* (4.4)*.*

REMARK 4.2. For the conjugate operator T^*_α, using $\overline{\theta_\alpha(x,z)} = \theta_{\bar\alpha}(x,\bar z)$ and $\overline{\phi_\alpha(x,z)} = \phi_{\bar\alpha}(x,\bar z)$, it follows that $m^{\pm}_{\bar\alpha}(\bar z) = \overline{m_\alpha(z)}$ where $m^{\pm}_{\bar\alpha}$ is the Titchmarsh-Weyl function for T^*_α.

The first part of the proof of the next theorem follows that of Theorem 10 of [8]. Also equations (4.6)-(4.7) hold in greater generality, e.g., see Remling [34].

THEOREM 4.3. *Assume the hypotheses of Theorem* 4.1*. Then the function m_α is meromorphic on $\mathbb{C}$, and its poles are the eigenvalues of T_α.*

Proof. The proof of [8] shows that for $z, z_0 \in \rho(T_\alpha)$,

$$m_\alpha(z) - m_\alpha(z_0) = (z_0 - z)\int_a^\infty w\psi_\alpha(\cdot,z)\psi_\alpha(\cdot,z_0)\,dx, \tag{4.6}$$

and m_α is differentiable at z_0 with ($\dot m_\alpha = dm_\alpha/dz$)

$$\dot m_\alpha(z_0) = -\int_a^\infty w\psi^2_\alpha(\cdot,z_0)\,dx. \tag{4.7}$$

The proof of this is omitted. It follows that m_α is analytic on $\rho(T_\alpha)$.

Now let $z_0 \in \mathbb{C}$ be arbitrary. Since ϕ_α satisfies the α boundary condition, and the eigenspace is one dimensional, we have that z_0 is an eigenvalue of T_α if and only if $\phi_\alpha(\cdot,z_0) \in \mathscr{L}^2_w[a,\infty)$.

Since m_α is analytic on $\rho(T_\alpha)$, the only possible singular points of m_α are the eigenvalues of T_α. By the connection formula (4.3) we can choose a boundary condition $\beta = \alpha + \pi/2$. Now let z_0 be an eigenvalue of T_α. Then z_0 is not an eigenvalue of

T_β since the $\mathscr{L}^2_w[a,\infty)$ solution space of $L[y] = z_0 y$ is one dimensional, and an eigenfunction cannot satisfy two independent boundary conditions. By the first part of this proof, m_β is analytic at z_0. Since the eigenvalues of T_α are isolated, see Remark 1.1, and $\beta = \alpha + \pi/2$ we have in a deleted neighborhood of z_0, using (4.1) and (4.3),

$$m_\beta = -1/m_\alpha, \quad \phi_\alpha = -\theta_\beta, \quad \theta_\alpha = \phi_\beta, \quad \psi_\alpha = -m_\alpha \psi_\beta. \tag{4.8}$$

Thus either $m_\alpha = -1/m_\beta$ has a pole z_0 (if $m_\beta(z_0) = 0$) or removable singularity at z_0 (if $m_\beta(z_0) \neq 0$).

Suppose m_α has a removable singularity at the eigenvalue z_0 of T_α. Then $\phi_\alpha = -\theta_\beta$ and $\psi_\beta = \theta_\beta + m_\beta(z_0)\phi_\beta$ are two linearly independent $\mathscr{L}^2_w[a,\infty)$ solutions of $(T - z_0)(y) = 0$, contrary to there being only one such solution. Therefore m_α has a pole at z_0. $\square$

REMARK 4.4. Note that the analyticity of m_α on $\rho(T_\alpha)$ does not use hypothesis (H).

5. The Titchmarsh-Weyl function at an eigenvalue

In this section we give a representation of m_α at a pole. For an operator with compact resolvent, the eigenvalues are isolated, and they are poles of the resolvent. It is an open problem which potentials give arise to simple poles, respectively, higher order poles.

Let $y(\cdot,z)$ be a solution of (1.1) with nontrivial initial conditions $y(a,z)$, $(py')(a,z)$ independent of z. We use the notation : $\cdot = d/dz$, $y^{[n]} = d^n y/dz^n$. Then $-(p\dot{y}')' + q\dot{y} = zw\dot{y} + wy$, $\dot{y}(a,z) = (p\dot{y}')(a,z) = 0$, and in general the following holds.

PROPOSITION 5.1. *Let $z \in \mathbb{C}$. Then for every positive integer n,*

$$y^{[n]}(a,z) = p(y^{[n]})'(a,z) = 0,$$

and

$$(a) \quad (L-z)(y^{[n]}) = ny^{[n-1]}, \quad (b) \quad (L-z)^n(y^{[n]}) = n!y. \tag{5.1}$$

Proof. The zero initial conditions hold since $y^{[n]}(a,z)$, $p(y^{[n]})'(a,z)$ are independent of z. Clearly both equations of (5.1) hold for $n = 1$. If $(L-z)(y^{[n]}) = ny^{[n-1]}$, then applying d/dz,

$$L(y^{[n+1]}) = \frac{d}{dz}(zy^{[n]}) + ny^{[n]} = zy^{[n+1]} + (n+1)y^{[n]}$$

which establishes (5.1) (a) by induction. If $(L-z)^n(y^{[n]}) = n!y$, then applying d/dz,

$$-n(L-z)^{n-1}(y^{[n]}) + (L-z)^n(y^{[n+1]}) = n!y^{[1]}.$$

Applying $(L-z)$ to both sides of this equation and using (5.1) (a) gives

$$-n(L-z)^n(y^{[n]}) + (L-z)^{n+1}(y^{[n+1]}) = n!(L-z)(y^{[1]}) = n!y,$$

which simplifies to (5.1) after the substitution $(L-z)^n(y^{[n]}) = n!y$ which is the induction hypothesis. $\square$

THEOREM 5.1. *For each integer $m \geqslant 0$, the functions $\{y^{[n]}\}_{n=0}^m$ are linearly independent.*

Proof. Suppose not and let m be the smallest integer so that $y^{[m+1]}$ is a linear combination of $\{y^{[n]}\}_{n=0}^m$, say,

$$y^{[m+1]} = c_0 y + c_1 \dot{y} + \ldots + c_m y^{[m]}.$$

Applying the operator $L-z$ to this equation gives from (5.1),

$$L(y^{[m+1]}) - zy^{[m+1]} = (m+1)y^{[m]} = c_0 \cdot 0 + c_1 y^{[1]} + \ldots + c_m m y^{[m-1]}.$$

This implies that $y^{[m]}$ is a linear combination of $\{y^{[n]}\}_{n=0}^{m-1}$ which is contradiction. $\square$

THEOREM 5.2. *Let $z \in \rho(T_\beta)$ and define h_β by*

$$h_\beta(x,z) = R_\beta(z)(\psi_\beta(\cdot,z))(x) = \int_a^\infty H_\beta(x.t.z)w(t)\psi_\beta(t,z)\,dt. \tag{5.2}$$

Then $h_\beta = \psi_\beta^{[1]}$ or $\psi_\beta^{[1]} = (T_\beta - z)^{-1}(\psi_\beta)$.

Proof. From (4.5) one has

$$h'_\beta(x,z) = -\int_a^x w(t)\psi'_\beta(x,z)\phi_\beta(t,z)\psi_\beta(t,z)\,dt - \int_x^\infty w(t)\phi'_\beta(x,z)\psi_\beta^2(t,z)\,dt. \tag{5.3}$$

From equations (4.1), (4.7), (5.2), and (5.3), we obtain

$$h_\beta(a,z) = (\sin\beta)\dot{m}(z), \quad h'_\beta(a,z) = -(\cos\beta)\dot{m}(z).$$

The relation $h_\beta = (T_\beta - z)^{-1}(\psi_\beta)$ also implies

$$-(ph'_\beta)' - qh_\beta = zwh_\beta + w\psi_\beta.$$

From $\psi_\beta = \theta_\beta + m_\beta\phi_\beta$, we obtain

$$\dot{\psi}_\beta(x,z) = \dot{\theta}_\beta(x,z) + \dot{m}_\beta(z)\phi_\beta(x,z) + m_\beta(z)\dot{\phi}_\beta(x,z)$$

and

$$\dot{\psi}'_\beta(x,z) = \dot{\theta}'_\beta(x,z) + \dot{m}_\beta(z)\phi'_\beta(x,z) + m_\beta(z)\dot{\phi}'_\beta(x,z).$$

The initial conditions of θ_β, θ'_β, ϕ_β, ϕ'_β are independent of z which implies

$$\dot{\theta}_\beta(a,z) = \dot{\theta}'_\beta(a,z) = \dot{\phi}_\beta(a,z) = \dot{\phi}_\beta(a,z) = 0$$

so that

$$\dot{\psi}_\beta(a,z) = \phi(a,z)\dot{m}(z) = (\sin\beta)\dot{m}(z), \quad \dot{\psi}'_\beta(a,z) = \phi'_\beta(a.z)\dot{m}(z) = -(\cos\beta)\dot{m}(z).$$

Since h_β, $\dot{\psi}$ satisfy the same nonhomogeneous equation with the same initial values, we have $h_\beta = \dot{\psi}$ by uniqueness of initial value problems. $\square$

REMARK 5.3. Let $z \in \rho(T_\beta)$. Combining Proposition 5.1 and Theorem 5.2 then yields for every positive integer n, $\psi_\beta^{[n]} \in \mathscr{L}_w^2[a,\infty)$, and

$$\psi_\beta^{[n]} = n(T_\beta - z)^{-1}(\psi_\beta^{[n-1]}), \quad \psi_\beta^{[n]} = n!(T_\beta - z)^{-n}(\psi_\beta). \tag{5.4}$$

Recall the algebraic multiplicity $\nu(z_0)$ of an eigenvalue z_0 of the operator T_α is the smallest positive integer k so that $\mathscr{N}((T_\alpha - z_0)^k) = \mathscr{N}((T_\alpha - z_0)^{k+1})$.

THEOREM 5.4. *Suppose z_0 is an isolated eigenvalue of T_α of algebraic multiplicity $\nu(z_0)$. Then $\nu(z_0)$ is the order of the pole of $m_\alpha(z)$ at z_0. Further the functions $\phi_\alpha(\cdot,z_0), \phi_\alpha^{[i]}(\cdot,z_0)$, $i = 1,\dots,\nu(z_0)-1$ form a Jordan basis of the generalized eigenspace for z_0.*

Proof. From (4.8) we see that the order of the pole of $m_\alpha(z)$ at z_0 is the same as the multiplicity of the zero of $m_\beta(z)$ at z_0, $\beta = \alpha + \pi/2$, which we define to be n. Also from (4.8), $\theta_\beta = -\phi_\alpha$. Then

$$m_\beta(z_0) = \dot{m}_\beta(z_0) = \ldots = m_\beta^{[n-1]}(z_0) = 0, \quad m_\beta^{[n]}(z_0) \neq 0. \tag{5.5}$$

From (5.5) and $\psi_\beta(x.z) = \theta_\beta(x,z) + m_\beta(z)\phi_\beta(x,z)$, and $\theta_\beta = -\phi_\alpha$, it follows from (4.8) that

$$\psi_\beta^{[i]}(x,z) = -\phi_\alpha^{[i]}(x,z), i = 0,\dots,n-1, \psi_\beta^{[n]}(x,z) = -\theta_\beta^{[n]}(x,z) + m^{[n]}(z)\phi_\beta(x,z). \tag{5.6}$$

Now $\phi_\beta(\cdot,z_0) \notin \mathscr{L}_w^2[a,\infty)$ since $\phi_\beta(\cdot,z_0) = \theta_\alpha(\cdot,z_0)$ and $\phi_\alpha(\cdot,z_0)$ are linearly independent and there is only one linearly independent $\mathscr{L}_w^2[a,\infty)$ solution of $L[y] = z_0 y$. Hence (5.6) implies $\theta_\beta^{[n]}(\cdot,z_0) \notin \mathscr{L}_w^2[a,\infty)$ as $m^{[n]}(z_0) \neq 0$ and $\psi_\beta^{[n]}(\cdot,z_0) \in \mathscr{L}_w^2[a,\infty)$. Thus by Remark 5.3 and (5.6), $\phi_\alpha^{[i]}(\cdot,z_0) \in D(T_\alpha)$ for $i = 0,\dots,n-1$. Further by Proposition 5.1, for $s = 0,\dots,n-1, k \leqslant s$,

$$(T_\alpha - z_0)^{s+1}(\phi_\alpha^{[k]}) = (T_\alpha - z_0)^{s-k+1}(T_\alpha - z_0)^k(\phi_\alpha^{[k]}) = (T_\alpha - z_0)^{s-k+1}(k!\phi_\alpha) = 0,$$

and $(s \geqslant 1)$,

$$(T_\alpha - z_0)^s(\phi_\alpha^{[s]}) = s!\phi_\alpha^{[s-1]} \neq 0.$$

This gives with proper set containment,

$$\mathscr{N}((T_\alpha - z_0)) \subset \mathscr{N}((T_\alpha - z_0)^2) \subset \ldots \subset \mathscr{N}((T_\alpha - z_0)^n)$$

since for $s = 0,\dots,n-1$,

$$\phi_\alpha, \phi_\alpha^{[1]}, \dots, \phi_\alpha^{[s]} \in \mathscr{N}((T_\alpha - z_0)^{s+1}), \phi_\alpha^{[s]} \notin \mathscr{N}((T_\alpha - z_0)^s).$$

Therefore $\nu(z_0) \geqslant n$ for T_α.

Suppose now $\nu(z_0) > n$, and without loss of generality, we can take $\nu(z_0) = n+1$. Thus there is a function $y \in D(T_\alpha)$ so that $(T_\alpha - z_0)^{n+1}(y) = 0$ and $u := (T_\alpha - z_0)^n(y) \neq 0$. Then $(T_\alpha - z_0)u = 0$ and hence $u = constant \times \phi_\alpha$ since dim $\mathscr{N}(T_\alpha - z_0) = 1$. For simplicity take $u = \phi_\alpha = -\theta_\beta$.

We now prove by induction that for $j = 0, \ldots, n-1$,

$$(T_\alpha - z_0)^{n-j}(y) \in \text{span}\,\{\phi_\alpha, \ldots, \phi_\alpha^{[j]}\}, \quad \text{with the coefficient of } \phi_\alpha^{[j]} \neq 0. \tag{5.7}$$

For $j = 0$, this is established above. Suppose for some $j \geqslant 0$, that

$$(T_\alpha - z_0)^{n-j}(y) = c_0\phi_\alpha + \ldots + c_j\phi_\alpha^{[j]}, \quad c_j \neq 0. \tag{5.8}$$

Then by Proposition 5.1, with $d_i = c_i/i$,

$$(T_\alpha - z_0)[(T_\alpha - z_0)^{n-j-1}(y) - \sum_{i=0}^{j} d_i\phi_\alpha^{[i+1]}] = \sum_{i=0}^{j} c_i\phi_\alpha^{[i]} - \sum_{i=0}^{j} id_i\phi_\alpha^{[i]} = 0$$

since $d_i = c_i/i$. Thus

$$(T_\alpha - z_0)^{n-j-1}(y) - \sum_{i=0}^{j} d_i\phi_\alpha^{[i+1]} = \text{constant } \times \phi_\alpha$$

which implies

$$(T_\alpha - z_0)^{n-j-1}(y) \in \text{span}\,\{\phi_\alpha, \ldots, \phi_\alpha^{[j+1]}\} \quad \text{coefficient of } \phi_\alpha^{[j+1]} \neq 0.$$

For $j = n-1$ (5.7) gives

$$(T_\alpha - z_0)(y) = \sum_{i=0}^{n-1} a_i\phi_\alpha^{[i]}. \tag{5.9}$$

Hence as $y \in D(T_\alpha)$,

$$(L - z_0)[y - \sum_{i=0}^{n-1}(a_i/i)\phi_\alpha^{[i+1]}] = (T_\alpha - z_0)(y) - \sum_{i=0}^{n-1}(a_i/i)i\phi_\alpha^{[i]} = 0$$

by (5.9). Thus

$$y - \sum_{i=0}^{n-1}(a_i/i)\phi_\alpha^{[i+1]} = a\theta_\alpha + b\phi_\alpha \tag{5.10}$$

for some constants a and b. Now y, ϕ_α satisfy the α boundary conditions, $\phi_\alpha^{[i+1]}$, $i = 0, \ldots, n-1$, satisfy the zero boundary conditions, and θ_α satisfies the β boundary conditions. Since the α and β boundary conditions are linearly independent, this gives $a = 0$. But with $a = 0$ (5.10) implies $\phi_\alpha^{[n]} \in \mathscr{L}_w^2[a,\infty)$ which is a contradiction. Thus $\nu(z_0) \leqslant n$ and hence $\nu(z_0) = n$. $\square$

We now examine the behavior of $(T_\alpha - z)^{-1}$ around an eigenvalue z_0 of T_α and give a representation of the resolvent there. Suppose z_0 is an eigenvalue of T_α of multiplicity n. By Theorem 5.4, $m_\alpha(z)$ can be expressed in a deleted neighborhood of z_0 as

$$m_\alpha(z) = \frac{c_{-n}}{(z-z_0)^n} + \ldots + \frac{c_{-1}}{(z-z_0)} + m_{0,\alpha}(z), \tag{5.11}$$

where $c_{-n} \neq 0$, and $m_{0,\alpha}(z)$ is analytic at z_0.

THEOREM 5.5. *Suppose z_0 is an isolated eigenvalue of T_α of multiplicity n. Then the Green's function H_α of (4.4) has the representation in a deleted neighborhood of z_0,*

$$H_\alpha(x,t,z) = \sum_{r=1}^{n} \frac{H_{\alpha,r}(x,t)}{(z-z_0)^r} + H_{\alpha,0}(x,t,z), \tag{5.12}$$

where $H_{\alpha,0}$ is analytic at z_0, and

$$H_{\alpha,r}(x,t) = \sum_{j=0}^{n-r} \theta_\beta^{[j]}(x,z_0) \sum_{s=j}^{n-r} c_{-r-s} \binom{s}{s-j} \theta_\beta^{[s-j]}(t,z_0), \tag{5.13}$$

where $\beta = \alpha + \pi/2$, c_{-j} are as in (5.11). Further $H_{\alpha,0}(x,t,z) = H_{\alpha,0}(t,x,z)$, $H_{\alpha,r}(x,t) = H_{\alpha,r}(t,x)$.

Proof. From the proof of Theorem 5.4, we have that

$$\psi_\beta^{[i]}(x,z) = \theta_\beta^{[i]}(x,z) = -\phi_\alpha^{[i]}(x,z), \quad i = 0,\ldots,n-1,$$

and that $\{\theta_\beta,\ldots,\theta_\beta^{[i-1]}\}$ is contained in the null space $\mathscr{N}((T_\alpha - z_0)^i)$. By (4.5) and (4.8), for $z \in \rho(T_\alpha)$,

$$\begin{aligned}(R_\alpha(z)f)(x) &= -\int_a^x \psi_\alpha(x,z)\phi_\alpha(t,z)w(t)f(t)\,dt - \int_x^\infty \phi_\alpha(x,z)\psi_\alpha(t,z)w(t)f(t)\,dt \\ &= -m_\alpha(z)(S_z f)(x), \\ (S_z f)(x) &:= \int_a^x \psi_\beta(x,z)\theta_\beta(t,z)w(t)f(t)\,dt + \int_x^\infty \psi_\beta(x,z)\theta_\beta(t,z)w(t)f(t)\,dt.\end{aligned} \tag{5.14}$$

Using only functions f of compact support we compute that $(S_z f)(x)$ is analytic at z_0 and with $\psi_\beta(x,z_0) = \theta_\beta(t,z_0)$, the two integrals in (5.14) add yielding,

$$(S_{z_0} f)(x) = \int_a^\infty \theta_\beta(x,z_0)\theta_\beta(t,z_0)w(t)f(t)\,dt,$$

and in general,

$$\frac{\partial^s}{\partial z^s}(S_z f)(x)\rfloor_{z=z_0} = \sum_{k=0}^{s} \binom{s}{r} \theta_\beta^{[s-k]}(x,z_0) \int_a^\infty \theta_\beta^{[k]}(t,z_0)w(t)f(t)\,dt, \tag{5.15}$$

so that

$$(S_z f)(x) = \sum_{s=0}^{n-1}\sum_{k=0}^{s}(z-z_)^s\binom{s}{r}\theta_\beta^{[s-k]}(x,z_0)\int_a^\infty \theta_\beta^{[k]}(t,z_0)w(t)f(t)\,dt + (z-z_0)^n(S_{0,z}f)(x), \tag{5.16}$$

where $(S_{0,z}f)(x)$ is analytic at z_0. Now substitute (5.5) and (5.16) into (5.14) to obtain

$$\begin{aligned}(R_\alpha(z)f)(x) = &\left[\sum_{i=1}^{n}\frac{c_{-i}}{(z-z_0)^i}\right]\\ &\times\left[\sum_{s=0}^{n-1}\sum_{k=0}^{s}(z-z_)^s\binom{s}{r}\theta_\beta^{[s-k]}(x,z_0)\int_a^\infty \theta_\beta^{[k]}(t,z_0)w(t)f(t)\,dt\right]\\ &+\text{analytic term,}\end{aligned} \tag{5.17}$$

so that

$$\begin{aligned}&\sum_{i=1}^{n}\frac{c_{-i}}{(z-z_0)^i}\times\sum_{s=0}^{n-1}\sum_{k=0}^{s}(z-z_)^s\binom{s}{r}\theta_\beta^{[s-k]}(x,z_0)\int_a^\infty \theta_\beta^{[k]}(t,z_0)w(t)f(t)\,dt\\ &=\sum_{i=1}^{n}\sum_{s=0}^{i-1}\frac{c_{-i}}{(z-z_0)^{i-s}}\sum_{k=0}^{s}\binom{s}{r}\theta_\beta^{[s-k]}(x,z_0)\int_a^\infty \theta_\beta^{[k]}(t,z_0)w(t)f(t)\,dt + \text{ analytic term}\end{aligned} \tag{5.18}$$

Interchanging the order of summation in the first two summands of (5.18) yields

$$\sum_{i=1}^{n}\sum_{s=0}^{i-1} = \sum_{s=0}^{n-1}\sum_{i=s+1}^{n}.$$

With the change of index $r=i-s$ and $\sum_{i=s+1}^{n}=\sum_{r=1}^{n-s}$, we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{s=0}^{i-1}\frac{c_{-i}}{(z-z_0)^{i-s}}\sum_{k=0}^{s}\binom{s}{r}\theta_\beta^{[s-k]}(x,z_0)\int_a^\infty \theta_\beta^{[k]}(t,z_0)w(t)f(t)\,dt,\\ &=\sum_{s=0}^{n-1}\sum_{r=1}^{n-s}\frac{c_{-r-s}}{(z-z_0)^{r}}\sum_{k=0}^{s}\binom{s}{r}\theta_\beta^{[s-k]}(x,z_0)\int_a^\infty \theta_\beta^{[k]}(t,z_0)w(t)f(t)\,dt,\\ &=\sum_{r=1}^{n}\sum_{s=0}^{n-r}\frac{c_{-r-s}}{(z-z_0)^{r}}\sum_{k=0}^{s}\binom{s}{r}\theta_\beta^{[s-k]}(x,z_0)\int_a^\infty \theta_\beta^{[k]}(t,z_0)w(t)f(t)\,dt.\end{aligned} \tag{5.19}$$

Finally, with $j=s-k$ below and one more interchange of summations, the coefficient of $1/(z-z_0)^r$ in (5.19) with the integral and $w(t)f(t)$ deleted is

$$\begin{aligned}\sum_{s=0}^{n-r}c_{-r-s}\sum_{k=0}^{s}\binom{s}{r}\theta_\beta^{[s-k]}(x,z_0)\theta_\beta^{[k]}(t,z_0) &= \sum_{j=0}^{n-r}c_{-r-s}\sum_{s=j}^{n-r}\binom{s}{s-j}\theta_\beta^{[j]}(x,z_0)\theta_\beta^{[s-j]}(t,z_0),\\ &=\sum_{j=0}^{n-r}\theta_\beta^{[j]}(x,z_0)\sum_{s=j}^{n-r}c_{-r-s}\binom{s}{s-j}\theta_\beta^{[s-j]}(t,z_0),\end{aligned} \tag{5.20}$$

which is (5.13).

The equation above shows the symmetry of $H_{\alpha,r}$, and H_β is symmetric by definition. Together these make $H_{\alpha,0}$ symmetric. $\square$

An examination of the proofs of Theorems 4.3 and 5.4 shows that if $T_0 - z$ has a closed range and $s = 2$, then $m_\alpha(z)$ satisfies locally the properties of Theorems 4.3 and 5.4 without invoking Theorem 2.2. In particular, $m_\alpha(z)$ has a pole at an eigenvalue of T_α. These results for the m-function hold in much more generality and have been shown by Remling [34], Brown et al [10], and Behncke and Hinton [3].

REMARK 5.6. Note that the term $\sum_{s=j}^{n-r} c_{-r-s}\binom{s}{s-j}\theta_\beta^{[s-j]}(t,z_0)$ of (5.20) is in the null space $\mathscr{N}((T_\alpha - z)^{n-r-j+1})$.

Note from equation (5.13) that $H_{\alpha,n}(x,t) = c_{-n}\theta_\beta(x,z_0)\theta_\beta(t,z_0)$, a one term equation. An illustration is for $n = 2$,

$$H_{\alpha,1}(x,t) = c_{-1}\theta_\beta(x,z_0)\theta_\beta(t,z_0) + c_{-2}[\dot\theta_\beta(x,z_0)\theta_\beta(t,z_0) + \theta_\beta(x,z_0)\dot\theta_\beta(t,z_0)].$$

The coefficients c_{-j} can be found by expanding the function m_β and using the relation $m_\beta(z) = -1/m_\alpha(z)$. If $m_\beta(z) = c_0 + c_1(z - z_0) + \dots$, then for a pole of order one, $c_0 = 0$ and $c_{-1} = -1/c_1 = -1/\dot m_\beta(z_0)$. For a pole of order two, $c_0 = c_1 = 0$, and $c_{-2} = -1/c_2 = -2/m_\beta^{[2]}(z_0)$, $c_{-1} = c_3/c_2^2 = 2m_\beta^{[3]}(z_0)/3(m_\beta^{[2]})(z_0)^2$. The values of $m_\beta^{[i]}(z)$ can be found be successively differentiating the expression (4.7),

$$\dot m_\beta(z) = -\int_a^\infty w(t)\psi_\beta(t,z)^2\,dt,$$

yielding

$$\ddot m_\beta(z) = -2\int_a^\infty w(t)\psi_\beta(t,z)\dot\psi_\beta(t,z)\,dt, \dots$$

From these relations certain orthogonality relations follow, e.g., if z_0 is a pole of order three of m_α, then

$$\int_a^\infty w(t)\psi_\beta(t,z)^2\,dt = \int_a^\infty w(t)\psi_\beta(t,z)\dot\psi_\beta(t,z)\,dt = 0.$$

This extends to

$$\int_a^\infty w(t)\psi_\beta^{[k]}(t,z)\psi_\beta^{[l]}(t,z)\,dt = 0, \quad k,l \leqslant \nu(z_0) - 2.$$

An alternate approach to Theorem 5.5 is to use contour integration, i.e., for $r = 0,\dots,n$

$$H_{\alpha,r}(x,t) = \frac{1}{2\pi i}\int_\Gamma (z - z_0)^{r-1} H_\alpha(x,t,z)\,dz$$

where Γ is a small circle around z_0. From

$$(T_\alpha - z_0)(H_{\alpha,r}(x,t)) = H_{\alpha,r+1}(x,t), \quad H_{\alpha,n+1} \equiv 0,$$

and the symmetry of $H_{\alpha,r}$, one can show $H_{\alpha,r}$ has the form

$$H_{\alpha,r}(x,t) = \sum_{i,j=0}^{n-r+1} a_{ij}\phi_\alpha^{[i]}(x)\phi_\alpha^{[j]}(t) \tag{5.21}$$

which is of the form (5.13). A similar approach has been used by Kemp [26] for the operator $-y''+qy$ on $(-\infty,\infty)$ with q satisfying $\int_{-\infty}^{\infty}(1+x^2)^{1/2}|q(x)|\,dx<\infty$. In this case there is an essential spectrum $[0,\infty)$. In Theorem 3.2 and Corollary 4.1 of Kemp an expansion of the Green's function is obtained which in addition to containing terms like (5.21) contains a term $H_{\alpha,0}$ which integrates over the essential spectrum. The explicit representation of H_α is an improvement of the results of Kemp for the discrete part of the resolvent. Our approach here is more elementary in that it consists of multiplying two series and collecting terms. This yields a more direct calculation of the coefficients a_{ij} in (5.21). However, with these strong conditions on q, Kemp is able to obtain a spectral resolution of the operator. We expect that these results for the discrete spectrum hold in much greater generality, because most computations are formal and no specific properties of the coefficients are used.

While it is an open problem which operators lead to higher order poles, potentials with analytic or rational coefficients will not have higher order poles, the proof of this may be quite complicated as the next example shows.

EXAMPLE 5.1. The complex square well. Suppose in (1.1) that $w=p=1$, and

$$q(x) = \begin{cases} c, & \text{for } 0\leqslant x\leqslant 1\\ 0, & \text{for } 1\leqslant x<\infty. \end{cases}$$

where $c\neq 0$ is a complex number. Assume z_0 is an eigenvalue of T_α of algebraic multiplicity $\nu_\alpha(z_0)>1$. Then with $\beta=\alpha+\pi/2$, we have $m_\beta(z_0)=\dot m_\beta(z_0)=0$. Since $-y''=zy$ has no $\mathscr{L}^2[0,\infty)$ solutions if $z\in[0,\infty)$, we can assume $z_0=s^2$ with $\operatorname{Im} s>0$. Then

$$\psi_\beta(x,z_0)=\theta_\beta(x,z_0)=-\phi_\alpha(x,z_0)=\begin{cases} de^{isx}, & x\geqslant 1,\\ ae^{is_1x}+be^{-is_1x}, & 0\leqslant x<1, \end{cases}$$

where $s_1^2=s^2-c$. Since $(T_\alpha-z_0)(\phi_\alpha)=(T_\alpha-z_0)^2(\dot\phi_\alpha)=0$, we have using $\dot\psi_\beta=-\dot\phi_\alpha$, continuity of $\psi_\beta,\psi_\beta',\dot\psi_\beta,\dot\psi_\beta'$ at $x=1, z=z_0$, and $ds/dz=1/2s, ds_1/dz=1/2s_1$, that

$$de^{is}=ae^{is_1}+be^{-is_1},\quad isde^{is}=is_1(ae^{is_1}-be^{-is_1}),$$

$$\frac{di}{2s}e^{is}=\frac{i}{2s_1}(ae^{is_1}-be^{-is_1}),$$

$$\frac{di}{2s}e^{is}(1+is)=\frac{i}{2s_1}(ae^{is_1}-be^{-is_1}+is_1(ae^{is_1}+be^{-is_1})).$$

Equating the first two equations gives

$$\frac{s_1}{s}=\frac{ae^{is_1}+be^{-is_1}}{ae^{is_1}-be^{-is_1}}, \tag{5.22}$$

and equating the second two gives

$$\frac{1}{1+is}=\frac{ae^{is_1}-be^{-is_1}}{ae^{is_1}-be^{-is_1}+is_1(ae^{is_1}+be^{-is_1})}. \tag{5.23}$$

Substituting (5.22) into (5.23) and simplifying yields

$$\frac{1}{1+is}=\frac{s}{s+is^2-ic}$$

which implies $c=0$ a contradiction. Thus all eigenvalues have algebraic multiplicity one.

The next example from [14, p. 2314] gives an operator with a higher order pole. So far no explicit examples for half line operators with higher order poles are known though the results of Lyantse [29]and Pavlov [33] state abstract results in this direction.

EXAMPLE 5.2. Here we show the operator $L[y]=-y''$ on $[0,1]$ with boundary conditions $y(0)=k_1y'(0), y(1)=k_2y'(1)$ may have a resolvent with a higher order pole. For this let the general solution of $L[y]=s^2y$, $z=s^2\neq 0$ be given by

$$y(x)=c_1\cos(sx)+c_2\sin(sx), \quad y'(x)=-c_1 s\sin(sx)+c_2 s\cos(sx).$$

In order for z to be an eigenvalue, there must be a nontrivial solution satisfying the boundary conditions. This gives

$$c_1=sk_1c_2, \quad c_1\cos s+c_2\sin s=k_2s(-c_1\sin s+c_2\cos s). \tag{5.24}$$

For (5.24) to have a nontrivial solution the determinant $\Delta(s)$ of the coefficients must be zero and this gives

$$\begin{aligned}\Delta(s)&=\sin s-k_2 s\cos s+k_1 s(\cos s+k_2 s\sin s)\\&=\sin s(1+ds^2)-cs\cos s=0,\end{aligned} \tag{5.25}$$

where $c=k_2-k_1, d=k_1k_2$. Since the boundary conditions are separated, the geometric multiplicity of an eigenvalue is one. Now $\Delta(s)$ occurs in the denominator in the construction of the Green's function, Hence the resolvent will have a pole of order at least two if also $\Delta'(s)=0$ when $\Delta(s)=0$. A short computation yields that

$$\Delta'(s)=(\cos s)(1+ds^2-c)+(\sin s)(2sd+cs). \tag{5.26}$$

If $\cos s\neq 0$, then the eigenvalue equation is

$$\Delta(s)=0 \Leftrightarrow \quad \tan s=\frac{s(k_2-k_1)}{1+k_2k_1s^2}=\frac{cs}{1+ds^2}$$

which is also given in [14, p. 2314]. It is noted in [14] that only finitely many eigenvalues are nonsimple. The asymptotic form of the eigenvalues is also derived.

From (5.25) and (5.26) a pole in the resolvent of order at least two if

$$\begin{aligned} 0 &= (1+ds^2)\sin s - cs\cos s \\ 0 &= (1+ds^2-c)\cos s + (2sd+cs)\sin s. \end{aligned} \tag{5.27}$$

Note that the equations (5.27) are linear in c, d. Thus we can assume a value s for our eigenvalue $z = s^2$, compute c, d from (5.27), and then solve for k_1, k_2 to produce an example of a pole of order two in the resolvent. For example with $s = \pi/2$, the equations in (5.27) become

$$0 = 1 + ds^2, \quad 0 = s(2d + c),$$

whose solution is $d = -4/\pi^2$, $c = 8/\pi^2$, and k_2 satisfies the equation $k_2^2 - (8/\pi^2)k_2 + 4/\pi^2 = 0$. Both k_1, k_2 are complex.

Examples of a boundary value problem on a compact interval $[a,b]$ with eigenvalues of finite algebraic multiplicity $\geqslant 3$ have been given by Chen and Lin [12].

6. The hypothesis H

We first prove on a compact interval with separated boundary conditions there are no eigenvalues of infinite algebraic multiplicity. We write the boundary conditions in the form

$$(BC): \alpha_1 y(a) + \beta_1 (py')(a) = 0,\ \alpha_2 y(b) + \beta_2 (py')(b) = 0,\ |\alpha_1| + |\beta_1| \neq 0,\ |\alpha_2| + |\beta_2| \neq 0.$$

PROPOSITION 6.1. *Let p, q, and w be continuous on $[a,b]$ with $w > 0$. Suppose $p = p_1 + ip_2$ and one of p_1, p_2 does not vanish on $[a,b]$. Define the operator A on $[a,b]$ by*

$$A[y] = L[y] = \frac{1}{w}[-(py')' + qy],$$

where the domain of A is given by

$$D(A) = \{y \in \mathscr{L}^2_w[a,b] : y,\ py' \in AC[a,b], A[y] \in \mathscr{L}^2_w[a,b],\ y \text{ satisfies the conditions } BC\}$$

Then A has no eigenvalues of infinite algebraic multiplicity.

Proof. First we note that if z_0 is an eigenvalue of A with eigenfunction y_0, then a solution of $A[y] = z_0 y$ independent of y_0 will not satisfy either of the boundary conditions BC as the solution set is one dimensional. Thus z_0 will be in the resolvent set of an operator with the same first boundary condition of A, but with a different condition at b. Such an operator will have a compact resolvent and hence empty essential spectrum. This implies A also would have empty essential spectrum as both operators are finite dimensional extensions of the minimal operator. Thus if z_0 is an eigenvalue of infinite algebraic multiplicity, then by Locker [28, p. 57], z is an eigenvalue of infinite algebraic multiplicity for all $z \in \mathbb{C}$. We now prove this leads to a contradiction as the conditions above bound the eigenvalues to a half plane.

For definiteness, suppose p_1 is not zero on $[a,b]$ and $p_1 > 0$. The other cases are similar. From $A[y_0] = z_0 y_0$ we multiply by $\bar{y}_0$ and integrate to yield

$$-(py_0')\bar{y_0}\big|_a^b + \int_a^b [p|y_0'|^2 + q|y_0|^2]dx = z_0 \int_a^b w|y_0|^2\,dx. \tag{6.1}$$

By the boundary conditions BC,

$$-(py_0')\bar{y_0}\big|_a^b = -\gamma_1|y(b)|^2 + \gamma_2|y(a)|^2$$

where $\gamma_i = -\alpha_i/\beta_i$ if $\beta_i \neq 0$ and $\gamma_i = 0$ if $\beta_i = 0$.

Taking $\int_a^b |y_0|^2\,dx = 1$, equation (6.1) yields by taking the real part,

$$(\operatorname{Re} z_0)w_{max} \geqslant -\operatorname{Re}\gamma_1|y_0(b)|^2 + \operatorname{Re}\gamma_2|y_0(a)|^2 + \int_a^b p_{1,min}|y'|^2\,dx + q_{1,min}(b-a), \tag{6.2}$$

where the subscripts max, min refer to the maximum, minimum values on $[a,b]$.

Choose c so that $|y_0(x)| \geqslant |y_0(c)|$ for all $x \in [a,b]$. Then $y_0(x)^2 - y_0(c)^2 = \int_c^x 2y_0y_0'\,ds$ implies by the Cauchy Schwarz inequality,

$$|y_0(x)|^2 \leqslant |y_0(c)|^2 + 2\Big(\int_a^b |y_0'|^2\,ds\Big)^{1/2} \leqslant \frac{1}{b-a} + 2\Big(\int_a^b |y_0'|^2\,ds\Big)^{1/2} \tag{6.3}$$

since $|y_0(c)|^2(b-a) \leqslant \int_a^b |y_0|^2\,dx = 1$.

From (6.2) and (6.3) we have an inequality of the form

$$(\operatorname{Re} z_0) \geqslant B_1 + B_2\Big(\int_a^b |y_0'|^2\,dx\Big)^{1/2} + B_3\Big(\int_a^b |y_0'|^2\,dx\Big) \tag{6.4}$$

with $B_3 > 0$. The right hand side of (6.4) is bounded below as it is a quadratic in the quantity $(\int_a^b |y_0'|^2\,dx)^{1/2}$ which completes the proof. □

Suppose under the conditions of Theorem 2.2 that z_0 is an eigenvalue of infinite algebraic multiplicity of T_α. By Locker [28, p. 57] then z is an eigenvalue of infinite algebraic multiplicity of T_α for all $z \in \mathbb{C}$ since $\sigma_{ess}(T_\alpha) = \emptyset$ by Theorem 3.2. Suppose $[a,b_0]$ is an interval on which $\operatorname{Re} p$ or $\operatorname{Im} p$ does not vanish and that for some $b \in (a,b_0]$ there is an eigenvalue of infinite algebraic multiplicity of some operator T_β on $[b,\infty)$. Since there is only one linearly independent $\mathscr{L}_w^2[a,\infty)$ solution of $L[y] = zy$ for all $z \in \mathbb{C}$, then we have on the interval $[a,b]$ and separated boundary conditions α and β an eigenvalue of infinite algebraic multiplicity contrary to the above Proposition. We have proved

THEOREM 6.1. *Under the conditions of Theorem 2.2, the set of a's for which some boundary condition yields an eigenvalue of infinite algebraic multiplicity is a set of isolated points.*

Further, if the condition that Re p or Im p does not vanish on $[a,\infty)$, then Proposition 6.1 proves that there is at most one point a for which an eigenvalue of infinite algebraic multiplicity exists.

We note also under the conditions of Theorem 2.2 that if z_0 is an eigenvalue of infinite algebraic multiplicity of T_α, then for $\beta \neq \alpha$, $z \in \rho(T_\beta)$ for all $z \in \mathbb{C}$ since there is only one linearly independent $\mathscr{L}_w^2[a,\infty)$ solution of $L[y] = zy$ for all $z \in \mathbb{C}$.

7. The operator (1.1) on $(-\infty,\infty)$

We consider now the singular second order operator

$$L[y] = \frac{1}{w}\left[(-py')' + qy\right], \quad -\infty < x < \infty, \tag{7.1}$$

where again w is real and $p = p_1 + ip_2 \neq 0, q = q_1 + iq_2 \neq 0$ are complex valued. To apply Theorem 2.2 at $-\infty$, we make the change of variable $v(x) = y(-x)$. Then

$$L[y] = zy \Leftrightarrow \hat{L}[v] = \frac{1}{\hat{w}}\left[(-\hat{p}v')' + \hat{q}v\right], \quad -\infty < x < \infty,$$

where $\hat{p}(x) = p(-x), \hat{q}(x) = q(-x), \hat{w}(x) = w(-x)$. We say Theorem 2.2 holds for L at $-\infty$ if it holds for $\hat{L}$ at ∞. For example, $-y'' + xy = 0$ becomes $-v'' - xv = 0$, and we see that Theorem 2.2 does not apply although Theorem 2.1 does.

The maximal operator T and an unclosed minimal operator T_0' are now defined by the action of L on the domains

$$D(T) = \{y \in \mathscr{L}_w^2(-\infty,\infty) : y, py' \in AC_{\text{loc}} \text{ and } L[y] \in \mathscr{L}_w^2(-\infty,\infty)\}$$

and

$$D(T_0') = \{y \in D(T) : y \text{ has compact support in } (-\infty,\infty)\}.$$

Again T_0' is closeable, and the closure will be indicated by T_0. As before, let T^+ denote the adjoint maximal operator.

THEOREM 7.1. *Assume at each endpoint $\pm\infty$ hypothesis H holds and that the conditions of Theorem 2.2 hold. Then $\sigma_{ess}(T) = \emptyset, T_0 = T,$ the eigenvalues of T are isolated, and z not an eigenvalue of T implies $z \in \rho(T)$.*

Proof. Under the conditions of Theorem 2.2 the minimal half line operators S_-, S_+ on $(-\infty,0]$, $[0,\infty)$, respectively, have empty essential spectrum by Theorem 3.2. By the decomposition principle, see Glazman [18, p. 10], $\sigma_{ess}(T) = \emptyset$. This gives that $T - z, z \in \mathbb{C}$, is a Fredholm operators so T has isolated eigenvalues by Locker [28, p. 56].

To show $T_0 = T$, note first that $y \in D(S_-)$ or in $D(S_+)$ satisfies $y(0) = (py')(0) = 0$. Further the each of the domains of T and T_0 is equal to the domain of $S_+ \oplus S_-$ plus the same two dimensional subspace of $D(T)$. Hence $T_0 = T$. $\square$

Following [8], we can define a resolvent for T. Hence by the above Theorem 7.1, T has isolated eigenvalues. Define for all $z \in \mathbb{C}$, the basis of solutions of $L[y] = zy$ by the initial conditions

$$\begin{bmatrix} \theta & \phi \\ p\theta' & p\phi' \end{bmatrix}(0,z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{7.2}$$

For $z \in \rho(T)$, define the functions $m_\pm(z)$ and functions $\psi_\pm$ as follows: let $\psi_+(\cdot,z) = \theta(\cdot,z) + m_+(z)\phi(\cdot,z)$ be the normalized $\mathscr{L}_w^2[0,\infty)$ solution of $L[y] = z$ and $\psi_-(\cdot,z) = \theta(\cdot,z) + m_-(z)\phi(\cdot,z)$ be the normalized $\mathscr{L}_w^2(-\infty,0]$ solution of $L[y] = zy$. As proved in section 5, the functions $m_\pm$ are meromorphic functions under the hypotheses of Theorem 7.1.

As in the symmetric case, we define a Green's function $H(x,y,z)$, for $z \in \rho(T)$, by (below it is noted that the meromorphic function $m_+(z) - m_-(z)$ is analytic on $z \in \rho(T)$):

$$H(x,y,z) = \begin{cases} -\frac{\psi_+(x,z)\psi_-(y,z)}{m_+(z)-m_-(z)}, & y \leqslant x, \\ -\frac{\psi_-(x,z)\psi_+(y,z)}{m_+(z)-m_-(z)}, & x < y. \end{cases} \tag{7.3}$$

Define $(R(z)f)(x)$ on $\mathscr{L}_w^2(-\infty,\infty)$ by, wirh $M = M(z) := [m_+(z) - m_-(z)]^{-1}$,

$$\begin{aligned} (R(z)f)(x) &= \int_{-\infty}^{\infty} H(x,y,z)w(y)f(y)\,dy \\ &= \int_{-\infty}^{x} -M\psi_+(x,z)\psi_-(y,z)w(y)f(y)\,dy \\ &\quad - \int_{x}^{\infty} M\psi_-(x,z)\psi_+(y,z)w(y)f(y)\,dy. \end{aligned} \tag{7.4}$$

It follows as in the proof of Theorem 15 of [8] that $R(z) = (T-z)^{-1}$ is the resolvent of T. This gives

THEOREM 7.2. *Assume at each endpoint $\pm\infty$ hypothesis H holds and that the conditions of Theorem 2.2 hold. Then the resolvent operator for T is given by (7.4) where H is given by (7.3).*

The proof of Theorem 16 in [8] also applies to give the following result.

THEOREM 7.3. *Assume at each endpoint $\pm\infty$ hypothesis H holds and that the conditions of Theorem 2.2 hold. Then the eigenvalues of T are given by:*

$$\textit{Spectrum } T = \mathscr{S}_1 \cup \mathscr{S}_2,$$

where

$$\mathscr{S}_1 = \{z : m_+, m_- \textit{ are analytic at } z \textit{ and } m_-(z) = m_+(z)\},$$

and

$$\mathscr{S}_2 = \{z : m_+, m_- \textit{ each have a pole at } z\}.$$

We also have the analog of Theorem 3.4.

THEOREM 7.4. *Assume at each endpoint $\pm\infty$ hypothesis H holds and that the conditions of Theorem 2.2 hold. Then the resolvent operator for T is compact. Further the operator T has a Schatten class $\mathscr{C}_t$ resolvent, $t \geqslant 2$, if*

$$\int_{-\infty}^{\infty} w(x)^2 |p(x)|^{-1/2} |q(x)|^{-3/2}\, dx < \infty \tag{7.5}$$

in which case for some constant C_t, the Schatten norm of the resolvent $R(z)$ of T is given by some constant C_t,

$$\|R(z)\|_t^t \leqslant C_t \int_{-\infty}^{\infty} w(x)^t |p(x)|^{-1/2} |q(x)|^{-t+1/2}\, dx. \tag{7.6}$$

Proof. The operator $R(z)$ of (7.4) can be written as the sum of two operators $R_1(z), R_2(z)$ where $R_1(z)$, respectively, $R_2(z)$, acts on functions f with support in $(-\infty,0]$, respectively, $[0,\infty)$. For $R_2(z)$, we have

$$(R_2(z)f)(x) = \begin{cases} \int_0^\infty -M\psi_+(x,z)\psi_-(y,z)w(y)f(y)\,dy = C(f)\psi_+(x,z),\ x<0, \\ \displaystyle\int_0^x -M\psi_+(x,z)\psi_-(y,z)w(y)f(y)\,dy \\ \qquad\qquad - \displaystyle\int_x^\infty M\psi_-(x,z)\psi_+(y,z)w(y)f(y)\,dy,\ x \geqslant 0, \end{cases} \tag{7.7}$$

where $C(f) = -\int_0^\infty M\psi_-(y,z)w(y)f(y)\,dy$. Since

$$(R_2(z)(f)(0) = -M\psi_-(0,z)\int_0^\infty \psi_+(y,z)w(y)f(y)\,dy,$$

$$(R_2(z)f)'(0) = -M\psi_-'(0,z)\int_0^\infty \psi_+(y,z)w(y)f(y)\,dy,$$

we see that the $x \geqslant 0$ part of $R_2(z)$ is the resolvent for an extension of the minimal operator for the interval $[0,\infty)$ with boundary condition

$$y(0)\psi_-'(0,z) - y'(0)\psi_-(0,z) = 0,$$

and is hence a compact operator by Corollary 3.1. The operator $C(f)$ is a bounded rank one operator and is thus compact. It follows then that $R_2(z)$ is a compact operator. Similarly $R_1(z)$ is compact making $R(z)$ compact.

It also follows that R_1 is Hilbert-Schmidt by Corollary 3.1 since by the condition (7.5),

$$\int_0^\infty w(x)^2 |p(x)|^{-1/2} |q(x)|^{-3/2}\, dx < \infty.$$

Similarly R_2 is Hilbert-Schmidt which gives R_z Hilbert-Schmidt since it is the sum of two Hilbert-Schmidt operators. The proof of the Schatten class $\mathscr{C}_t$ part follows from similar considerations. $\square$

EXAMPLE 7.1. Consider the example $T[y] = -y'' + axy$, Im $a \neq 0, -\infty < x < \infty$. For $a \neq 0$ real the spectrum of T is $(-\infty, \infty)$. This follows from the fact that for a real T is selfadjoint and there are no $\mathscr{L}^2[0,\infty)$, $(a < 0)$ or $\mathscr{L}^2(-\infty, 0]$, $(a > 0)$, solutions of $T[y] = zy$, z real, as follows from the asymptotic solutions. Theorem 7.1 gives that $T = T_0$ and $\sigma_{ess}(T) = \emptyset$. Thus $\sigma(T) = \sigma_p(T)$. Herbst [22] has shown that the Schrodinger operator in n-dimensions with potential ax_1 has an empty spectrum. With $|\arg a - \pi/2| < \pi/3$ this can be seen directly. Transformation of the equation $T[y] = zy$ with $t = a^{-2/3}(ax - z)$ leads to

$$\frac{d^2y}{dt^2} - ty = 0,$$

the Airy equation. The solutions are the Airy functions $Ai(t)$ and $Bi(t)$. These have the asymptotics for large t,

$$Ai(t) \approx \frac{1}{2\sqrt{\pi}} t^{-1/4} \exp\left(-\frac{2}{3} t^{3/2}\right), \quad Bi(t) \approx \frac{1}{\sqrt{\pi}} t^{-1/4} \exp\left(\frac{2}{3} t^{3/2}\right).$$

For $|\arg a - \pi/2| < \pi/2$ the Airy function Ai is square integrable on $[0, \infty)$, while Bi is square integrable on $(-\infty, 0]$. At $x = 0 \Leftrightarrow t = -a^{-2/3} z$ these functions don't match [1, p. 446]. The nonmatching condition results from the fact that the wronskian $W(Ai, Bi) \neq 0$ so that $Ai'/Ai \neq Bi'/Bi$. Thus the spectrum is empty.

Note that Theorem 2.2 applies to the Airy operator. Let T_+, T_- be the J-selfadjoint extensions of the corresponding minimal half line operators S_+, S_-, respectively, defined by the boundary condition $y(0) = 0$. By Theorem 3.4 T_+ and T_- have Hilbert-Schmidt resolvents and by Theorem 7.4 T has a Hilbert-Schmidt resolvent. Earlier Herbst [22, Theorem II.3] has shown that T has a compact resolvent. By Theorem 12 of [8] each of T_+, T_- has a nonempty point spectrum. Let $S = T_+ \oplus T_-$. Then S has nonempty point spectrum. The resolvents of S and T differ by a rank one operator, i.e., the finite rank perturbation destroys all eigenvalues. This cannot happen with selfadjoint or normal operators where a rank l spectral projection $P(\Delta)$ will keep a dimension between $l - k$ and $l + k$ under a rank k perturbation. For Sturm-Liouville operators this is known as the interlacing property.

The example above is in contrast to the operator $L[y] = -y'' + ix^3 y$ on $(-\infty, \infty)$. This operator has real spectrum which consists of positive eigenvalues tending to infinity. See the discussion in Giordanelli and Graf [19] and references.

For $L[y] = -y'' + i|x|^n y$, $n > 2/3$, on $(-\infty, \infty)$, the theory of [8] shows that L is Hilbert-Schmidt and the eigenfunctions and associated eigenfunctions are complete in $\mathscr{L}^2(-\infty, \infty)$. Further the Dirichlet condition holds so that if z is an eigenvalue with eigenfunction y, then

$$\int_{-\infty}^{\infty} [|y'|^2 + i|x|^n |y|^2] \, dx = z \int_{-\infty}^{\infty} |y|^2 \, dx.$$

This equation shows that both the real and imaginary parts of z are positive.

8. The complex Wigner-von Neumann potential

In this section we examine for two cases the problem of eigenvalues embedded in the essential spectrum. The classic example being given by von Neumann and Wigner in 1929 [37]. Here we consider the operator for two choices of the potential q,

$$L[y] = -y'' - q(t)y,\ 0 \leqslant t < \infty,\ q_1(t) = g(t)\sin(\lambda t),\ q_2(t) = g(t)e^{i\lambda t},\ g \notin \mathscr{L}[a,\infty), \tag{8.1}$$

where λ is real and the complex valued function g also satisfies

$$g(t), tg'(t), t^2 g''(t) = O(t^{-\alpha}),\ \alpha > 1/2,\ \text{Re}\ g(t), \text{Im}\ g(t) \to 0 \text{ monotonically as } t \to \infty. \tag{8.2}$$

Thus $\text{Re}\ g(t), \text{Im}\ g(t)$ are eventually of one sign. In the case $q \in \mathscr{L}[a,\infty)$ then it is well known there are no positive eigenvalues. Since $g(t) \to 0$ as $t \to \infty$, the maximal operator T for (8.1) is a relatively compact perturbation of the maximal operator for $L[y] = -y''$; hence the essential spectrum of T is $[0,\infty)$, see Goldberg [20, p. 166]. Since the equation $-y'' = zy, z \notin [0,\infty)$, satisfies an exponential dichotomy and $g(t) \to 0$ as $t \to \infty$, then so does $-y'' + q(t)y = zy, z \notin [0,\infty)$. (Corollary 3.1 of [23]) Thus for $z \notin [0,\infty)$, nullity $T - z =$ nullity $T^* - z = 1$, and all $\mathscr{C}$-selfadjoint operators T_α, which are restrictions of T,are given by imposing a boundary condition of the form (1.7). Further for $z \notin [0,\infty), z \in \rho(T_\alpha)$ or z is an eigenvalue of T_α.

If $z \in (0,\infty)$, is an eigenvalue, scaling allows us to assume $z = 1$ so that the eigenvalue equation becomes

$$Y' = \begin{pmatrix} 0 & 1 \\ -1-q & 0 \end{pmatrix} Y, \quad Y = \begin{pmatrix} y \\ y' \end{pmatrix}.$$

Following Harris and Lutz [21], this equation is transformed by

$$T_1 = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}, \quad T_1 Z_1 = Y$$

into

$$Z_1' = \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \frac{q}{2}\begin{pmatrix} i & -1 \\ -1 & -i \end{pmatrix}\right] Z_1. \tag{8.3}$$

Splitting off the diagonal terms by $T_2 Z_2 = Z_1, T_2 =$ diag (e^{it}, e^{-it}) gives

$$Z_2' = \frac{q}{2}\begin{pmatrix} i & -e^{-2it} \\ -e^{2it} & -i \end{pmatrix} Z_2 =: AZ_2. \tag{8.4}$$

The so called $(1+Q)$-transformation has the form $T = (1+Q)$ and $Z_2 = (1+Q)Z_3$ leads to

$$Z_3' = (1+Q)^{-1}[-Q' + A(1+Q)]Z_3. \tag{8.5}$$

For this one needs that Q is small, so that $(1+Q)^{-1} = 1 - Q + Q^2 + \dots$. The $(1+Q)$ transformation is use to eliminate unwanted terms of A by counterterms of Q'. Using the identity $(1+Q)^{-1} = 1 - Q + (1+Q)^{-1}Q^2$, we may collect terms in the form

$$\begin{aligned}(1+Q)^{-1}&[-Q' + A(1+Q)] \\ &= -Q' + A + QQ' + AQ - QA - QAQ + (1+Q)^{-1}Q^2[-Q' + A(1+Q)], \quad (8.6)\end{aligned}$$

which is used for grouping of terms of different order.

The non-resonant case $\lambda \neq 2$ is particularly easy. Here Q is chosen so that Q' compensates the off-diagonal terms of A. Thus

$$Q = \begin{pmatrix} 0 & q_{12} \\ q_{21} & 0 \end{pmatrix}, \quad q_{12}' = A_{12}, \quad q_{21}' = A_{21}$$

with $q_{12}(t) = \int_t^\infty -A_{12}(s)ds$, $q_{21}(t) = \int_t^\infty -A_{21}(s)ds$. Then $q_{12}(t) = O(t^{-\alpha})$, $q_{21}(t) = O(t^{-\alpha})$, so that (8.5) reduces to

$$Z_3' = \left[\frac{q}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + R\right] Z_3, \quad R \in \mathscr{L}[a, \infty). \tag{8.7}$$

because QA, AQ are of order $t^{-2\alpha}$, and all expressions in (8.7) that involve more than two factors like $Q^2A, QAQ, \ldots$ are of order $t^{-3\alpha}$. The solutions of (8.7) are given by

$$Z_3(t) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + o(1)\right] \text{ diag } (\exp(ih(t)), \exp(-ih(t))), \quad h(t) = \int_a^t \frac{1}{2} q(s)\, ds. \tag{8.8}$$

Since q is conditinally integrable, the full solution of $Y(t) = T_1T_2(1+Q)Z_3(t)$ can, after some simplification, be written

$$y_1(t) = \cos(t + h(t)) + o(1), \quad y_2(t) = \sin(t + h(t)) + o(1).$$

In the resonant case $\lambda = 2$, q_1 and q_2 require different approaches. The classical case arises for q_1. In this case, (8.4) is transformed by

$$\tilde{Z}_3 = T_3 Z_2, \quad T_3 = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix},$$

to

$$\tilde{Z}_3' = \frac{q_1}{2}\begin{pmatrix} -\sin 2t & -1 - \cos 2t \\ 1 - \cos 2t & \sin 2t \end{pmatrix}. \tag{8.9}$$

Since the terms $q_1(t)(1 \pm \cos 2t)$ are conditionally integrable, these expressions can be removed by a $(1+Q)$-transformation as above. The fundamental matrix of (8.9) then has the form

$$\tilde{Z}_3(t) = [1 + o(1)]\text{diag}(e^{G(t)}, e^{-G(t)}), \quad G(t) = \frac{1}{2}\int_a^t q_1(s)\sin 2s\, ds.$$

Thus the solutions in this case are given by

$$y_1(t) = \rho(t)(1 + o(1))\cos t, \quad y_2(t) = \rho(t)(1 + o(1))\sin t, \tag{8.10}$$

where $\rho(t) = \exp(\frac{1}{2}\int_a^t q_1(s)\sin 2s\,ds)$. For $q_1(t) = ct^{-1}\sin(\lambda t)$, there are solutions of the form

$$y_1(t) = t^{c/4}(\cos t + o(1)), \quad y_2(t) = t^{-c/4}(\sin t + o(1)).$$

For $\mathrm{Re}\,|c| > 2$, one of these functions is square integrable and yields the eigenvalue $z = 1$ for a suitable boundary condition. Quantum mechanics explains these bound states as arising from the resonance of the eigenfunction with the potential.

For $q_2(t) = g(t)\exp(2it)$ a different approach is needed though again all conditionally integrable terms can and will be eliminated . First remove the A_{21} element by Q with only the $q_{21} \neq 0$. Thus (8.5) becomes with $q_{11} = q_{12} = q_{22} = 0$,

$$Z_3' = [-Q' + A + AQ - QA - QAQ]Z_3.$$

Integration by parts shows that $(q/2)\exp(2it)$ is conditionally integrable so we choose $q_{21}(t) = -\int_t^\infty (q(s)/2)\exp(2is)\,ds$ so as to make $q_{21}' + A_{21} = 0$. Note also integration by parts yields $|q_{21}(t)| = O(t^{-\alpha})$ by (8.2). Then

$$-Q' + A = \frac{q}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{g}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$R := AQ - QA - QAQ = \frac{q}{2}\begin{pmatrix} -q_{21}e^{-2it} & 0 \\ -2iq_{21} + q_{21}^2 e^{-2it} & q_{21}e^{-2it} \end{pmatrix}$$

The elements of R are of order $O(t^{-2\alpha})$. We now perform a second $1 + \tilde{Q}$ transformation of the same type with $\tilde{q}_{21} + R_{21} = 0$. This results in the system

$$Z_3' = \left[\frac{q}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{g}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \tilde{R}\right] Z_3, \tag{8.11}$$

where now all elements of $\tilde{R}$ are of order $O(t^{-3\alpha})$.

Set $p(t) = \exp((-i/2)\int_t^\infty q(s)\,ds)$ and make the transformation $Z_3 = \mathrm{diag}\,(p, 1/p)Z_4$ to obtain

$$Z_4' = \left[\begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \tilde{R}_{11} & 0 \\ \tilde{R}_{21}p^2 & \tilde{R}_{22} \end{pmatrix}\right] Z_4 \tag{8.12}$$

with $\mu(t) = (-g(t)/2)\exp(i\int_t^\infty q(s)\,ds$. Define $\sigma(t) = \int_a^t \mu(s)\,ds$. Theorem 1.10.1 of Eastham [15] applies if

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} \tilde{R}_{11} & 0 \\ \tilde{R}_{21}p^2 & \tilde{R}_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sigma^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{R}_{11} & 0 \\ \sigma\tilde{R}_{21}p^2 & \tilde{R}_{22} \end{pmatrix} \in \mathscr{L}[a,\infty), \tag{8.13}$$

and $|\sigma(x)/\sigma(t)| \geqslant K > 0$ for $a \leqslant t \leqslant x < \infty$. Since the elements of $\tilde{R}$ are of order $O(t^{-3\alpha})$, then $\tilde{R}_{11}, \tilde{R}_{22} \in \mathscr{L}[a,\infty)$. Again integration by parts shows that $\int_t^\infty q(s)\,ds = O(t^{-\alpha})$. Thus $p(t) = 1 + O(t^{-\alpha})$ and

$$\mu(t) = -\frac{g(t)}{2}(1 + O(t^{-\alpha})) = -\frac{g(t)}{2} + O(t^{-2\alpha}).$$

This gives $|\sigma(t)| = O(t^{1-\alpha})$, and since $\tilde{R}_{21} = O(t^{-3\alpha})$, we see that $\sigma\tilde{R}_{21}p^2 \in \mathscr{L}[a,\infty)$. Further since the real and complex parts of g are monotone, $g \notin \mathscr{L}[a,\infty)$, and $\sigma(t) = -\int_a^t g(s)/2\,ds +$ a bounded function , it follows that for some K, $|\sigma(x)/\sigma(t)| \geqslant K > 0$ for $a \leqslant t \leqslant x < \infty$.

This yields by Theorem 1.10.1 of Eastham [15] a solution of (8.12) with the asymptotic form

$$Z_4 = \begin{pmatrix} \sigma & 1 \\ 1 & \sigma^{-1} \end{pmatrix} (1 + o(1)). \tag{8.14}$$

This gives a fundamental matrix Y for $L[y] = y$ with the asymptotic form

$$Y = T_1 T_2 \begin{pmatrix} p\sigma & p \\ 1/p & 1/p\sigma \end{pmatrix} (1 + o(1)) = T_1 T_2 \begin{pmatrix} \sigma & 1 \\ 1 & 1/\sigma \end{pmatrix} (1 + o(1)).$$

For there to be an $\mathscr{L}^2[a.\infty)$ solution of $L[y] = y$, there must be a linear combination of σ, 1 or of $1/\sigma$, 1 which is in $\mathscr{L}^2[a.\infty)$. This does not happen since $g \notin \mathscr{L}[a,\infty)$. Thus for no T_α does $z = 1$ appear as an embedded eigenvalue as in contrast to the classical case.

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Don Hinton
Mathematics Department
University of Tennessee
Knoxville, TN 37996, USA
e-mail: hinton@math.utk.edu

Horst Behncke
Fachbereich Mathematik/Informatik
Universitat Osnabruck
49069 Osnabruck, Germany
e-mail: sabine.schroeder@uni-osnabrueck.de