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A HIAI–LIN TYPE LOG–MAJORIZATION VIA BLOCK–MATRICES

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Abstract. A Hiai-Lin type log-majorization, which is a simultaneous extension of [3, Corollary 3.1] and [4, Corollaries 3.3 and 3.4], is obtained via a block-matrix technique.

1. Introduction

A capital letter, such as T, stands for an $n\times n$ matrix. $T\geqslant O$ means that T is a positive semidefinite matrix; $T>O$ means that T is a positive definite matrix.

Recall that for $X,Y\geqslant O$, the log-majorization $X \underset{(\log)}{\succ} Y$ means that

$$\begin{cases} \prod_{i=1}^{k}\lambda_i(X)\geqslant \prod_{i=1}^{k}\lambda_i(Y); & k=1,2,\cdots,n-1\\ \prod_{i=1}^{k}\lambda_i(X)= \prod_{i=1}^{k}\lambda_i(Y), & k=n \end{cases}$$

where $\lambda_1(X)\geqslant\lambda_2(X)\geqslant\cdots\geqslant\lambda_n(X)$ are the eigenvalue of X in decreasing order counting multiplicities.

For $A,B>O$ and $\alpha\in[0,1]$, the α-power mean or α-weighted geometric mean of A and B, which is an operator mean in the Kubo-Ando sense, denoted by $A\sharp_\alpha B$, means that

$$A\sharp_\alpha B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}},$$

and it can be extended to $A,B\geqslant O$ by

$$\lim_{\varepsilon\to\infty}(A+\varepsilon I)\sharp_\alpha(B+\varepsilon I),$$

where I is the $n\times n$ identity matrix.

Recently, F. Hiai and M. Lin obtained a log-majorization in [2, Theorem 2.5] as follows.

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THEOREM 1.1. (Hiai-Lin log-majorization, [2]) *If* $A,B \geqslant O$, *then*

$$A^{\frac{1}{2}}BA^{\frac{1}{2}} \underset{(\log)}{\succ} (A\sharp_{\alpha}B)^{\frac{1}{2}}(A\sharp_{1-\alpha}B)(A\sharp_{\alpha}B)^{\frac{1}{2}} \tag{1.1}$$

holds for $\alpha \in [0,1]$.

As an extension of Hiai-Lin log-majorization, R. Lemos and G. Soares proved the following result in [3, Corollary 3.1].

THEOREM 1.2. ([3]) *If* $A,B,X \geqslant O$, *then*

$$A^{\frac{1}{2}}XBXA^{\frac{1}{2}} \underset{(\log)}{\succ} (A\sharp_{\alpha}B)^{\frac{1}{2}}X(A\sharp_{1-\alpha}B)X(A\sharp_{\alpha}B)^{\frac{1}{2}} \tag{1.2}$$

holds for $\alpha \in [0,1]$.

Lemos and Soares proved Theorem 1.2 as a corollary of an eigenvalue inequality, involving of matrix connections. Very recently, we showed another extensions of Hiai-Lin log-majorization in [4, Corollaries 3.3 and 3.4] as follows.

THEOREM 1.3. ([4, Corollary 3.3]) *If* $A,B > O$, *then*

$$(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{-r} \underset{(\log)}{\succ} (A\sharp_{t-r-1}B)^{\frac{1}{2}}A^{-1-r}(A\sharp_{1-t}B)A^{-1-r}(A\sharp_{t-r-1}B)^{\frac{1}{2}} \tag{1.3}$$

holds for $t \in [0,\frac{1}{2}]$ *and* $1 \geqslant -r \geqslant 1-t \geqslant \frac{1}{2}$.

THEOREM 1.4. ([4, Corollary 3.4]) *If* $A,B > O$, *then (1.3) holds for* $t \in [\frac{1}{2},1]$ *and* $1 \geqslant -r \geqslant 1-t \geqslant 0$.

Put $\alpha = t$ and $\beta = -r$ in Theorem 1.3 and Theorem 1.4. It is obvious to obtain the following theorem together with them.

THEOREM 1.5. *If* $A,B > O$, *then*

$$(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\beta} \underset{(\log)}{\succ} (A\sharp_{\alpha+\beta-1}B)^{\frac{1}{2}}A^{-1+\beta}(A\sharp_{1-\alpha}B)A^{-1+\beta}(A\sharp_{\alpha+\beta-1}B)^{\frac{1}{2}} \tag{1.4}$$

holds for $\alpha,\beta \in [0,1]$ *and* $\alpha+\beta \geqslant 1$.

In this paper, we will show a Hiai-Lin type log-majorization which is a simultaneus extension of Theorem 1.2 and Theorem 1.5, by block-matrix technique.

2. Main result

In this section, we will state the main result. Then we will introduce some lemmas refer to block-matrix. Lastly, we will prove the main result.

First, let us state the main result as follows.

THEOREM 2.1. *If* $A, B, X > O$, *then*

$$(A^{\frac{1}{2}} X B X A^{\frac{1}{2}})^{\beta} \underset{(\log)}{\succ} (A \sharp_{\alpha+\beta-1} B)^{\frac{1}{2}} (A^{-1} \sharp_{\beta} X)(A \sharp_{1-\alpha} B)(A^{-1} \sharp_{\beta} X)(A \sharp_{\alpha+\beta-1} B)^{\frac{1}{2}} \quad (2.1)$$

holds for $\alpha, \beta \in [0,1]$ *and* $\alpha + \beta \geqslant 1$.

In order to prove the main result, we introduce some lemmas here.

LEMMA 2.1. (Löwner-Heinz Inequality, [1, Theorem 4.2.1]) *If* $A \geqslant B \geqslant O$, *then* $A^{\alpha} \geqslant B^{\alpha}$ *holds for* $\alpha \in [0,1]$.

LEMMA 2.2. ([1, Theorem 1.3.3]) *For* $A, B > O$,

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant O$$

is equivalent to $B \geqslant X^* A^{-1} X$.

LEMMA 2.3. *For* $A, X > O$, *if*

$$\begin{bmatrix} A & X \\ X & I \end{bmatrix} \geqslant O,$$

then

$$\begin{bmatrix} A^{1-\alpha} & X^{\beta} \\ X^{\beta} & A^{\alpha+\beta-1} \end{bmatrix} \geqslant O,$$

where $\alpha, \beta \in [0,1]$ *and* $\alpha + \beta \geqslant 1$.

Proof. If

$$\begin{bmatrix} A & X \\ X & I \end{bmatrix} \geqslant O,$$

according to Lemma 2.2, $I \geqslant X A^{-1} X$ holds. It follows that $A \geqslant X^2$. Notice that $\alpha + \beta - 1 \in [0,1]$ and $1 - \alpha \in [0,1]$. Applying Löwner-Heinz Inequality twice, we have

$$A^{\alpha+\beta-1} \geqslant X^{2(\alpha+\beta-1)} = X^{\beta} X^{2(\alpha-1)} X^{\beta} \geqslant X^{\beta} A^{\alpha-1} X^{\beta}. \quad (2.2)$$

Thus,

$$\begin{bmatrix} A^{1-\alpha} & X^{\beta} \\ X^{\beta} & A^{\alpha+\beta-1} \end{bmatrix} \geqslant O$$

holds by Lemma 2.2. □

THEOREM 2.2. *For* $A, B \geqslant O$ *and* $X > O$, *if*

$$\begin{bmatrix} B & X \\ X & A \end{bmatrix} \geqslant O,$$

then

$$\begin{bmatrix} A \sharp_{1-\alpha} B & A \sharp_{\beta} X \\ A \sharp_{\beta} X & A \sharp_{\alpha+\beta-1} B \end{bmatrix} \geqslant O,$$

where $\alpha,\beta \in [0,1]$ *and* $\alpha+\beta \geqslant 1$.

Proof. Without loss of generality, we may consider $A,B > O$, otherwise we can relpace A and B by $A+\varepsilon I$ and $B+\varepsilon I$.

Notice that

$$\begin{bmatrix} A^{-\frac{1}{2}} & O \\ O & A^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} B & X \\ X & A \end{bmatrix} \begin{bmatrix} A^{-\frac{1}{2}} & O \\ O & A^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} A^{-\frac{1}{2}}BA^{-\frac{1}{2}} & A^{-\frac{1}{2}}XA^{-\frac{1}{2}} \\ A^{-\frac{1}{2}}XA^{-\frac{1}{2}} & I \end{bmatrix}. \tag{2.3}$$

Thus,

$$\begin{bmatrix} B & X \\ X & A \end{bmatrix} \geqslant O$$

is equivalent to

$$\begin{bmatrix} A^{-\frac{1}{2}}BA^{-\frac{1}{2}} & A^{-\frac{1}{2}}XA^{-\frac{1}{2}} \\ A^{-\frac{1}{2}}XA^{-\frac{1}{2}} & I \end{bmatrix} \geqslant O.$$

According to Lemma 2.3,

$$\begin{bmatrix} (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha} & (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{\beta} \\ (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{\beta} & (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha+\beta-1} \end{bmatrix} \geqslant O$$

holds.

Therefore,

$$\begin{aligned} &\begin{bmatrix} A\sharp_{1-\alpha}B & A\sharp_{\beta}X \\ A\sharp_{\beta}X & A\sharp_{\alpha+\beta-1}B \end{bmatrix} \\ &= \begin{bmatrix} A^{\frac{1}{2}} & O \\ O & A^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\alpha} & (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{\beta} \\ (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{\beta} & (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha+\beta-1} \end{bmatrix} \begin{bmatrix} A^{\frac{1}{2}} & O \\ O & A^{\frac{1}{2}} \end{bmatrix} \end{aligned}$$

is positive semidefinite. □

COROLLARY 2.1. *For* $A,B \geqslant O$ *and* $X > O$, *if*

$$\begin{bmatrix} B & X \\ X & A \end{bmatrix} \geqslant O,$$

then

$$\begin{bmatrix} A\sharp_{1-\alpha}B & X \\ X & A\sharp_{\alpha}B \end{bmatrix} \geqslant O$$

if $\alpha \in [0,1]$.

Next, we will present the proof of Theorem 2.1.

Proof of Theorem 2.1. Notice that

$$I \geqslant A^{\frac{1}{2}}XBXA^{\frac{1}{2}}. \tag{2.4}$$

is equivalent to $A^{-1} \geqslant XBX$ and to

$$\begin{bmatrix} B^{-1} & X \\ X & A^{-1} \end{bmatrix} \geqslant O \tag{2.5}$$

according to Lemma 2.2.

According to Theorem 2.2, (2.4) implies

$$\begin{bmatrix} A^{-1}\sharp_{1-\alpha}B^{-1} & A^{-1}\sharp_{\beta}X \\ A^{-1}\sharp_{\beta}X & A^{-1}\sharp_{\alpha+\beta-1}B^{-1} \end{bmatrix} \geqslant O. \tag{2.6}$$

Similarly, (2.6) is equivalent to

$$A^{-1}\sharp_{\alpha+\beta-1}B^{-1} \geqslant (A^{-1}\sharp_{\beta}X)(A^{-1}\sharp_{1-\alpha}B^{-1})^{-1}(A^{-1}\sharp_{\beta}X), \tag{2.7}$$

and (2.7) is equivalent to

$$I \geqslant (A\sharp_{\alpha+\beta-1}B)^{\frac{1}{2}}(A^{-1}\sharp_{\beta}X)(A\sharp_{1-\alpha}B)(A^{-1}\sharp_{\beta}X)(A\sharp_{\alpha+\beta-1}B)^{\frac{1}{2}}. \tag{2.8}$$

At last, we will prove that the equality of the determinants of the matrices in the right hand side and the left hand side of (2.1).

Notice that $\det X\sharp_{\gamma}Y = (\det X)^{1-\gamma}(\det Y)^{\gamma}$. We have that

$$\begin{aligned}
&\det((A\sharp_{\alpha+\beta-1}B)^{\frac{1}{2}}(A^{-1}\sharp_{\beta}X)(A\sharp_{1-\alpha}B)(A^{-1}\sharp_{\beta}X)(A\sharp_{\alpha+\beta-1}B)^{\frac{1}{2}})\\
=&(\det A\sharp_{\alpha+\beta-1}B)^{\frac{1}{2}}(\det A^{-1}\sharp_{\beta}X)(\det A\sharp_{1-\alpha}B)(\det A^{-1}\sharp_{\beta}X)(\det A\sharp_{\alpha+\beta-1}B)^{\frac{1}{2}}\\
=&(\det A\sharp_{\alpha+\beta-1}B)(\det A^{-1}\sharp_{\beta}X)^{2}(\det A\sharp_{1-\alpha}B)\\
=&(\det A)^{2-\alpha-\beta}(\det B)^{\alpha+\beta-1}((\det A^{-1})^{1-\beta}(\det X)^{\beta})^{2}(\det A)^{\alpha}(\det B)^{1-\alpha}\\
=&(\det A)^{\beta}(\det B)^{\beta}(\det X)^{2\beta}\\
=&((\det A)^{\frac{1}{2}}(\det X)(\det B)(\det X)(\det A)^{\frac{1}{2}})^{\beta}\\
=&\det(A^{\frac{1}{2}}XBXA^{\frac{1}{2}})^{\beta}.
\end{aligned}$$

By the well-known antisymmetric tensor power technique, (2.1) holds obviously according to the fact that (2.4) implies (2.8). □

REMARK 2.1. It is easy to prove that Theorem 2.2 can be derived from Theorem 2.1, if we replacd A by A^{-1} and B by B^{-1} in (2.1). Therefore, Theorem 2.2 and Theorem 2.1 are equivalent if $A,B > O$.

REMARK 2.2. If we put $X = I$ in Theorem 2.1, then it is just Theorem 1.5.

REMARK 2.3. If we put $\beta = 1$ in Theorem 2.1, then it is just Theorem 1.2.

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