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A NOTE ON BAPAT'S q–PERMANENT CONJECTURE

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Abstract. Ravindra Bapat conjectured the q-permanent of a non-diagonal Hermitian positive definite matrix is a strictly increasing (in q) interpolation between the determinant ($q=-1$) and the permanent ($q=1$). We prove that this is true for non-diagonal positive definite matrices if and only if it is true for singular positive semidefinite matrices without a zero row. Thus we conjecture the q-permanent of a non-diagonal Hermitian positive semidefinite matrix without a zero row is strictly increasing on $[-1,1]$. We prove this extended conjecture in the rank-one case and the 3-by-3 case.

The q-permanent of a n-by-n matrix $A=(a_{ij})$ is defined by

$$p_q(A) := \sum_{\sigma\in S_n} a_{1\sigma(1)}\cdots a_{n\sigma(n)} q^{l(\sigma)},$$

where $l(\sigma)$ is the number of inversions of the permutation σ in the symmetric group S_n. The polynomial $p_q(A)$ interpolates between the determinant ($q=-1$) and the permanent ($q=1$). Ravindra Bapat conjectured that when A is a non-diagonal (Hermitian) positive definite (PD) matrix, $p_q(A)$ is a strictly increasing function of q for $q\in[-1,1]$, and he proved this conjecture for 3-by-3 matrices [2]. Proofs have also been given in special cases [13], including for tridiagonal matrices [9]. Other aspects of $p_q(A)$ have been explored [1, 4, 10, 12], but even for $n=4$ the conjecture remains difficult [3]. Indeed, the fundamental fact that $p_q(A)\geqslant 0$ for $q\in[-1,1]$ was not easy to establish, and comes from work of Bożejko and Speicher on generalized Brownian motions [5]. They showed that for $q\in(-1,1)$,

$$\langle g_1\otimes g_2\otimes\cdots\otimes g_n, h_1\otimes h_2\otimes\cdots\otimes h_n\rangle_q = \sum_{\sigma\in S_n} q^{l(\sigma)}\langle g_1,h_{\sigma(1)}\rangle\cdots\langle g_n,h_{\sigma(n)}\rangle$$

defines an inner product on the full Fock space $\mathscr{F}=\bigoplus_{k\geqslant 0}\mathscr{H}^{\otimes k}$, where $\mathscr{H}$ is a complex separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\mathscr{H}^0=\mathbb{C}$. Thus if A is the Gram matrix [8] of non-zero vectors $v_1,v_2,\ldots,v_n\in\mathscr{H}$ and $q\in[-1,1]$, then

$$p_q(A) = \langle v_1\otimes v_2\otimes\cdots\otimes v_n, v_1\otimes v_2\otimes\cdots\otimes v_n\rangle_q \geqslant 0.$$

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There is also a recursive definition,

$$\begin{aligned}\langle g_1\otimes g_2\otimes\cdots\otimes g_n, h_1\otimes h_2\otimes\cdots\otimes h_n\rangle_q\\ =\sum_{k=1}^{n} q^{k-1}\langle g_1,h_k\rangle\langle g_2\otimes\cdots\otimes g_n, h_1\otimes\cdots\otimes h_{k-1}\otimes h_{k+1}\otimes\cdots\otimes h_n\rangle_q,\end{aligned}$$

which can be useful to explain, for example, a fact that we will use in what follows: Given $w, v_1, v_2, \ldots, v_n \in \mathscr{H}$ with w orthogonal to each v_i,

$$\begin{aligned}&\langle v_1\otimes v_2\otimes\cdots\otimes v_{k-1}\otimes(v_k+w)\otimes v_{k+1}\otimes\cdots\otimes v_n,\\ &\qquad v_1\otimes v_2\otimes\cdots\otimes v_{k-1}\otimes(v_k+w)\otimes v_{k+1}\otimes\cdots\otimes v_n\rangle_q\\ &\qquad=\langle v_1\otimes v_2\otimes\cdots\otimes v_n, v_1\otimes v_2\otimes\cdots\otimes v_n\rangle_q\\ &\quad+\langle v_1\otimes v_2\otimes v_{k-1}\otimes w\otimes v_{k+1}\otimes\cdots\otimes v_n, v_1\otimes v_2\otimes v_{k-1}\otimes w\otimes v_{k+1}\otimes\cdots\otimes v_n\rangle_q.\end{aligned}$$

We will first consider the relationship between PD and positive semidefinite (PSD) matrices with respect to Bapat's conjecture.

THEOREM 1. *Bapat's q-permanent conjecture is true for non-diagonal* PD *matrices only if it is true for all singular* PSD *matrices that have no zero row.*

Proof. We will prove the contrapositive. Let n be fixed and let A be an n-by-n singular PSD matrix with no zero row. Then A is the Gram matrix of non-zero linearly dependent vectors $v_1, v_2, \ldots, v_n$ in $\mathscr{H}=\mathbb{C}^n$. Since $v_1, \ldots, v_n$ are linearly dependent, there exists i such that v_i is in the span of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$, and so A is non-diagonal. Let

$$P(q) := \langle v_1\otimes v_2\otimes\cdots\otimes v_n, v_1\otimes v_2\otimes\cdots\otimes v_n\rangle_q,$$

and assume $P'(a) < 0$ for some $a \in (-1,1)$. Let $w \in \mathbb{C}^n$ be a vector orthogonal to each v_i. Using

$$Q(q) := \langle v_1\otimes v_2\otimes\cdots\otimes v_{i-1}\otimes w\otimes v_{i+1}\otimes\cdots\otimes v_n, v_1\otimes v_2\otimes\cdots\otimes v_{i-1}\otimes w\otimes v_{i+1}\otimes\cdots\otimes v_n\rangle_q,$$

and choosing $\varepsilon > 0$ small enough that $\varepsilon^2|Q'(a)| < |P'(a)|$, we have $R'(a) < 0$ where

$$\begin{aligned}R(q) := \langle &v_1\otimes v_2\otimes\cdots\otimes v_{i-1}\otimes(v_i+\varepsilon w)\otimes v_{i+1}\otimes\cdots\otimes v_n,\\ &v_1\otimes v_2\otimes\cdots\otimes v_{i-1}\otimes(v_i+\varepsilon w)\otimes v_{i+1}\otimes\cdots\otimes v_n\rangle_q.\end{aligned}$$

Thus, if the conjecture is not true for some n-by-n singular PSD matrix of rank r that has no zero row, it also fails for a PSD matrix of rank $r+1$ that has no zero row. Since A was non-diagonal, so is the Gram matrix of $v_1, v_2, \ldots, v_{i-1}, v_i+\varepsilon w, v_{i+1}, \ldots, v_n$. Using induction, the conjecture is not true for some non-diagonal n-by-n PD matrix. $\square$

Theorem 1 suggests an extension of the original conjecture to PSD matrices. For PSD matrices, having a non-constant q-permanent is equivalent to the matrix having non-zero diagonal entries and (if the matrix is PD) not being a diagonal matrix. We will call such a matrix QPSD.

CONJECTURE. If A is a QPSD matrix, then $p_q(A)$ is a strictly increasing function of q for $q \in [-1,1]$.

Having established that QPSD matrices could create counterexamples to the original conjecture, we can ask if there are classes of QPSD matrices for which the extended conjecture can be proved.

THEOREM 2. *The extended conjecture is true for rank-one* QPSD *matrices.*

Proof. If e is a unit vector and $c_1, c_2, \dots, c_n$ are non-zero complex numbers, then

$$\langle c_1 e \otimes c_2 e \otimes \cdots \otimes c_n e, c_1 e \otimes c_2 e \otimes \cdots \otimes c_n e \rangle_q = |c_1||c_2|\cdots|c_n| \prod_{k=1}^{n-1}(1+q+\cdots+q^k).$$

Define $f_m(q) := (1+q+\cdots+q^m)$ and $h_m(q) = f_{m-1}(q) f_m(q)$. Then for all m, both $f_m(q)$ and $h_m(q)$ are non-negative for $q \in [-1,1]$ and strictly increasing for $q \geqslant 0$, so it remains to consider $q < 0$. For $q \neq 1$ we find

$$f_m(q) = \frac{1-q^{m+1}}{1-q}$$

and

$$h_m(q) = \frac{(1-q^m)(1-q^{m+1})}{(1-q)^2},$$

so that, for $q \neq 1$, we can write

$$f_m'(q) = \frac{f_m(q) - (m+1)q^m}{1-q}$$

and

$$h_m'(q) = \frac{2-3q^m+2q^{2m}}{(1-q)^3} - q^{m-1}\frac{m-(m-1)q^2}{(1-q)^3} + \frac{(2m-1)q^{2m}}{(1-q)^2}$$

and see $f_m'(q) > 0$ for $q \in [-1,0]$ when m is odd and $h_m'(q) > 0$ for $q \in [-1,0]$ when m is even. Since we can write $\prod_{k=1}^{n-1}(1+q+\cdots+q^k)$ as $\prod_{j=1}^{(n-1)/2} h_{2j}(q)$ if n is odd or $f_{n-1}(q)\prod_{j=1}^{(n-2)/2} h_{2j}(q)$ if n is even, we find that for $q \in [-1,1]$, $\prod_{k=1}^{n-1}(1+q+\cdots+q^k)$ is a product of non-negative strictly increasing functions and hence is also strictly increasing for $q \in [-1,1]$. $\square$

REMARK. Carlos da Fonseca has conjectured that for every non-diagonal PD matrix A there is an $r < -1$ such that $p_q(A)$ is strictly increasing on (r,∞) [6]. This is not true in the QPSD case as $\prod_{k=1}^{j}(1+q+\cdots+q^k)$ shows when $j=3$ (and seems to be the case for $j \geqslant 3$ and $j \equiv 0$ or $j \equiv 3$ modulo 4). However, these examples seem not to lead to counterexamples to da Fonseca's conjecture, as numerical evidence suggests $p_q(J+\varepsilon I)$, for $\varepsilon > 0$ where J is the matrix of all ones and I is the identity matrix, does satisfy the conjecture.

We can also prove a stronger converse to Theorem 1:

THEOREM 3. *If the extended conjecture is true for all* QPSD *matrices of nullity* 1*, the extended conjecture is true for all* QPSD *matrices.*

Proof. Assume the extended conjecture is true for all QPSD matrices of nullity 1. By the proof of Theorem 1 the only possible counterexample to the extended conjecture would be a non-diagonal PD matrix. Proceed by induction on n, first noting the result is true for $n=2$. Let A be an n-by-n non-diagonal PD matrix with $n \geqslant 3$, so that A is the Gram matrix of non-zero linearly independent vectors $v_1, v_2, \ldots, v_n$. We can write

$$\begin{aligned}
&\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n, v_1 \otimes v_2 \otimes \cdots \otimes v_n \rangle_q \\
=&\langle (v_1' + w) \otimes v_2 \otimes \cdots \otimes v_n, (v_1' + w) \otimes v_2 \otimes \cdots \otimes v_n \rangle_q \\
=&\langle v_1' \otimes v_2 \otimes \cdots \otimes v_n, v_1' \otimes v_2 \otimes \cdots \otimes v_n \rangle_q + \langle w \otimes v_2 \otimes \cdots \otimes v_n, w \otimes v_2 \otimes \cdots \otimes v_n \rangle_q \\
=&\langle v_1' \otimes v_2 \otimes \cdots \otimes v_n, v_1' \otimes v_2 \otimes \cdots \otimes v_n \rangle_q + \langle w, w \rangle \langle v_2 \otimes v_3 \otimes \cdots \otimes v_n, v_2 \otimes v_3 \otimes \cdots \otimes v_n \rangle_q,
\end{aligned}$$

where w is non-zero and orthogonal to $v_2, \ldots, v_n$ and v_1' is in the span of $v_2, v_3, \ldots, v_n$. If $v_1' = 0$, then the Gram matrix of $v_2, v_3, \ldots, v_n$ is an $(n-1)$-by-$(n-1)$ non-diagonal PD matrix, which by the induction hypothesis has strictly increasing q-permanent on $[-1,1]$. If $v_1' \neq 0$ then the Gram matrix of $v_1', v_2, v_3, \ldots, v_n$ is an n-by-n matrix of rank $n-1$, which by assumption has strictly increasing q-permanent on $[-1,1]$. The Gram matrix of $v_2, v_3, \ldots, v_n$ may or may not be non-diagonal, but the q-permanent of A is either the sum of two strictly increasing functions or one strictly increasing function and a constant. $\square$

Thanks to Theorem 1 and Bapat's original result [2], the extended conjecture is true for 3-by-3 matrices. Using Theorem 3 allows for a simpler proof: we can assume without loss of generality we are dealing with the Gram matrix of vectors $e_1, ae_1 + ce_2, be_1 + de_2$ where e_1 and e_2 are orthonormal and a, b, c, and d are complex numbers such that $|a|^2 + |c|^2$ and $|b|^2 + |d|^2$ are non-zero. In that case, the derivative

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}q}\left(\langle e_1 \otimes (ae_1 + ce_2) \otimes (be_1 + de_2), e_1 \otimes (ae_1 + ce_2) \otimes (be_1 + de_2) \rangle_q\right) \\
&\qquad = \left|(1+q)a\overline{b} + c\overline{d}\right|^2 + |a|^2|b|^2(1+q)^2 + |ad + bcq|^2 + (|a|^2 + 2|c|^2)|b|^2 q^2
\end{aligned}$$

is non-negative for all q.

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