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OPERATOR REPRESENTATIONS OF K–FRAMES: BOUNDEDNESS AND STABILITY

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Abstract. In this paper, a K-frame $\{f_k\}_{k\in Z}$ for a Hilbert space H, with the form $\{T^k f_0\}_{k\in Z}$ for an operator T is analyzed. Some conditions under which a K-frame can be represented by an operator and then investigate the properties of this operator are discussed. More Specifically, a necessary and sufficient condition for a K-frame that has an operator representation can be obtained by a K-dual. Furthermore, we find the boundedness of the operator T has an integral relationship with the operator K when a K-frame can be represented by an operator T. In addition, the stability of operator representation is studied. We prove that the stability and boundedness are preserved under certain restrictions on the perturbation condition. A pretty small perturbation will heavily affect the property of being representable by an operator if $\mu > 0$, and an example is used to illustrate it. Furthermore, some elements from a subspace of H are used to perturb a K-frame, and then some useful stability results are obtained.

1. Introduction

Frames in Hilbert space were first introduced by Duffin and Schaeffer to deal with some problems concerned with the nonharmonic Fourier series in 1952 [c.f. [1]]. More specifically, frames are defined as follows.

DEFINITION 1.1.. Let H be a separable Hilbert space. A collection of vectors $F = \{f_i\}_{k\in Z} \subset H$ is called a frame for H if there are constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{k\in Z} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2,$$

for any $f \in H$. Constants A and B are called lower and upper frame bounds, respectively. In particular, if $A = B$, then the frame is called a tight frame for H. Especially, if $A = B = 1$, then the tight frame is called a Parseval frame for H.

It is well known that frames can span the whole Hilbert space as redundant bases. The robust, stable and non-unique representations of vectors in a Hilbert space via a frame lead to its many applications such as signal and image processing [6], quantization [16], the capacity of transmission channels [21], coding theory [4], and data transmission technology [20], etc. ([5], [7], [9], [23], [24]). With the development of

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frames theory, some special frames such as fusion frames [10], g-frames [2], group-frame [22], K-frames [8] are proposed. In this paper, K-frames are mainly studied.

DEFINITION 1.2. Let H be a separable Hilbert space and K be a bounded linear operator from H to H. A collection of vectors $F = \{f_i\}_{k\in Z} \subset H$ is called a K-frame for H if there are constants $0 < A \leqslant B < \infty$ such that

$$A\|K^*f\|^2 \leqslant \sum_{k\in Z} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2,$$

for any $f \in H$. Constants A and B are called lower and upper K-frame bounds, respectively. In particular, if $A\|K^*f\|^2 = \sum_{k\in Z} |\langle f, f_i\rangle|^2$, for any $f \in H$, the K-frame is called a tight K-frame for H. Especially, if $A = 1$, the tight K-frame is called a Parseval K-frame for H.

Since K-frames first introduced by Găvruţa [11], many scholars have studied K-frames in the literature. For example, Li Liang et al. studied a relationship between K-frames and operator K [14]. They used a synthesis operator and an operator K to characterize the optimal boundary of K-frames. Xiangchun Xiao et al. discussed the interchangeability of two Bessel sequences for K-frames and the stability of a more general perturbation for K-frames were given in [3].

Recently Ole Christensen et al. investigated the operator representation of frames in [13] and [19] in which the bound of the operator and the stability of operator representations were discussed. It is known that the frame operator of a K-frame may be invertible, so some properties of frames are not preserved for K-frames ([15], [18]). In this paper, we first consider a K-frame $\{f_k\}_{k\in Z}$ in a Hilbert space H which is formed by $\{f_k\}_{k\in Z} = \{T^k f_0\}_{k\in Z}$ for an operator $T\colon\ span\{f_k\}_{k\in Z} \to span\{f_k\}_{k\in Z}$. We call the K-frame $\{f_k\}_{k\in Z}$ is represented via the operator T. Our motivations to study K-frames of this form, which are similarly related to (Fourier) orthonormal basis, a group representation, and Gabor systems mentioned in [13]. Besides these properties, K-frames have their additional features compared to frames. Therefore, we extend the investigation to the operator representation of K-frames, and some properties of the operator K are considered in our proof.

For the rest of this paper, a necessary condition for K-frames having an operator representation for a bounded linear operator is obtained at the beginning of Section 2. Then we find the set $\{T \in B(H) |$ there is a $\varphi \in H$ such that $\{T^n\varphi\}$ is a K-frame$\}$ is not open with that necessary condition. Furthermore, a necessary and sufficient condition for T bounded is obtained. In addition, for an operator representation of Parseval K-frames, the bound of the operator T is more special than ordinary K-frames. At the end of section 2, K-duals are used to explore the necessary and sufficient conditions for K-frames to have a representation for the bounded operator. An example is used to show the application of this property. In section 3, the stability of an operator representation is discussed. We prove that under certain restrictions on the perturbation condition, stability and boundedness are preserved. Then these elements $\tilde{\varphi}$ from a subspace of H with sufficiently small norm are used to perturb a K-frame $\{T^k\varphi\}_{k\in Z}$, and we can obtain a $P_{Q(R(K))}$ K-frame $\{T^k(\varphi+\tilde{\varphi})\}_{k\in Z}$.

In the rest of this introduction, we will collect some background knowledge of frame theory and operator theory.

Firstly, the following symbols will be used.

H	The Hilbert space
I_H	The identity operator on H
$B(X,Y)$	The set of bounded linear operators from X to Y
$B(H)$	The set of bounded linear operators from H to H
$Q^\dagger$	The pseudo inverse for Q
Q^{-1}	The inverse for Q
$R(T)$ or R_T	The range of T
$N(T)$ or N_T	The kernel of T
T^*	The adjoint operator of T
$\|f\|$	The norm of f
$\|T\|$	The operator norm of operator T

Next, we review the definition of the Bessel sequence.

DEFINITION 1.3. Let H be a separable Hilbert space. A collection of vectors $F=\{f_i\}_{k\in Z}\subset H$ is called a Bessel sequence for H if there is a constant $0<B$ such that

$$\sum_{k\in Z}|\langle f,f_i\rangle|^2\leqslant B\|f\|^2,$$

for any $f\in H$. And the constant B is called Bessel bound for the Bessel sequence.

In addition, similar to the dual frame, we introduce the definition of the K-dual.

DEFINITION 1.4. For a K-frame $F=\{f_i\}_{i\in Z}\subset H$, $K\in B(H)$, if there is a sequence $G=\{g_i\}_{i\in Z}$ such that

$$Kf=\textstyle\sum_{i\in Z}\langle f,g_i\rangle f_i,$$

and

$$K^*f=\textstyle\sum_{i\in Z}\langle f,f_i\rangle g_i$$

for any $f\in H$. Then $G=\{g_i\}_{i\in Z}$ is a K-dual of $F=\{f_i\}_{i\in Z}$.

Then we introduce the following operators which will often appear throughout this paper.

DEFINITION 1.5. Let $F=\{f_i\}_{k\in Z}\subset H$ be a Bessel sequence for H.

(I) The analysis operator of F is defined by

$$T:H\rightarrow\ell^2, Tf=\{\langle f,f_i\rangle\}_{i\in Z}.$$

(II) The synthesis operator of F is defined by

$$T^*:\ell^2\rightarrow H, T^*\{c_i\}_{i\in Z}=\sum_{i\in Z}c_if_i.$$

(III) The frame operator of F is defined by

$$S:H\rightarrow H, Sf=T^*Tf=\sum_{i\in Z}\langle f,f_i\rangle f_i.$$

Sometimes the analysis operator of F is defined by

$$T : H \to \ell^2, Tf = \sum_{i\in Z} \langle f, f_i\rangle e_i,$$

where $\{e_i\}_{k\in Z}$ is a canonical basis of ℓ^2.

DEFINITION 1.6. (c.f. [14]) Let X, Y be Banach spaces, $Q \in B(X,Y)$ be a closed-value operator. If $Q^+ \in B(Y,X)$ such that $QQ^\dagger Q = Q$, then $Q^\dagger$ is called a pseudo inverse for Q. Obviously, for any $f \in R(Q)$, we have $QQ^\dagger f = f$.

2. Representations of K-frames

In this section, we mainly introduce the operator representation of K-frames with some operators and some applications of the operator representation.

Firstly, we present the following two very important lemmas that will be used later.

LEMMA 2.1. (c.f. [12]) *Consider any sequence $\{f_k\}_{k=1}^\infty$ for which span$\{f_k\}_{k=1}^\infty$ is infinite-dimensional. Then the following are equivalent:*

(i) $\{f_k\}_{k=1}^\infty$ is linearly independent.

(ii) There is a linear operator T: span$\{f_k\}_{k=1}^\infty \to$ span$\{f_k\}_{k=1}^\infty$ such that

$$\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=1}^\infty.$$

Then according to the proof of Lemma 2.1, we can obtain Lemma 2.2 easily.

LEMMA 2.2. *Consider any sequence $\{f_k\}_{k\in Z}$ for which span$\{f_k\}_{k\in Z}$ is infinite-dimensional. Then the following are equivalent:*

(i) $\{f_k\}_{k\in Z}$ is linearly independent.

(ii) There is a linear operator T: span$\{f_k\}_{k\in Z} \to$ span$\{f_k\}_{k\in Z}$ such that

$$\{f_k\}_{k\in Z} = \{T^n f_1\}_{n\in Z}.$$

Now we give a necessary condition for a K-frame $\{f_k\}_{k\in Z}$ for H to have an operator representation on the form $\{T^k f_0\}_{k\in Z}$ for a bounded linear operator T.

PROPOSITION 2.1. *If a K-frame $\{f_k\}_{k\in Z}$ for H have the form $\{T^k f_0\}_{k\in Z}$ for a bounded linear operator T: span$\{f_k\}_{k\in Z} \to$ span$\{f_k\}_{k\in Z}$. Then $\|T\| \geqslant 1$.*

Proof. Let A, B denote the K-frame bounds for $\{f_k\}_{k\in Z}$, respectively, then for any $f \neq 0$, we have

$$\begin{aligned} A\|K^* f\|^2 &\leqslant \sum_{k\in Z} |\langle f, T^k f_0\rangle|^2 \\ &= \sum_{k\in Z} |\langle (T^n)^* f, T^{k-n} f_0\rangle|^2 \\ &= \sum_{k\in Z} |\langle (T^n)^* f, T^k f_0\rangle|^2 \\ &\leqslant B\|(T^n)^* f\|^2 \leqslant B\|T\|^{2n}\|f\|^2. \end{aligned}$$

Therefore

$$A\|K^*f\|^2 \leqslant B\|T\|^{2n}\|f\|^2,$$

thus

$$\frac{A}{B}\frac{\|K^*f\|^2}{\|f\|^2} \leqslant \|T\|^{2n}.$$

According to different values of $\frac{\|K^*f\|}{\|f\|}$, we discuss them separately. If $\frac{\|K^*f\|}{\|f\|} \geqslant \sqrt{\frac{B}{A}}$, we obtain

$$1 \leqslant \frac{A}{B}\frac{\|K^*f\|^2}{\|f\|^2} \leqslant \|T\|^{2n}.$$

Hence $\|T\| \geqslant 1$.

If $\frac{\|K^*f\|}{\|f\|} \leqslant \sqrt{\frac{B}{A}}$, since

$$\frac{A}{B}\frac{\|K^*f\|^2}{\|f\|^2} \leqslant \|T\|^{2n},$$

for all $n \in N$, we have $\|T\| \geqslant 1$.

Above all, we obtain $\|T\| \geqslant 1$. □

In fact, Proposition 2.1 gives us a lower bound for the operator T. Then we use an example to show the application of the lower bound. More specifically, we use Proposition 2.1 to show the set of bounded operators T such that the sequence $\{T^n\varphi\}_{n\in Z}$ is a K-frame for some $\varphi \in H$ does not form an open set.

EXAMPLE 2.1. Consider any K-frame $\{f_k\}_{k\in Z}$ for H on the form $\{T^n\varphi\}_{n\in Z}$ for a linear operator $T\colon H \to H$ and $\|T\| = 1$. Now, given any $\varepsilon \in (0,1)$, define the operator W

$$W : H \to H \text{ and } W = (1-\varepsilon)T.$$

Then we obtain

$$\|T - W\| = \|\varepsilon T\| = \varepsilon.$$

But

$$\|W\| = \|(1-\varepsilon)T\| = 1-\varepsilon < 1.$$

According to Proposition 2.1, we know $\{W^n\varphi\}_{n\in Z}$ is not a K-frame for any $\varphi \in H$.

This implies that the set

$$\{T \in B(H) |\ \text{there exists } \varphi \in H \text{ such that } \{T^n\varphi\} \text{ is a K-frame}\}$$

is not open.

On the other hand, we want to know the upper bound of the operator T when a K-frame $\{f_k\}_{k\in Z}$ has an operator representation on the form $\{T^k f_0\}_{k\in Z}$. Hence in the following proposition, we discuss the upper bound of the operator T. Furthermore, we obtain a necessary and sufficient condition to make T bounded. Before that, we need to introduce the definition of the right-shift operator.

DEFINITION 2.1. Define the right-shift operator on $\ell^2(Z)$ by

$$\tau : \ell^2(Z) \to \ell^2(Z), \tau\{c_k\}_{k\in Z} = \{c_{k-1}\}_{k\in Z}.$$

In this case, we call that a space H is invariant under the right-shift operator τ if $\tau(H) \subseteq H$.

PROPOSITION 2.2. *Let $\{f_k\}_{k\in Z}$ be a K-frame for H, with K-frame bounds A, B, respectively. If the K-frame has a representation $\{T^k f_0\}_{k\in Z}$ for a linear operator T: span$\{f_k\}_{k\in Z} \to$ span$\{f_k\}_{k\in Z}$. Then the operator T is bounded if and only if the kernel N_U of synthesis operator U is invariant under the right-shift operator τ.*

In the affirmative case, if the operator K^* is bijective then

$$1 \leqslant \|T\| \leqslant \sqrt{\tfrac{B}{A}}\|(K^*)^{-1}\|.$$

Proof. First, if T is a bounded linear operator, then for any $\{c_k\}_{k\in Z} \in N_U$, we have

$$U\tau\{c_k\}_{k\in Z} = \textstyle\sum_{k\in Z} c_{k-1} f_k = \sum_{k\in Z} c_k f_{k+1} = T\sum_{k\in Z} c_k f_k = T0 = 0.$$

That is $\tau(N_U) \in N_U$. Hence we obtain N_U is invariant under the right-shift operator τ.

Next if N_U is invariant under right-shift operator τ. Then for any $f \in$ span$\{f_k\}_{k\in Z}$,

$$f = \textstyle\sum_{k=m}^{n} c_k f_k,$$

where $c_k = 0$ for $k > n$ and $k < m$.

So

$$\{c_k\}_{k\in Z} = \{d_k\}_{k\in Z} + \{r_k\}_{k\in Z},$$

where $\{d_k\}_{k\in Z} \in N_U$ and $\{r_k\}_{k\in Z} \in N_U^{\perp}$.

Since $\{d_k\}_{k\in Z} \in N_U$, we obtain

$$\textstyle\sum_{k\in Z} d_k f_k = 0.$$

Since N_U is invariant under right-shift, we get

$$\textstyle\sum_{k\in Z} d_k f_{k+1} = 0.$$

Furthermore

$$\begin{aligned}\|Tf\|^2 = \|T\sum_{k=m}^{n} c_k f_k\|^2 = \|\sum_{k=m}^{n} c_k f_{k+1}\|^2 \\ = \|\sum_{k\in Z}(d_k + r_k) f_{k+1}\|^2 = \|\sum_{k\in Z} r_k f_{k+1}\|^2.\end{aligned}$$

Since $\{f_k\}_{k\in Z}$ is a Bessel sequence, we have

$$\begin{aligned}\|\sum_{k\in Z} r_k f_{k+1}\|^2 &\leqslant (\sum_{k\in Z}\|r_k f_{k+1}\|)^2\\ &\leqslant \sum_{k\in Z}\|r_k\|^2\sum_{k\in Z}\|f_{k+1}\|^2\\ &\leqslant B\sum_{k\in Z}\|r_k\|^2.\end{aligned}$$

That is

$$\|Tf\|^2 \leqslant B\sum_{k\in Z}\|r_k\|^2.$$

For the synthesis operator U, we know

$$R_{U^*} = \overline{R_{U^*}} = N_U^{\perp},$$

since R_{U^*} is closed.

Moreover, we know $N_U^{\perp}$ is consists of all sequences of the form $\{\langle f, f_k\rangle\}_{k\in Z}$, $f\in H$.

Then for any $f\in H$,

$$\begin{aligned}(\sum_{k\in Z}|\langle f, f_k\rangle|^2)^2 = |\langle Sf, f\rangle|^2 &\leqslant \|Sf\|^2\|f\|^2\\ &= \|Sf\|^2\|(K^*)^{-1}K^*f\|^2\\ &\leqslant \|Sf\|^2\|(K^*)^{-1}\|^2\|K^*f\|^2\\ &\leqslant \frac{1}{A}\|Sf\|^2\|(K^*)^{-1}\|^2\sum_{k\in Z}|\langle f, f_k\rangle|^2.\end{aligned}$$

This implies that

$$\sum_{k\in Z}|\langle f, f_k\rangle|^2 \leqslant \tfrac{1}{A}\|Sf\|^2\|(K^*)^{-1}\|^2 = \tfrac{c}{A}\|(K^*)^{-1}\|^2\|U\{\langle f, f_k\rangle\}_{k\in Z}\|^2.$$

So for any $\{r_k\}_{k\in Z}\in N_U^{\perp}$,

$$\sum_{k\in Z}\|r_k\|^2 \leqslant \tfrac{1}{A}\|(K^*)^{-1}\|^2\|U\{r_k\}_{k\in Z}\|^2 = \tfrac{1}{A}\|(K^*)^{-1}\|^2\|\sum_{k\in Z} r_k f_k\|^2.$$

Then we obtain

$$\begin{aligned}\|Tf\|^2 &\leqslant \frac{B}{A}\|(K^*)^{-1}\|^2\|\sum_{k\in Z} r_k f_k\|^2\\ &= \frac{B}{A}\|(K^*)^{-1}\|^2\|\sum_{k\in Z} c_k f_k\|^2\\ &= \frac{B}{A}\|(K^*)^{-1}\|^2\|f\|^2.\end{aligned}$$

That is

$$\|T\| \leqslant \sqrt{\tfrac{B}{A}}\|(K^*)^{-1}\|.$$

Furthermore,

$$1 \leqslant \|T\| \leqslant \sqrt{\tfrac{B}{A}}\|(K^*)^{-1}\|. \quad \square$$

Particularly, for a special K-frame, we want to know the operator representation of a Parseval K-frame. From the following Corollary 2.3, we find the bound of the operator T, so that a Parseval K-frame has an operator representation. In this case, we also find the bound of operator T is more special than an ordinary K-frame.

COROLLARY 2.3. *Consider a Parseval K-frame $\{f_k\}_{k\in Z}$ which on the form $\{T^k f_0\}_{k\in Z}$ for a linear operator T: span$\{f_k\}_{k\in Z} \rightarrow$ span$\{f_k\}_{k\in Z}$. Similar to the normal K-frame, we have, the operator T is bounded if and only if the kernel N_U of synthesis operator U is invariant under right-shifts.*

The difference is, in the affirmative case, if the operator K^* is bijective, then we have

$$1 \leqslant \|T\| \leqslant \|(K^*)\|\|(K^*)^{-1}\|.$$

Proof. The proof of sufficiency is the same as the Proposition 2.2. Then, on the other hand, we have

$$\begin{aligned}\|Tf\|^2 &= \|\sum_{k\in Z} r_k f_{k+1}\|^2 \\ &\leqslant \sum_{k\in Z}\|r_k\|^2 \sum_{k\in Z}\|f_{k+1}\|^2,\end{aligned}$$

for any $f \in H$.

It is easy to check that

$$\begin{aligned}\|f_{k+1}\|^4 = \|\langle f_{k+1}, f_{k+1}\rangle\|^2 &= \sum_{i\in Z}\|\langle f_{k+1}, f_{i+1}\rangle\|^2 \\ &= \|K^* f_{k+1}\|^2 \leqslant \|K^*\|^2\|f_{k+1}\|^2.\end{aligned}$$

Hence

$$\|Tf\|^2 \leqslant \|K^*\|^2 \textstyle\sum_{k\in Z}\|r_k\|^2.$$

Then given any $f \in H$, we obtain

$$\textstyle\sum_{k\in Z}|\langle f, f_k\rangle|^2 \leqslant \|Sf\|^2\|(K^*)^{-1}\|^2.$$

So for any $\{r_k\}_{k\in Z} \in N_U^{\perp}$, we have

$$\textstyle\sum_{k\in Z}\|r_k\|^2 \leqslant \|(K^*)^{-1}\|^2\|\sum_{k\in Z} r_k f_k\|^2.$$

Furthermore, $1 \leqslant \|T\| \leqslant \|(K^*)\|\|(K^*)^{-1}\|.$ $\square$

Similarly, we find that if a K-frame has an operator representation for a linear bounded operator T, then the kernel N_U of synthesis operator U is also invariant under the left-shift τ^*.

COROLLARY 2.4. *If a K-frame on the form $\{f_k\}_{k\in Z} = \{T^k f_0\}_{k\in Z}$ for a linear bounded operator T, then the kernel N_U of synthesis operator U is invariant under the left-shift operator τ^*.*

Proof. Since T is a bounded linear operator, then for any $\{c_k\}_{k\in Z} \in N_U$, we have

$$\begin{aligned} U\tau^*\{c_k\}_{k\in Z} &= \sum_{k\in Z} c_{k+1} f_k = \sum_{k\in Z} c_k f_{k+1} \\ &= T\sum_{k\in Z} c_k f_k = T0 = 0. \end{aligned}$$

Thus $\tau^*(N_U) \in N_U$, so N_U is invariant under left-shifts. □

The next proposition also shows a necessary and sufficient condition for a K-frame $\{f_k\}_{k\in Z}$ to have an operator representation $\{T^k f_0\}_{k\in Z}$ for the bounded operator T. It is worth noting that the Proposition 2.5 combines a K-frame with a K-dual.

PROPOSITION 2.5. *Consider a K-frame $\{f_k\}_{k\in Z}$ with K-frame bounds A, B, respectively. Let T be a bounded linear operator. Then the following conclusions are equivalent:*

(i) The K-frame has a representation $\{f_k\}_{k\in Z} = \{T^k f_0\}_{k\in Z}$ for the bounded operator T.

(ii) For a K-dual frame $\{g_k\}_{k\in Z}$, we obtain

$$f_{j+1} = \Sigma_{k\in Z}\langle K^\dagger f_j, g_k\rangle f_{k+1},$$

where $K^\dagger$ is the pseudo inverse of K.

Proof. (i)⇒(ii) Since $\{g_k\}_{k\in Z}$ is a K-dual for $\{f_k\}_{k\in Z}$, we obtain

$$Kf = \Sigma_{k\in Z}\langle f, g_k\rangle f_k,$$

for all $f \in H$.

Thus

$$f = KK^\dagger f = \Sigma_{k\in Z}\langle K^\dagger f, g_k\rangle f_k,$$

for all $f \in H$.

Hence

$$\begin{aligned} f_{j+1} = Tf_j &= T\sum_{k\in Z}\langle K^\dagger f_j, g_k\rangle f_k \\ &= \sum_{k\in Z}\langle K^\dagger f_j, g_k\rangle T f_k \\ &= \sum_{k\in Z}\langle K^\dagger f_j, g_k\rangle f_{k+1}. \end{aligned}$$

(ii)⇒(i) Let T: H → H,

$$Tf = \Sigma_{k\in Z}\langle K^\dagger f, g_k\rangle f_{k+1},$$

for all $f \in H$.

Obviously,

$$Tf_j = \sum_{k\in Z}\langle K^\dagger f_j, g_k\rangle f_{k+1} = f_{j+1}.$$

Thus $Tf_j = f_{j+1}$. So $\{f_k\}_{k\in Z} = \{T^k f_0\}_{k\in Z}$. $\square$

According to Proposition 2.5, we can determine whether a K-frame has representation by a bounded operator T. More specifically, we can assume that K-frame has a bounded operator representing by calculating the operator in Proposition 2.5 for two different K-dual frames, and show that they are not equal. We use the following example to show that.

EXAMPLE 2.6. Let $\{e_k\}_{k\in Z}$ be an orthonormal basis for a Hilbert space H. It is easy to check that $\{Ke_1, Ke_1, Ke_2\ldots\}$ is a K-frame for H. Then we consider the K-dual of $\{Ke_1, Ke_1, Ke_2\ldots\}$.

Since

$$Kf = \sum_{k\in Z}\langle f, e_k\rangle Ke_k,$$

we obtain that sequences $\{0, e_1, e_2\ldots\}$ and $\{e_1, 0, e_2\ldots\}$ are both K-dual of $\{Ke_1, Ke_1, Ke_2\ldots\}$.

Then we assume that K-frame has a bounded operator representing via operator T. Firstly, we consider the K-dual $\{0, e_1, e_2\ldots\}$.

In this case, we obtain that

$$T_1 f = \sum_{k\in Z}\langle K^\dagger f, e_k\rangle e_{k+1},$$

for any $f \in H$, where $K^\dagger$ is the pseudo inverse for K.

Then we consider the K-dual $\{e_1, 0, e_2\ldots\}$. In this case, similarly, we have that

$$T_2 f = \langle K^\dagger f, e_1\rangle e_1 + \sum_{k\in Z\setminus\{1\}}\langle K^\dagger f, e_k\rangle e_{k+1}.$$

Since $T_1 \neq T_2$, thus the K-frame does not have a representation in terms of a bounded linear operator.

3. Stability of the K-frame representations

In recent years, research on the stability of various frames has attracted the attention of many scholars. Hence in this section, we mainly discuss the stability of K-frame operator representations.

Firstly, we introduce a lemma, which plays an essential role in the discussion of the stability of K-frame representations.

LEMMA 3.1. (c.f. [3]) *Suppose that $\{f_k\}_{k\in Z}$ is a K-frame for a Hilbert space H, with K-frame bounds A, B, respectively. And there exist $\alpha, \beta, \gamma \in [0, +\infty)$, such that max$\{\alpha + \gamma\sqrt{A^{-1}}\|K^+\|, \beta\} < 1$. Furthermore, if $\{g_k\}_{k\in Z} \subset H$ and satisfy*

$$\|\sum_{k\in Z} c_k(f_k - g_k)\| \leqslant \alpha\|\sum_{k\in Z} c_k f_k\| + \beta\|\sum_{k\in Z} c_k g_k\| + \gamma(\sum_{k\in Z}|c_k|^2)^{\frac{1}{2}},$$

for any sequence $\{c_i\}_{k\in Z}$. *Then* $\{g_k\}_{k\in Z}\subset H$ *is a* $P_{Q(R(K))}$ K*-frame for* H*, where* $P_{Q(R(K))}$ *is the orthogonal projection operator from* H *to* $Q(R(K))$. *And* $Q=UT^*$*, where* T*,* U *are the synthesis operators for* $\{f_k\}_{k\in Z}$ *and* $\{g_k\}_{k\in Z}$*, respectively.*

Then we use Lemma 3.1 to discuss the stability of the K-frame representations.

PROPOSITION 3.2. *Consider a* K*-frame* $\{f_k\}_{k\in Z}$ *for* H *which has an operator representation* $\{T^k f_0\}_{k\in Z}$. *Let* $\{g_k\}_{k\in Z}$ *be a sequence in* H. *Assume that there are constants* $\lambda_1,\lambda_2\in[0,1]$ *such that*

$$\|\sum_{k\in Z}c_k(f_k-g_k)\|\leqslant\lambda_1\|\sum_{k\in Z}c_kf_k\|+\lambda_2\|\sum_{k\in Z}c_kg_k\|$$

for all sequence $\{c_k\}_{k\in Z}$. *Then we obtain* $\{g_k\}_{k\in Z}$ *is a* $P_{Q(R(K))}$ K*-frame for a Hilbert space* H. *Furthermore,* $\{g_k\}_{k\in Z}$ *can be represented as* $\{g_k\}_{k\in Z}=\{V^kg_0\}_{k\in Z}$ *for a linear operator* V. *In this case, if* T *is bounded, then* V *is also bounded.*

Proof. First, according to Lemma 3.1, we know $\{g_k\}_{k\in Z}$ is a $P_{Q(R(K))}$ K-frame for H.

Then by Lemma 2.2, we obtain that $\{f_k\}_{k\in Z}$ is linearly independent. Since $\lambda_1,\lambda_2\in[0,1]$, we have that

$$\textstyle\sum_{k\in Z}c_kf_k=0\Leftrightarrow\sum_{k\in Z}c_kg_k=0.$$

Hence $\{g_k\}_{k\in Z}$ is linearly independent too.

Therefore according to Lemma 2.2, we know there is a linear operator V such that

$$\{g_k\}_{k\in Z}=\{V^kg_0\}_{k\in Z}.$$

Thus the sequence $\{g_k\}_{k\in Z}$ also has an operator representation via a linear operator V.

Furthermore, we will show the linear operator V is bounded according to the boundedness of linear operator T. First of all, let U, W be the synthesis operator for $\{f_k\}_{k\in Z}$ and $\{g_k\}_{k\in Z}$, respectively.

Moreover, consider the sequence $\{c_k\}_{k\in Z}\in N_W$. Obviously, $\{c_k\}_{k\in Z}\in N_U$. By Proposition 2.2, the operator T is bounded if and only if the kernel N_U of the synthesis operator U is invariant under the right-shift operator τ. Thus

$$\textstyle\sum_{k\in Z}c_{k-1}f_k=0.$$

So

$$\textstyle\sum_{k\in Z}c_{k-1}g_k=0.$$

By using the Lemma 2.2 again, we have the kernel N_W of the synthesis operator W is invariant under the right-shift operator τ. That is to say, V is bounded. □

The following corollary is also an application of Lemma 3.1. In Corollary 3.2, we use these elements from a subspace of H to perturb a K-frame. And get a useful stability result.

COROLLARY 3.2. *Let $\{f_k\}_{k\in Z}$ be a K-frame for a Hilbert space H, and have an operator representation $\{T^k\varphi\}_{k\in Z}$ for a bounded linear operator T and some $\varphi \in H$. Then let A denote a lower K-frame bound of $\{f_k\}_{k\in Z}$. Furthermore, assume that $V \subset H$ is invariant under operator T and there exists $\mu \in [0,1]$ such that*

$$\|T\tilde{\varphi}\| \leqslant \mu\|\tilde{\varphi}\|,$$

for all $\tilde{\varphi} \in V$. If $\|K^+\| < 1$, then the following hold: $\{T^k(\varphi+\tilde{\varphi})\}_{k\in Z}$ is a $P_{Q(R(K))}$ K-frame for all $\tilde{\varphi} \in V$ for which $\|\tilde{\varphi}\| < \sqrt{A(1-\mu^2)}$.

Proof. For all $\tilde{\varphi} \in V$, we have

$$\begin{aligned}\|\sum_{k\in Z} c_k(T^k(\varphi+\tilde{\varphi}) - T^k\varphi)\| &\leqslant \sum_{k\in Z}\|c_k(T^k(\varphi+\tilde{\varphi}) - T^k\varphi)\| \\ &= \sum_{k\in Z}\|c_k(T^k\tilde{\varphi})\| \\ &\leqslant \|\tilde{\varphi}\|\sum_{k\in Z}\|c_k\|\mu^k \\ &\leqslant \|\tilde{\varphi}\|(\sum_{k\in Z}\|c_k\|^2)^{\frac{1}{2}}(\sum_{k\in Z}\mu^{2k})^{\frac{1}{2}} \\ &= \frac{\|\tilde{\varphi}\|}{\sqrt{1-\mu^2}}(\sum_{k\in Z}\|c_k\|^2)^{\frac{1}{2}} \\ &\leqslant \sqrt{A}(\sum_{k\in Z}\|c_k\|^2)^{\frac{1}{2}},\end{aligned}$$

for any c_i.

Thus

$$\|\textstyle\sum_{k\in Z} c_k(T^k(\varphi+\tilde{\varphi}) - T^k\varphi)\| \leqslant \sqrt{A}(\sum_{k\in Z}\|c_k\|^2)^{\frac{1}{2}}.$$

Since $\|K^+\| < 1$, then according to Lemma 3.1, we have $\{T^k(\varphi+\tilde{\varphi})\}_{k\in Z}$ is a $P_{Q(R(K))}$ K-frame. □

It is easy to check that the condition of Proposition 3.2 is a particular case of the following,

$$\|\textstyle\sum_{k\in Z} c_k(f_k - g_k)\| \leqslant \lambda_1\|\sum_{k\in Z} c_k f_k\| + \lambda_2\|\sum_{k\in Z} c_k g_k\| + \mu(\sum_{k\in Z}|c_k|^2)^{\frac{1}{2}}.$$

However if $\mu > 0$, the perturbation condition does not preserve the property of being representable by an operator. Now we use the following example to show that.

EXAMPLE 3.3. Let $\{e_k\}_{k\in Z}$ be an orthonormal basis for a Hilbert space H. Then it is easy to prove that family

$$\{f_k\}_{k\in I} = \{e_k\}_{k\in Z}\bigcup\{\alpha\textstyle\sum_{j=1}^{\infty}\frac{e_j}{2^j}\}$$

is a linearly independent frame for any choice of $\alpha > 0$.

Furthermore, we can construct a K-frame for H as follows by using the above frame

$$\{Kf_k\}_{k\in I} = \{Ke_k\}_{k\in Z} \bigcup \{\alpha \sum_{j=1}^{\infty} \frac{Ke_j}{2^j}\},$$

where K is an injection linear operator. Hence

$$\{Kf_k\}_{k\in I} = \{Ke_k\}_{k\in Z} \bigcup \{\alpha \sum_{j=1}^{\infty} \frac{Ke_j}{2^j}\}$$

is a linearly independent K-frame.

For $1 > \alpha > 0$, the family $\{Kg_k\}_{k\in I} = \{Ke_k\}_{k\in Z} \bigcup \{0\}$ is a perturbation of $\{Kf_k\}_{k\in I}$. However, regardless of how small we choose α, the family $\{Kg_k\}_{k\in I}$ is not linearly independent. Hence, the sequence $\{Kg_k\}_{k\in I}$ can not be represented on the form $\{W^k\varphi\}_{k\in Z}$.

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