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OPERATOR PARAMETERIZATIONS OF FRAME GENERATORS AND GENERALIZED DUAL PAIR OF FRAME GENERATORS OF UNITARY SYSTEMS

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Abstract. In this paper, we first give a condition such that a countable unitary system has complete wandering vectors, which plays the key role to operator parameterize frame generators for unitary systems in the literatures. Then we continue to study more general operator parameterizations and their applications. Based on the invertibility of some parameterizing operators, we introduce the concept of generalized dual pair of frame generators for a unitary system, which is a natural generalization of dual pair of frame generators. Then we characterize generalized dual pair of frame generators and some interesting properties and constructions of generalized dual frame generator pairs are also studied.

1. Introduction

A *frame* for a Hilbert space H is a sequence $(f_i)_{i\in I}$ of elements in H such that there are constants $A,B>0$ satisfying

$$A\|f\|^2 \leqslant \sum_{i\in I}|\langle f,f_i\rangle|^2 \leqslant B\|f\|^2, \text{ for all } f\in H.$$

The numbers A and B are called the *lower* and the *upper* frame bounds respectively. The frame is a *tight frame* if $A=B$ and a *normalized tight frame* if $A=B=1$. If only require the right inequality to be satisfied, then $(f_i)_{i\in I}$ is called a *Bessel sequence* for H. If $(f_i)_{i\in I}$ is a frame as well as a Schauder basis of H, then $(f_i)_{i\in I}$ is called a *Riesz basis* for H.

The concept of frame first appeared in the late 1940's and early 1950's (see [4, 13]) during the study on nonharmonic Fourier series. Now frame has been applied in many fields such as signal processing, data compression and so on. There are tons of literatures on this topic, see [1, 2, 8, 11, 12]. In applications, the most useful frames are structured frames. The typical structured frames are wavelet frames and Gabor frames, frame $\{2^{\frac{n}{2}}f(2^nx-k) : n,k\in Z\}\subset L^2(R)$ is called a wavelet frame and frame $\{e^{2\pi max}g(x-nb) : m,n\in Z\}\subset L^2(R)$ is called a Gabor frame where a,b are two given

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positive real numbers. Abstractly, both of them are generated by a countable actions of unitary operators on some functions.

In [3, 8], X. Dai, D. Han and D. Larson introduced an unified way to study the structured frames, where a *unitary system* is defined to be a set of unitary operators $\mathscr{U}$ acting on a Hilbert space H which contains the identity operator I of $B(H)$, the set of all linear bounded operators on H. A *Bessel generator* of $\mathscr{U}$ is defined to be a vector $x \in H$ with the property that $\mathscr{U}x := \{Ux : U \in \mathscr{U}\}$ is Bessel sequence for H. In this paper, we use $\mathscr{B}(\mathscr{U})$ to denote the set of all Bessel generators of a unitary system $\mathscr{U}$. If $x \in H$ is a unit vector and $\mathscr{U}x$ forms an orthonormal basis for H, then x is called a *complete wandering vector* for $\mathscr{U}$. The set of all complete wandering vectors of a unitary system $\mathscr{U}$ is denoted as $\mathscr{W}(\mathscr{U})$. In case that $\mathscr{U}x$ forms a frame for H, we call x a *frame generator* for $\mathscr{U}$. The set of all frame generators of a unitary system $\mathscr{U}$ is denoted as $\mathscr{F}(\mathscr{U})$. The concepts of *normalized tight frame generator* and *Riesz generator* for $\mathscr{U}$ are defined similarly and we use $\mathscr{N}\mathscr{T}\mathscr{F}(\mathscr{U})$ and $\mathscr{R}(\mathscr{U})$ to denote them respectively.

In [3], X. Dai and D. Larson studied the complete wandering vectors for a unitary system, one of most important results is the operator parameterization of all complete wandering vectors of a unitary system. In [8], D. Han and D. Larson followed the similar ideas developed in [3] to study the frame generators for a unitary system. They generalized the operator parameterizations on complete wandering vectors to frame generators of a unitary system. Motivated by [3, 8], the authors in [5, 9, 10] studied the wandering vectors for some special unitary systems. In [6, 7], the author considered the multi-wandering vectors and multi-frame vectors for unitary systems and obtained some similar results. However, all the results of these literatures rely on the assumption that there exist complete wandering vectors for the unitary systems. But it was pointed out in [3] that not all countable unitary system has complete wandering vectors. In this paper, we first give a condition such that a countable unitary system has complete wandering vectors. Then we continue to study more general operator parameterizations and their applications. Finally, based on the invertibility of some parameterizing operators, we introduce the concept of generalized dual frame generator pairs for a unitary system, which is a natural generalization of dual frame generator pairs. Some interesting properties and constructions of generalized dual frame generator pairs are also studied.

In order to understand our main results, let's recall that the *local commutant* of $\mathscr{U}$ at ψ is defined to be the set $\mathscr{C}_\psi(\mathscr{U}) = \{T \in B(H) : TU\psi = UT\psi, \forall U \in \mathscr{U}\}$. There are also three important operators associated with each $\eta \in \mathscr{B}(\mathscr{U})$. The *analysis operator* associated with η is defined to be the operator $\theta_\eta(x) = \sum_{U\in\mathscr{U}} \langle x, U\eta\rangle e_U$, where $\{e_U\}_{U\in\mathscr{U}}$ is the standard orthonormal basis of Hilbert space $l^2(\mathscr{U})$ and θ_η^* is called the *synthesis operator* associated with η. The *frame operator* associated with η is defined to be $S_\eta = \theta_\eta^*\theta_\eta$. Throughout the whole paper, we use $\mathbb{U}(S)$ to denote the set of unitary operators in $S \subset B(H)$.

2. Operator parameterizations

Since some operator parameterizations of frame generators of unitary systems heavily rely on the existence of complete wandering vectors, in this section, we first give a condition such that a countable unitary system has complete wandering vectors. Then we study more on the operator parameterizations of frame generators of unitary systems and their generalizations and some applications.

THEOREM 2.1. *Suppose $\mathscr{U} = \{U_j\}_{j\in\mathscr{Z}}$ is a countable unitary system for Hilbert space H with $U_0 = I$. Let $\{e_j\}_{j\in\mathscr{Z}}$ be an orthonormal basis for H. Let $S \in B(H)$ be the bilateral shift of multiplicity one, i.e., $Se_j = e_{j+1}$ for all $j \in \mathscr{Z}$. Then $\mathscr{W}(\mathscr{U}) \neq \emptyset$ if and only if there exists a unitary operator $T \in B(H)$ such that for any $m \in \mathscr{Z}$, $T^*S^m e_0 = U_m T^* e_0$, i.e., U_m is locally unitary equivalent to S^m at e_0 for any $m \in \mathscr{Z}$.*

Proof. $\Longrightarrow$: Suppose that $\mathscr{W}(\mathscr{U}) \neq \emptyset$. Let $\psi \in \mathscr{W}(\mathscr{U})$. Then $\{U_j\psi\}_{j\in\mathscr{Z}}$ is an orthonormal basis for H. Since $\{e_j\}_{j\in\mathscr{Z}}$ is also an orthonormal basis of H, there exists a unitary operator $T \in B(H)$ such that $TU_m\psi = e_m$ for all $m \in \mathscr{Z}$. Hence $TU_0\psi = e_0$, i.e., $T\psi = e_0$. So $\psi = T^*e_0$. Hence $TU_mT^*e_0 = e_m = S^m e_0$, which implies that $T^*S^m e_0 = U_m T^* e_0$.

$\Longleftarrow$: Suppose that there exists a unitary operator $T \in B(H)$ such that $T^*S^m e_0 = U_m T^* e_0$ for all $m \in \mathscr{Z}$. Let $\psi = T^*e_0$. Then $U_m\psi = T^*S^m e_0 = T^*e_m$ for all $m \in \mathscr{Z}$. Since T^* is a unitary operator, $\{U_m\psi\}_{m\in\mathscr{Z}}$ is an orthonormal basis for H. So $\psi \in \mathscr{W}(\mathscr{U})$. Hence $\mathscr{W}(\mathscr{U}) \neq \emptyset$. $\square$

EXAMPLE 2.2. Let $\{e_n\}_{n=-\infty}^{+\infty}$ be an orthonormal basis for a separable Hilbert space H. Let U be the bilateral shift of multiplicity one, i.e., $Ue_n = e_{n+1}$, and $\mathscr{U} = \{U^n : n \in \mathscr{Z}\}$. Then it is easy to check that any $e_n \in \mathscr{W}(\mathscr{U})$. So $\mathscr{W}(\mathscr{U}) \neq \emptyset$. Let $T = U$. Then T is a unitary operator in $B(H)$ and it is obvious that $T^*S^m e_0 = U_m T^* e_0$ for all $m \in \mathscr{Z}$.

Under the assumption that $\mathscr{W}(\mathscr{U}) \neq \emptyset$, the following two lemmas operator parameterize the set of complete wandering vectors and the set of Bessel vectors of a unitary system respectively.

LEMMA 2.3. [3] *Let $\mathscr{U}$ be a unitary system in $B(H)$. Suppose $\psi \in \mathscr{W}(\mathscr{U})$. Then*

$$\mathscr{W}(\mathscr{U}) = \{V\psi : V \in \mathbb{U}(\mathscr{C}_\psi(\mathscr{U}))\}.$$

Moreover, the correspondence

$$V \longrightarrow V\psi, \mathbb{U}(\mathscr{C}_\psi(\mathscr{U})) \longrightarrow \mathscr{W}(\mathscr{U})$$

is one-to-one.

LEMMA 2.4. [8] *Suppose $\mathscr{U}$ is a unitary system for H. Let $\psi \in \mathscr{W}(\mathscr{U})$ and $\eta \in H$. Then*

(i) $\eta \in \mathscr{B}(\mathscr{U})$ if and only if there exists a unique bounded operator $T_\eta \in \mathscr{C}_\psi(\mathscr{U})$ such that $\eta = T_\eta \psi$.

(ii) $\eta \in \mathscr{F}(\mathscr{U})$ if and only if there exists a unique $T_\eta \in \mathscr{C}_\psi(\mathscr{U})$ and T_η is surjective such that $\eta = T_\eta \psi$.

(iii) $\eta \in \mathscr{N}\mathscr{T}\mathscr{F}(\mathscr{U})$ if and only if there exists a unique $T_\eta \in \mathscr{C}_\psi(\mathscr{U})$ and T_η is a co-isometry such that $\eta = T_\eta \psi$.

(iv) $\eta \in \mathscr{R}(\mathscr{U})$ if and only if there exists a unique $T_\eta \in \mathscr{C}_\psi(\mathscr{U})$ and T_η is an invertible operator such that $\eta = T_\eta \psi$.

The following lemma is of interest itself, which will also simplify our proofs of main results in sequel.

LEMMA 2.5. *Let $\mathscr{S}$ be a subset of $B(H)$ and $x \in H$. If $V \in \mathscr{C}_x(\mathscr{S})$, then*

$$\mathscr{C}_{Vx}(\mathscr{S}) = \{A \in B(H) : AV \in \mathscr{C}_x(\mathscr{S})\}.$$

Proof. Suppose that $T \in \mathscr{C}_{Vx}(\mathscr{S})$. Then for any $S \in \mathscr{S}$,

$$TVSx = TSVx = STVx.$$

So $TV \in \mathscr{C}_x(\mathscr{S})$. Conversely, if $AV \in \mathscr{C}_x(\mathscr{S})$, then for any $S \in \mathscr{S}$ we have $AVSx = SAVx$. But $AVSx = ASVx$. So $ASVx = SAVx$ for all $S \in \mathscr{S}$. It follows that $A \in \mathscr{C}_{Vx}(\mathscr{S})$. □

By Lemma 2.5, the following known result follows.

COROLLARY 2.6. *Suppose $\mathscr{S} \subset B(H)$ and $x \in H$. If $V \in \mathscr{C}_x(\mathscr{S})$ is invertible, then $\mathscr{C}_{Vx}(\mathscr{S}) = \mathscr{C}_x(\mathscr{S})V^{-1}$.*

The following result shows that the unique operator parameterizing the Bessel generators of a unitary system in Lemma 2.4 plays the role of synthesis operator.

LEMMA 2.7. *Suppose $\mathscr{U}$ is a unitary system, $\psi \in \mathscr{W}(\mathscr{U})$ and $\eta \in \mathscr{B}(\mathscr{U})$. Let S_η be the frame operator of $\{U\eta\}_{U \in \mathscr{U}}$ and T_η be the unique operator in $\mathscr{C}_\psi(\mathscr{U})$ such that $\eta = T_\eta \psi$. Then $S_\eta = T_\eta T_\eta^*$.*

Proof. Since $\psi \in \mathscr{W}(\mathscr{U})$, i.e., $\{U\psi : U \in \mathscr{U}\}$ forms an orthonormal basis of H, for any $f \in H$, we have

$$f = \sum_{U \in \mathscr{U}} \langle f, U\psi \rangle U\psi.$$

Since $T_\eta \in \mathscr{C}_\psi(\mathscr{U})$ and $\eta = T_\eta \psi$, we have that

$$\begin{aligned} S_\eta f &= \sum_{U \in \mathscr{U}} \langle f, U\eta \rangle U\eta = \sum_{U \in \mathscr{U}} \langle f, UT_\eta \psi \rangle UT_\eta \psi \\ &= \sum_{U \in \mathscr{U}} \langle f, T_\eta U\psi \rangle T_\eta U\psi = T_\eta \sum_{U \in \mathscr{U}} \langle T_\eta^* f, U\psi \rangle U\psi \\ &= T_\eta T_\eta^* f. \end{aligned}$$

So $S_\eta = T_\eta T_\eta^*$. $\square$

COROLLARY 2.8. *Suppose that $\mathscr{U}$ is a unitary system and $\psi \in \mathscr{W}(\mathscr{U}), \eta \in \mathscr{F}(\mathscr{U})$. Let T_η be the unique operator in $\mathscr{C}_\psi(\mathscr{U})$ such that $\eta = T_\eta \psi$. If T_η is a normal operator, then $\eta \in \mathscr{R}(\mathscr{U})$.*

Proof. Since T_η is a normal operator, $S_\eta = T_\eta T_\eta^* = T_\eta^* T_\eta$. Since S_η is invertible, it follows that T_η is an invertible operator. So $\eta \in \mathscr{R}(\mathscr{U})$ by Lemma 2.4. $\square$

The following result provides a general approach to generate new frame generators for a unitary system from given ones.

THEOREM 2.9. *Suppose that $\mathscr{U}$ is a unitary system and $\psi \in \mathscr{W}(\mathscr{U})$. Let $\xi_1, \xi_2 \in \mathscr{F}(\mathscr{U})$ and T_{ξ_1}, T_{ξ_2} be surjective operators in $\mathscr{C}_\psi(\mathscr{U})$ such that $\xi_1 = T_{\xi_1}\psi$, $\xi_2 = T_{\xi_2}\psi$. If $T_{\xi_1} T_{\xi_2}^* = 0$ and $A \in \mathscr{C}_{\xi_1}(\mathscr{U}), B \in \mathscr{C}_{\xi_2}(\mathscr{U})$, then $A\xi_1 + B\xi_2 \in \mathscr{F}(\mathscr{U})$ if and only if $AA^* + BB^*$ is invertible.*

Proof. Since

$$A \in \mathscr{C}_{\xi_1}(\mathscr{U}) = \mathscr{C}_{T_{\xi_1}\psi}(\mathscr{U}),$$

$AT_{\xi_1} \in \mathscr{C}_\psi(\mathscr{U})$ by Lemma 2.5. Similarly, $BT_{\xi_2} \in \mathscr{C}_\psi(\mathscr{U})$. So $AT_{\xi_1} + BT_{\xi_2} \in \mathscr{C}_\psi(\mathscr{U})$. Since

$$A\xi_1 + B\xi_2 = AT_{\xi_1}\psi + BT_{\xi_2}\psi = (AT_{\xi_1} + BT_{\xi_2})\psi,$$

$A\xi_1 + B\xi_2 \in \mathscr{F}(\mathscr{U})$ if and only if $AT_{\xi_1} + BT_{\xi_2}$ is surjective by Lemma 2.4. Since $T_{\xi_1} T_{\xi_2}^* = 0$, we have

$$(AT_{\xi_1} + BT_{\xi_2})(AT_{\xi_1} + BT_{\xi_2})^* = AT_{\xi_1} T_{\xi_1}^* A^* + BT_{\xi_2} T_{\xi_2}^* B^*.$$

Suppose a_1, b_1 and a_2, b_2 are lower and upper frame bounds of $\{U\xi_1 : U \in \mathscr{U}\}$ and $\{U\xi_2 : U \in \mathscr{U}\}$ respectively. Then

$$\min\{a_1,a_2\}(AA^* + BB^*) \leqslant AT_{\xi_1} T_{\xi_1}^* A^* + BT_{\xi_2} T_{\xi_2}^* B^* \leqslant max\{b_1,b_2\}(AA^* + BB^*).$$

So

$$\min\{a_1,a_2\}(AA^* + BB^*) \leqslant (AT_{\xi_1} + BT_{\xi_2})(AT_{\xi_1} + BT_{\xi_2})^* \leqslant max\{b_1,b_2\}(AA^* + BB^*).$$

It follows that $AT_{\xi_1} + BT_{\xi_2}$ is surjective if and only if $AA^* + BB^*$ is invertible. Hence $A\xi_1 + B\xi_2 \in \mathscr{F}(\mathscr{U})$ if and only if $AA^* + BB^*$ is invertible. $\square$

Specially, for the normalized tight frame generators, we have the following result.

COROLLARY 2.10. *Suppose that $\mathscr{U}$ is a unitary system and $\psi \in \mathscr{W}(\mathscr{U})$. Let $\xi_1, \xi_2 \in \mathscr{N}\mathscr{T}\mathscr{F}(\mathscr{U})$ and T_{ξ_1}, T_{ξ_2} be co-isometrical operators in $\mathscr{C}_\psi(\mathscr{U})$ such that $\xi_1 = T_{\xi_1}\psi$, $\xi_2 = T_{\xi_2}\psi$. If $T_{\xi_1} T_{\xi_2}^* = 0$ and $A \in \mathscr{C}_{\xi_1}(\mathscr{U}), B \in \mathscr{C}_{\xi_2}(\mathscr{U})$ then $A\xi_1 + B\xi_2 \in \mathscr{N}\mathscr{T}\mathscr{F}(\mathscr{U})$ if and only if $AA^* + BB^* = I$.*

Since for any operator $T \in \mathscr{B}(H)$, TT^* is invertible if and only if T is surjective. In the above results, let $B = 0$, then we have the following corollary, which characterizes the operator multipliers of frame generators and normalized tight frame generators of unitary systems.

COROLLARY 2.11. *Suppose $\mathscr{U}$ is a unitary system for H with $\mathscr{W}(\mathscr{U}) \neq \emptyset$.*
(i) If $\eta \in \mathscr{F}(\mathscr{U})$ and $T \in \mathscr{C}_\eta(\mathscr{U})$, then $T\eta \in \mathscr{F}(\mathscr{U})$ if and only if T is surjective.
(ii) If $\eta \in \mathscr{N}\mathscr{T}\mathscr{F}(\mathscr{U})$ and $T \in \mathscr{C}_\eta(\mathscr{U})$, then $T\eta \in \mathscr{N}\mathscr{T}\mathscr{F}(\mathscr{U})$ if and only if T is a co-isometry.

LEMMA 2.12. *Suppose $\mathscr{U}$ is a unitary system for H and $\psi \in \mathscr{W}(\mathscr{U})$. Let $\eta \in \mathscr{B}(\mathscr{U})$ and T_η is the unique operator in $\mathscr{C}_\psi(\mathscr{U})$ such that $\eta = T_\eta \psi$. If $T \in \mathscr{C}_\eta(\mathscr{U})$, then $T\eta \in \mathscr{B}(\mathscr{U})$ and $T_{T\eta} = TT_\eta$.*

Proof. Since $\eta \in \mathscr{B}(\mathscr{U})$, there exists a unique operator $T_\eta \in \mathscr{C}_\psi(\mathscr{U})$ such that $\eta = T_\eta \psi$ by Lemma 2.4(i). So $T\eta = TT_\eta \psi$. Since $T \in \mathscr{C}_\eta(\mathscr{U})$, for any $U \in \mathscr{U}$, we have

$$TT_\eta U\psi = TUT_\eta \psi = TU\eta = UT\eta = UTT_\eta \psi,$$

it follows that $TT_\eta \in \mathscr{C}_\psi(\mathscr{U})$. So $T\eta \in \mathscr{B}(\mathscr{U})$ and $T_{T\eta} = TT_\eta$ by Lemma 2.4(i). □

The following is the characterization of operator multipliers for Riesz basis generators of unitary systems.

THEOREM 2.13. *Suppose $\mathscr{U}$ is a unitary system for H with $\mathscr{W}(\mathscr{U}) \neq \emptyset$. Let $\eta \in \mathscr{R}(\mathscr{U})$ and $T \in \mathscr{C}_\eta(\mathscr{U})$. Then $T\eta \in \mathscr{R}(\mathscr{U})$ if and only if T is invertible.*

Proof. Since $\eta \in \mathscr{R}(\mathscr{U})$, T_η is invertible by Lemma 2.4 (iv). Since $T_{T\eta} = TT_\eta$ by Lemma 2.12, it follows that $T_{T\eta}$ is invertible if and only if T is invertible. So $T\eta \in \mathscr{R}(\mathscr{U})$ if and only if T is invertible. □

Suppose $S \subset B(H)$. Let $\mathscr{S}\mathscr{U}\mathscr{J}(S)$ denote the surjective operators in S, $\mathscr{C}\mathscr{O}\mathscr{I}(S)$ denote the co-isometry operators in S and $\mathscr{I}\mathscr{N}\mathscr{V}(S)$ denote the invertible operators in S. Now we have the following operator parameterization properties, which can be viewed as a generalization of the operator parameterization property for complete wandering vectors of unitary systems in Lemma 2.3.

THEOREM 2.14. *Suppose $\mathscr{U}$ is a unitary system for H with $\mathscr{W}(\mathscr{U}) \neq \emptyset$.*
(i) If $\eta \in \mathscr{F}(\mathscr{U})$, then the map $T_1 : \mathscr{S}\mathscr{U}\mathscr{J}(\mathscr{C}_\eta(\mathscr{U})) \longrightarrow \mathscr{F}(\mathscr{U})$ such that $T_1(V) = V\eta$ is a one to one map.
(ii) If $\eta \in \mathscr{N}\mathscr{F}\mathscr{T}(\mathscr{U})$, then the map $T_2 : \mathscr{C}\mathscr{O}\mathscr{I}(\mathscr{C}_\eta(\mathscr{U})) \longrightarrow \mathscr{N}\mathscr{F}\mathscr{T}(\mathscr{U})$ such that $T_2(V) = V\eta$ is a one to one map.
(iii) If $\eta \in \mathscr{R}(\mathscr{U})$, then $\mathscr{R}(\mathscr{U}) = \{V\eta : V \in \mathscr{I}\mathscr{N}\mathscr{V}(\mathscr{C}_\eta(\mathscr{U}))\}$ and the map $T_3 : \mathscr{I}\mathscr{N}\mathscr{V}(\mathscr{C}_\eta(\mathscr{U})) \longrightarrow \mathscr{R}(\mathscr{U})$ such that $T_3(V) = V\eta$ is a one to one and onto map.

Proof. By Corollary 2.11 and Theorem 2.13, the maps T_1, T_2, T_3 are well-defined. If $V_1\eta = V_2\eta$, then $UV_1\eta = UV_2\eta$ for any $U \in \mathscr{U}$. Since $V_i \in \mathscr{C}_\eta(\mathscr{U})$ $(i = 1,2,3)$, we have $V_1U\eta = V_2U\eta$ for any $U \in \mathscr{U}$. Since $\{U\eta : U \in \mathscr{U}\}$ is a frame, $V_1 = V_2$. It follows that T_i $(i = 1,2,3)$ are one to one maps. This finishes the proofs of (i) and (ii). We now prove $\mathscr{R}(\mathscr{U}) = \{V\eta : V \in \mathscr{INV}(\mathscr{C}_\eta(\mathscr{U}))\}$ and the map T_3 is onto. For any $\xi \in \mathscr{R}(\mathscr{U})$, since $\{U\eta : U \in \mathscr{U}\}$ and $\{U\xi : U \in \mathscr{U}\}$ are Riesz bases of H, there exist Riesz lower bounds A_η, A_ξ and Riesz upper bounds B_η, B_ξ such that for any $x \in H$,

$$A_\eta \|x\|^2 \leqslant \sum_{U \in \mathscr{U}} |\langle x, U\eta\rangle|^2 \leqslant B_\eta \|x\|^2$$

and

$$A_\xi \|x\|^2 \leqslant \sum_{U \in \mathscr{U}} |\langle x, U\xi\rangle|^2 \leqslant B_\xi \|x\|^2.$$

For any $x \in H$, $x = \sum_{U \in \mathscr{U}} \langle x, S_\eta^{-1}U\eta\rangle U\eta$ since $\{S_\eta^{-1}U\eta : U \in \mathscr{U}\}$ is the dual Riesz basis of $\{U\eta : U \in \mathscr{U}\}$. Let $V : H \longrightarrow H$ be the operator defined by

$$Vx = \sum_{U \in \mathscr{U}} \langle x, S_\eta^{-1}U\eta\rangle U\xi.$$

Then

$$\begin{aligned}\|Vx\| &= \|\sum_{U \in \mathscr{U}} \langle x, S_\eta^{-1}U\eta\rangle U\xi\| = \|\theta_\xi^*(\{\langle x, S_\eta^{-1}U\eta\}_{U \in \mathscr{U}})\| \\ &\leqslant \|\theta_\xi^*\| \|\{\langle x, S_\eta^{-1}U\eta\}_{U \in \mathscr{U}}\| \leqslant \sqrt{B_\xi/A_\eta} \cdot \|x\|.\end{aligned}$$

So $V \in B(H)$ and $VU\eta = U\xi$ for any $U \in \mathscr{U}$, in particular, let $U = I$ we get $V\eta = \xi$. So $VU\eta = U\xi = UV\eta$ for any $U \in \mathscr{U}$, which implies that $V \in \mathscr{C}_\eta(\mathscr{U})$. Since $\{S_\xi^{-1}U\xi : U \in \mathscr{U}\}$ is the dual Riesz basis of $\{U\xi : U \in \mathscr{U}\}$, for any $y \in H$, $\{\langle y, S_\xi^{-1}U\xi\rangle : U \in \mathscr{U}\} \in l^2(\mathscr{U})$. So $\sum_{U \in \mathscr{U}} \langle y, S_\xi^{-1}U\xi\rangle U\eta$ is convergent. Let $z = \sum_{U \in \mathscr{U}} \langle y, S_\xi^{-1}U\xi\rangle U\eta$, then $Vz = y$. So V is surjective. If $Vx = 0$, i.e., $\sum_{U \in \mathscr{U}} \langle x, S_\eta^{-1}U\eta\rangle U\xi = 0$, since $\{U\xi : U \in \mathscr{U}\}$ is a Riesz basis, then $\langle x, S_\eta^{-1}U\eta\rangle = 0$ for all $U \in \mathscr{U}$. It follows that $x = 0$. Thus V is injective. So $V \in \mathscr{INV}(\mathscr{C}_\eta(\mathscr{U}))$ and $V\eta = \xi$. So $\mathscr{R}(\mathscr{U}) = \{V\eta : V \in \mathscr{INV}(\mathscr{C}_\eta(\mathscr{U}))\}$ and the map T_3 is one to one and onto. □

If $T \in \mathscr{U}'$, then $T \in \mathscr{C}_\eta(\mathscr{U})$ for any $\eta \in H$. So we have the following simple facts.

COROLLARY 2.15. *Suppose $\mathscr{U}$ is a unitary system for H with $\mathscr{W}(\mathscr{U}) \neq \emptyset$ and $T \in \mathscr{U}'$. Then*

(i) $T\mathscr{F}(\mathscr{U}) \subset \mathscr{F}(\mathscr{U})$ if only if T is surjective.

(ii) $T\mathscr{W}(\mathscr{U}) \subset \mathscr{W}(\mathscr{U})$ if and only if T is unitary.

(iii) $T\mathscr{NTF}(\mathscr{U}) \subset \mathscr{NTF}(\mathscr{U})$ if and only if T is co-isometry.

(iv) $T\mathscr{R}(\mathscr{U}) \subset \mathscr{R}(\mathscr{U})$ if and only if T is invertible.

For any $x,y,z \in H$, $x \otimes y(z) = \langle z,y\rangle x$ denotes a rank one operator. It is well known that for any $\xi,\eta \in \mathscr{B}(\mathscr{U})$, the operator $T = \sum_{U\in\mathscr{U}} U\xi \otimes U\eta$ is well-defined bounded operator in $B(H)$. And the properties of T is closed related with the properties of the Bessel generators. In fact, $T = \theta_\xi^*\theta_\eta$. the following facts are direct consequences of the well-known result that a Bessel sequence is a frame if and only if its synthesis operator is surjective and it is a Riesz basis if and only if its synthesis operator is invertible. For convenience, we collect them into the following lemma.

LEMMA 2.16. *Suppose $\xi,\eta \in \mathscr{B}(\mathscr{U})$. Let $T = \sum_{U\in\mathscr{U}} U\xi \otimes U\eta$.*
(1) If $T = I$, then (ξ,η) is a pair of dual frame generators of $\mathscr{U}$.
(2) If T is surjective, then $\eta \in \mathscr{F}(\mathscr{U})$.
(3) If T^* is surjective, then $\xi \in \mathscr{F}(\mathscr{U})$.
(4) If T is invertible, then $\eta,\xi \in \mathscr{F}(\mathscr{U})$. Moreover, if $T^{-1} \in \mathscr{C}_\xi(\mathscr{U})$, then $T^{-1}\xi \in \mathscr{F}(\mathscr{U})$ and $(T^{-1}\xi,\eta)$ is a pair of dual frame generators for $\mathscr{U}$; If $(T^*)^{-1} \in \mathscr{C}_\eta(\mathscr{U})$, then $(T^*)^{-1}\eta \in \mathscr{F}(\mathscr{U})$ and $(\xi,(T^*)^{-1}\eta)$ is a pair of dual frame generators for $\mathscr{U}$.
(5) If $\xi,\eta \in \mathscr{R}(\mathscr{U})$, then T is invertible.

The following is a result which generalizes Theorem 2.9 without $\mathscr{W}(\mathscr{U}) \neq \emptyset$.

THEOREM 2.17. *Suppose that $\mathscr{U}$ is a unitary system for H and $\xi,\eta_1,\eta_2 \in \mathscr{B}(\mathscr{U})$. Let $T_i = \sum_{U\in\mathscr{U}} U\eta_i \otimes U\xi$ $(i = 1,2)$. Suppose $T_iT_i^*(i = 1,2)$ are invertible and $T_1T_2^* = 0$, $A \in \mathscr{C}_{\eta_1}(\mathscr{U})$ and $B \in \mathscr{C}_{\eta_2}(\mathscr{U})$. If $AA^* + BB^*$ is invertible, then $A\eta_1 + B\eta_2 \in \mathscr{F}(\mathscr{U})$.*

Proof. Since $T_1T_1^*$ and $T_2T_2^*$ are invertible, there exist positive numbers a_1 and a_2 such that $a_1I \leqslant T_1T_1^*$ and $a_2I \leqslant T_2T_2^*$. Since $A \in \mathscr{C}_{\eta_1}(\mathscr{U}), B \in \mathscr{C}_{\eta_2}(\mathscr{U})$, for any $x \in H$, we have

$$\begin{aligned}(AT_1 + BT_2)x &= \sum_{U\in\mathscr{U}} \langle x,U\xi\rangle AU\eta_1 + \sum_{U\in\mathscr{U}} \langle x,U\xi\rangle BU\eta_2\\ &= \sum_{U\in\mathscr{U}} \langle x,U\xi\rangle UA\eta_1 + \sum_{U\in\mathscr{U}} \langle x,U\xi\rangle UB\eta_2\\ &= \sum_{U\in\mathscr{U}} \langle x,U\xi\rangle U(A\eta_1 + B\eta_2).\end{aligned}$$

So, $AT_1 + BT_2 = \sum_{U\in\mathscr{U}} U(A\eta_1 + B\eta_2) \otimes U\xi$. It is easy to check that $A\eta_1 + B\eta_2 \in \mathscr{B}(\mathscr{U})$. So it is sufficient to show that $AT_1 + BT_2$ is surjective by Lemma 2.16 (2). Since $T_1T_2^* = 0$, we have

$$(AT_1 + BT_2)(AT_1 + BT_2)^* = AT_1T_1^*A^* + BT_2T_2^*B^*.$$

Since

$$\min\{a_1,a_2\} \cdot (AA^* + BB^*) \leqslant AT_1T_1^*A^* + BT_2T_2^*B^*,$$

we have

$$\min\{a_1,a_2\}\cdot(AA^*+BB^*)\leqslant(AT_1+BT_2)(AT_1+BT_2)^*.$$

So $(AT_1+BT_2)(AT_1+BT_2)^*$ is invertible, which implies that AT_1+BT_2 is surjective. $\square$

REMARK 2.18. (i). In the above Theorem, if $\xi\in\mathscr{W}(\mathscr{U})$, then T_i $(i=1,2)$ is the parameterizing operator T_{η_i} of η_i $(i=1,2)$. Thus Theorem 2.17 generalizes the result in Theorem 2.9.

3. Generalized dual pair of frame generators

From the above section, we know that for $\xi,\eta\in\mathscr{B}(\mathscr{U})$, the operator

$$T=\sum_{U\in\mathscr{U}}U\xi\otimes U\eta,$$

plays an important role in studying the properties of the Bessel generators. And we know that if $\xi,\eta\in\mathscr{R}(\mathscr{U})$ then T is invertible, however if T is invertible, it is not guaranteed that $\xi,\eta\in\mathscr{R}(\mathscr{U})$. In this section, we concentrate on studying the properties of the Bessel generators when T is invertible. We will introduce the concept of generalized dual frame generator of a frame generator and pairs of generalized dual frame generators of a unitary system and give some characterizations and constructions of such pairs.

DEFINITION 3.1. Suppose that $\mathscr{U}$ is a unitary system for H and $\eta_1,\eta_2\in\mathscr{B}(\mathscr{U})$. Let $T=\sum_{U\in\mathscr{U}}U\eta_1\otimes U\eta_2$. If $T\in\mathscr{U}'$ and T is invertible, then η_2 is called a generalized dual frame generator of η_1.

REMARK 3.2. (1) Since $T=\sum_{U\in\mathscr{U}}U\eta_1\otimes U\eta_2$ is in $\mathscr{U}'$ and is invertible if and only if $T^*=\sum_{U\in\mathscr{U}}U\eta_2\otimes U\eta_1$ is in $\mathscr{U}'$ and is invertible, η_1 is a generalized dual frame generator of η_2 if and only if η_2 is a generalized dual frame generator of η_1.

(2) When η_1 is a generalized dual frame generator of η_2, we called (η_1,η_2) is a pair of generalized dual frame generators of $\mathscr{U}$.

(3) If $T=I$, then (η_1,η_2) is a pair of dual frame generators. So pair of generalized dual frame generators is a generalization of pair of dual frame generators.

(4) If $\mathscr{U}$ is a unitary group and $\xi,\eta\in\mathscr{R}(\mathscr{U})$, then it is easy to see that $\theta_\xi,\theta_\eta\in\mathscr{U}'$ and are invertible. So $T=\sum_{U\in\mathscr{U}}U\xi\otimes U\eta=\theta_\xi^*\theta_\eta\in\mathscr{U}'$ and is invertible, which implies that (ξ,η) is a pair of generalized dual frames generators of $\mathscr{U}$.

It is well-known that if a pair of Bessel generators is a pair of dual frame generators, then both of them are in fact frame generators. The following results generalized this fact to the case of pairs of generalized dual frame generators of a unitary system. The following results also tell us that pairs of generalized dual frame generators are closely related to pairs of dual frame generators, so they also have potential applications in expansion of elements in Hilbert spaces.

THEOREM 3.3. *Suppose $\mathscr{U}$ is a unitary system, $\eta_1,\eta_2 \in \mathscr{B}(\mathscr{U})$.*

(i) If (η_1,η_2) is a pair of generalized dual frame generators of $\mathscr{U}$, then $\eta_1,\eta_2 \in \mathscr{F}(\mathscr{U})$.

(ii) (η_1,η_2) is a pair of generalized dual frame generators of $\mathscr{U}$ if and only if there exists a unique invertible operator $A \in \mathscr{U}'$ such that $(A\eta_1,\eta_2)$ is a pair of dual frame generators of $\mathscr{U}$.

Proof. (i) is direct consequence of Lemma 2.16 (4).

(ii) $\Longrightarrow$: Since (η_1,η_2) is a pair of generalized dual frame generators of $\mathscr{U}$, $T=\sum_{U\in\mathscr{U}} U\eta_1\otimes U\eta_2$ is invertible and is in $\mathscr{U}'$. Let $A=T^{-1}$, then $A\in\mathscr{U}'$ is invertible and $(A\eta_1,\eta_2)$ is a pair of dual frame generators of $\mathscr{U}$ by (4) of Lemma 2.16. If $A,B\in\mathscr{U}'$ are invertible operators such that $(A\eta_1,\eta_2)$ and $(B\eta_1,\eta_2)$ are pairs of dual frames, then

$$\begin{aligned}\sum_{U\in\mathscr{U}} UA\eta_1\otimes U\eta_2 &= A\sum_{U\in\mathscr{U}} U\eta_1\otimes U\eta_2 = AT = I\\ &= \sum_{U\in\mathscr{U}} UB\eta_1\otimes U\eta_2 = B\sum_{U\in\mathscr{U}} U\eta_1\otimes U\eta_2 = BT.\end{aligned}$$

So $AT=BT$. Since T is invertible, we have $A=B$. So the operator A such that $A\in\mathscr{U}'$ is invertible and $(A\eta_1,\eta_2)$ is a pair of dual frame generators is unique.

$\Longleftarrow$: Since $(A\eta_1,\eta_2)$ is a pair of dual frame generators of $\mathscr{U}$ and $A\in\mathscr{U}'$ is invertible,

$$\begin{aligned}I &= \sum_{U\in\mathscr{U}} UA\eta_1\otimes U\eta_2 = \sum_{U\in\mathscr{U}} AU\eta_1\otimes U\eta_2\\ &= A\sum_{U\in\mathscr{U}} U\eta_1\otimes U\eta_2.\end{aligned}$$

So $A^{-1}=\sum_{U\in\mathscr{U}} U\eta_1\otimes U\eta_2$. Since $A\in\mathscr{U}'$, it is easy to see that $A^{-1}\in\mathscr{U}'$. So (η_1,η_2) is a pair of generalized dual frame generators of $\mathscr{U}$. □

COROLLARY 3.4. *Suppose that $\mathscr{U}$ is a unitary system for H and (ξ,η) is a pair of generalized dual frame generators of $\mathscr{U}$. Then there exists a dual frame generator η^D of η such that for any dual frame generator θ of ξ, (η^D,θ) is a pair of generalized dual frame generators of $\mathscr{U}$.*

Proof. Let $T=\sum_{U\in\mathscr{U}} U\xi\otimes U\eta$. Since (ξ,η) is a pair of generalized dual frame generators of $\mathscr{U}$, $T\in\mathscr{U}'$ is invertible. So $(T^{-1}\xi,\eta)$ is a pair of dual frame generators of $\mathscr{U}$ by Lemma 2.16(4). Let $\eta^D=T^{-1}\xi$. For any $\theta\in H$ such that (ξ,θ) is a pair of dual frame generators of $\mathscr{U}$, we have $A=\sum_{U\in\mathscr{U}} U\xi\otimes U\theta=I$. So

$$\begin{aligned}\sum_{U\in\mathscr{U}} U\eta^D\otimes U\theta &= \sum_{U\in\mathscr{U}} UT^{-1}\xi\otimes U\theta\\ &= \sum_{U\in\mathscr{U}} T^{-1}U\xi\otimes U\theta = T^{-1}\sum_{U\in\mathscr{U}} U\xi\otimes U\theta = T^{-1}.\end{aligned}$$

Since $T \in \mathscr{U}'$, $T^{-1} \in \mathscr{U}'$. So (η^D, θ) is a pair of generalized dual frame generators of $\mathscr{U}$. $\square$

Let $\{e_U\}_{U\in\mathscr{U}}$ be the canonical orthonormal basis of Hilbert space $l^2(\mathscr{U})$. Suppose $\eta \in \mathscr{B}(\mathscr{U})$. Then the analysis operator θ_η of $\{U\eta\}_{U\in\mathscr{U}}$ is defined by $\theta_\eta = \sum_{U\in\mathscr{U}} e_U \otimes U\eta$. In order to study the generalized dual frame generators, we give the structure of dual frame generator and generalized dual frame generator of a frame generator in the following theorem.

THEOREM 3.5. *Suppose $\mathscr{U}$ is a unitary system and $\eta \in \mathscr{F}(\mathscr{U})$, $\xi \in H$. If $\theta_\eta \in \mathscr{U}'$, then:*

(i) (ξ,η) is a pair of dual frame generators of $\mathscr{U}$ if and only if there exists $\theta \in \mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\theta\otimes U\eta = 0$ such that $\xi = S_\eta^{-1}\eta+\theta$.

(ii) (ξ,η) is a pair of generalized dual frame generators of $\mathscr{U}$ if and only if there exists an invertible operator $T\in\mathscr{U}'$ and $\theta\in\mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\theta\otimes U\eta = 0$ such that $\xi = TS_\eta^{-1}\eta+\theta$.

Proof. Since $\theta_\eta \in \mathscr{U}'$, it is easy to see that $\theta_\eta^* \in \mathscr{U}'$. So $S_\eta = \theta_\eta^*\theta_\eta \in \mathscr{U}'$, which implies that $S_\eta^{-1} \in \mathscr{U}'$. Since for any $U \in \mathscr{U}$ we have $S_\eta^{-1}U\eta = US_\eta^{-1}\eta$, the canonical dual frame of $\{U\eta : U\in\mathscr{U}\}$ is $\{US_\eta^{-1}\eta : U \in \mathscr{U}\}$, which follows that the canonical dual frame of $\{U\eta : U\in\mathscr{U}\}$ is generated by $S_\eta^{-1}\eta$.

(i) $\Longrightarrow$: Let $\theta = \xi - S_\eta^{-1}\eta$, then it is easy to see that $\theta\in\mathscr{B}(\mathscr{U})$. Since (ξ,η) and $(S_\eta^{-1}\eta,\eta)$ are pairs of dual frame generators of $\mathscr{U}$, we have

$$\sum_{U\in\mathscr{U}} U\xi\otimes U\eta = I = \sum_{U\in\mathscr{U}} U(S_\eta^{-1}\eta)\otimes U\eta.$$

So

$$\begin{aligned}\sum_{U\in\mathscr{U}} U\theta\otimes U\eta &= \sum_{U\in\mathscr{U}} U(\xi - S_\eta^{-1}\eta)\otimes U\eta \\ &= \sum_{U\in\mathscr{U}} U\xi\otimes U\eta - \sum_{U\in\mathscr{U}} U(S_\eta^{-1}\eta)\otimes U\eta = 0.\end{aligned}$$

$\Longleftarrow$: Since $\xi = S_\eta^{-1}\eta+\theta$ with $\theta\in\mathscr{B}(\mathscr{U})$, it is easy to see that $\xi\in\mathscr{B}(\mathscr{U})$. Since $\sum_{U\in\mathscr{U}} U\theta\otimes U\eta = 0$ and $\sum_{U\in\mathscr{U}} US_\eta^{-1}\eta\otimes U\eta = I$, we have

$$\sum_{U\in\mathscr{U}} U\xi\otimes U\eta = \sum_{U\in\mathscr{U}} US_\eta^{-1}\eta\otimes U\eta + \sum_{U\in\mathscr{U}} U\theta\otimes U\eta = I+0 = I.$$

So (ξ,η) is a pair of dual frame generators of $\mathscr{U}$.

(ii) $\Longrightarrow$: Since (ξ,η) is a pair of generalized dual frame generators of $\mathscr{U}$, by (ii) of Theorem 3.3, there exists an invertible operator $A\in\mathscr{U}'$ such that $(A\xi,\eta)$ is a pair of dual frame generators of $\mathscr{U}$. So there exists $\beta\in\mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\beta\otimes U\eta = 0$

such that $A\xi = S_\eta^{-1}\eta + \beta$ by (i). It follows that $\xi = A^{-1}S_\eta^{-1}\eta + A^{-1}\beta$. Let $T = A^{-1}$ and $\theta = A^{-1}\beta$. Then $\theta \in \mathscr{B}(\mathscr{U})$ and

$$\begin{aligned}\sum_{U\in\mathscr{U}} U\theta\otimes U\eta &= \sum_{U\in\mathscr{U}} UA^{-1}\beta\otimes U\eta\\ &= \sum_{U\in\mathscr{U}} A^{-1}U\beta\otimes U\eta = A^{-1}\sum_{U\in\mathscr{U}} U\beta\otimes U\eta = A^{-1}(0) = 0,\end{aligned}$$

which complete the proof.

$\Longleftarrow$: Since $\xi = TS_\eta^{-1}\eta + \theta$ with $T \in \mathscr{U}'$ is invertible and $\theta \in \mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\theta\otimes U\eta = 0$, $T^{-1}\xi = S_\eta^{-1}\eta + T^{-1}\theta$. Since $T \in \mathscr{U}'$, it is easy to see that $T^{-1} \in \mathscr{U}'$. So $T^{-1}\theta \in \mathscr{B}(\mathscr{U})$. Since

$$\sum_{U\in\mathscr{U}} UT^{-1}\theta\otimes U\eta = \sum_{U\in\mathscr{U}} T^{-1}U\theta\otimes U\eta = T^{-1}\sum_{U\in\mathscr{U}} U\theta\otimes U\eta = 0,$$

which follows that $(T^{-1}\xi,\eta)$ is a pair of dual frame generators of $\mathscr{U}$ by (i). So (ξ,η) is a pair of generalized dual frame generators of $\mathscr{U}$ by Theorem 3.3 (ii). □

REMARK 3.6. The condition that $\theta_\eta \in \mathscr{U}'$ is not very strong, in fact, if $\mathscr{U}$ is a unitary group or even a group-like unitary system, for example, the Gabor system, then $\theta_\eta \in \mathscr{U}'$ is guaranteed.

COROLLARY 3.7. *Suppose $\mathscr{U}$ is a unitary system and $\eta \in \mathscr{F}(\mathscr{U})$ with its analysis operator $\theta_\eta \in \mathscr{U}'$ and $\xi \in \mathscr{B}(\mathscr{U})$. Denote $S_\eta^{-1}\eta$ as η^*, which is the frame generator of the canonical dual frame of $\{U\eta : U \in \mathscr{U}\}$. Then (ξ,η) is a pair of generalized dual frame generators of $\mathscr{U}$ if and only if (ξ,η^*) is a pair of generalized dual frame generators of $\mathscr{U}$.*

Proof. It is easy to check that the frame operator $S_{\eta^*} = S_\eta^{-1}$ and $S_\eta \in \mathscr{U}'$, So $S_{\eta^*} \in \mathscr{U}'$. If (ξ,η) is a pair of generalized dual frame generators of $\mathscr{U}$, then there exists an invertible operator $A \in \mathscr{U}'$ and $\beta \in \mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\beta\otimes U\eta = 0$ such that $\xi = AS_\eta^{-1}\eta + \beta$ by Theorem 3.3 (ii). So

$$\xi = AS_\eta^{-1}(S_\eta S_\eta^{-1}\eta) + \beta = (AS_\eta^{-1})S_\eta\eta^* + \beta = (AS_\eta^{-1})S_{\eta^*}^{-1}\eta^* + \beta.$$

Since $AS_\eta^{-1} \in \mathscr{U}'$ and

$$\begin{aligned}\sum_{U\in\mathscr{U}} U\beta\otimes U\eta^* &= \sum_{U\in\mathscr{U}} U\beta\otimes US_\eta^{-1}\eta\\ &= \sum_{U\in\mathscr{U}} U\beta\otimes S_\eta^{-1}U\eta = (\sum_{U\in\mathscr{U}} U\beta\otimes U\eta)S_\eta^{-1} = 0,\end{aligned}$$

(ξ,η^*) is a pair of generalized dual frame generators of $\mathscr{U}$ by Theorem 3.3 (ii). It is easy to see that $(\eta^*)^* = \eta$. So (ξ,η) is a pair of generalized dual frame generators of $\mathscr{U}$ if and only if (ξ,η^*) is a pair of generalized dual frame generators of $\mathscr{U}$. □

REMARK 3.8. It is well-known that for a given frame, its two different dual frame never similar. However, according to the above corollary, its two different generalized dual frames may be similar,

In the following corollaries, as applications of Theorem 3.5, we give several different ways to construct pairs of generalized dual frame generators of a unitary system.

COROLLARY 3.9. *Suppose $\mathscr{U}$ is a unitary system and $\eta \in \mathscr{F}(\mathscr{U})$ with its analysis operator $\theta_\eta \in \mathscr{U}'$. If (θ_1, η) and (θ_2, η) are pairs of dual frame generators of $\mathscr{U}$, then $(\theta_1 + \theta_2, \eta)$ is a pair of generalized dual frame generators of $\mathscr{U}$.*

Proof. Since (θ_1, η) and (θ_2, η) are pairs of dual frame generators of $\mathscr{U}$, by (i) of Theorem 3.5, there exist $\beta_1, \beta_2 \in \mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\beta_1 \otimes U\eta = \sum_{U\in\mathscr{U}} U\beta_2 \otimes U\eta = 0$ such that $\theta_1 = S_\eta^{-1}\eta + \beta_1$ and $\theta_2 = S_\eta^{-1}\eta + \beta_2$. So $\theta_1 + \theta_2 = 2S_\eta^{-1}\eta + (\beta_1 + \beta_2)$. Since $\theta_\eta \in \mathscr{U}'$, it is easy to see that $S_\eta \in \mathscr{U}'$. Since

$$\sum_{U\in\mathscr{U}} U(\beta_1 + \beta_2) \otimes U\eta = \sum_{U\in\mathscr{U}} U\beta_1 \otimes U\eta + \sum_{U\in\mathscr{U}} U\beta_2 \otimes U\eta = 0,$$

it follows that $(\theta_1 + \theta_2, \eta)$ is a pair of generalized dual frame generators of $\mathscr{U}$ by Theorem 3.3 (ii). $\square$

COROLLARY 3.10. *Suppose $\mathscr{U}$ is a unitary system and $\eta \in \mathscr{F}(\mathscr{U})$ with its analysis operator $\theta_\eta \in \mathscr{U}'$. If (θ, η) is a pair of dual frame generators of $\mathscr{U}$, then $(A\theta, \eta)$ is a pair of generalized dual frame generators of $\mathscr{U}$ for any invertible operator $A \in \mathscr{U}'$.*

Proof. Since $\theta_\eta \in \mathscr{U}'$, it is easy to see that $S_\eta \in \mathscr{U}'$. Since (θ, η) is a pair of dual frame generators of $\mathscr{U}$, $\theta = S_\eta^{-1}\eta + \beta$ with $\beta \in \mathscr{B}(\mathscr{U})$ and $\sum_{U\in\mathscr{U}} U\beta \otimes U\eta = 0$ by (i) of Theorem 3.5. Suppose $A \in \mathscr{U}'$ is an invertible operator. Then $A\theta = AS_\eta^{-1}\eta + A\beta$. It is to check that $A\beta \in \mathscr{B}(\mathscr{U})$ and

$$\sum_{U\in\mathscr{U}} UA\beta \otimes U\eta = \sum_{U\in\mathscr{U}} AU\beta \otimes U\eta = A \sum_{U\in\in\mathscr{U}} U\beta \otimes U\eta = 0.$$

So $(A\theta, \eta)$ is a pair of generalized dual frame generators of $\mathscr{U}$ by (ii) of Theorem 3.5. $\square$

COROLLARY 3.11. *Suppose $\mathscr{U}$ is a unitary system and $\eta \in \mathscr{F}(\mathscr{U})$ with its analysis operator $\theta_\eta \in \mathscr{U}'$. If (θ_1, η) is a pair of generalized dual frame generators of $\mathscr{U}$ and (θ_2, η) is a pair of dual frame generators of $\mathscr{U}$, then there is a number $M > 0$ such that $(\theta_1 + C\theta_2, \eta)$ is a pair of generalized dual frame generators of $\mathscr{U}$ for any $C > M$.*

Proof. Since (θ_1, η) is a pair of generalized dual frame generators of $\mathscr{U}$, there exists an invertible operator $T \in \mathscr{U}'$ and $\beta_1 \in \mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\beta_1 \otimes U\eta = 0$ such that $\theta_1 = TS_\eta^{-1}\eta + \beta_1$ by (ii) of Theorem 3.5. Since (θ_2, η) is a pair of dual frame generators of $\mathscr{U}$, $\theta_2 = S_\eta^{-1}\eta + \beta_2$ for some $\beta_2 \in \mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\beta_2 \otimes U\eta = 0$ by (i) of Theorem 3.5. Let $M = \|T\|$. Then $T + CI = C(I + \frac{1}{C}T)$ is invertible whenever $C > M = \|T\|$. Since $T \in \mathscr{U}'$, it is easy to check that $T + CI \in \mathscr{U}'$ and $\beta_1 + C\beta_2 \in \mathscr{B}(\mathscr{U})$ and

$$\sum_{U\in\mathscr{U}} U(\beta_1 + C\beta_2) \otimes U\eta = \sum_{U\in\mathscr{U}} U\beta_1 \otimes U\eta + C\sum_{U\in\mathscr{U}} U\beta_2 \otimes U\eta = 0.$$

Since

$$\theta_1 + C\theta_2 = TS_\eta^{-1}\eta + \beta_1 + CS_\eta^{-1}\eta + C\beta_2 = (T + CI)S_\eta^{-1}\eta + (\beta_1 + C\beta_2),$$

$(\theta_1 + C\theta_2, \eta)$ is a pair of generalized dual frame generators of $\mathscr{U}$ for any $C > M$ by (ii) of Theorem 3.5. □

COROLLARY 3.12. *Suppose $\mathscr{U}$ is a unitary system and $\eta \in \mathscr{F}(\mathscr{U})$ with its analysis operator $\theta_\eta \in \mathscr{U}'$. If $A, B \in \mathscr{U}'$ are invertible and (ξ, η) is a pair of generalized dual frame generators of $\mathscr{U}$, then $(A\xi, B\eta)$ is a pair of generalized dual frame generators of $\mathscr{U}$.*

Proof. Since $\theta_\eta \in \mathscr{U}'$, it is easy to see that $S_\eta \in \mathscr{U}'$. So $S_\eta^{-1} \in \mathscr{U}'$. Since (ξ, η) is a pair of generalized dual frame generators of $\mathscr{U}$, there exists an invertible operator $T \in \mathscr{U}'$ and $\beta \in \mathscr{B}(\mathscr{U})$ with $\sum_{U\in\mathscr{U}} U\beta \otimes U\eta = 0$ such that $\xi = TS_\eta^{-1}\eta + \beta$ by (ii) of Theorem 3.5. Since $B \in \mathscr{U}'$ is invertible, it is easy to check that $B\eta \in \mathscr{F}(\mathscr{U})$. Since $A, B, S_\eta^{-1} \in \mathscr{U}'$, $ATS_\eta^{-1}B^{-1} \in \mathscr{U}'$. Since

$$\sum_{U\in\mathscr{U}} UA\beta \otimes U\eta = \sum_{U\in\mathscr{U}} AU\beta \otimes U\eta = A\sum_{U\in\mathscr{U}} U\beta \otimes U\eta = 0,$$

and

$$A\xi = ATS_\eta^{-1}\eta + A\beta = ATS_\eta^{-1}B^{-1}B\eta + A\beta = (ATS_\eta^{-1}B^{-1})B\eta + A\beta,$$

$(A\xi, B\eta)$ is a pair of generalized dual frame generators of $\mathscr{U}$ by (ii) of Theorem 3.5. □

REFERENCES

[1] C. K. CHUI, *An Introduction to Wavelets*, Acad. Press, New York, 1992.

[2] I. DAUBECHIES, *Ten Lectures on Wavelets*, CBMS 61, SIAM, 1992.

[3] X. DAI, D. LARSON, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc. 134 (640) (1998).

[4] R. J. DUFFIN, A. C. SHAFFER, *A class of nonharmonic Fourier Series*, Trans. Amer. Math. Soc., 72 (1952), 341–366.

[5] J. P. GABARDO, D. HAN, *Frame representations for group-like unitary operator systems*, J. Operator Theory, 49 (2003), 1–22.

[6] X. GUO, *Wandering r-tuples for unitary systems*, J. Math. Anal. Appl. 374 (2011) 722–728.

[7] X. GUO, *Multi-frame vectors for unitary systems*, Indian J. Pure Appl. Math., 43 (4), 2012, 391–409.

[8] D. HAN AND D. LARSON, *Bases, frames and group representations*, Memoirs of Amer. Math. Soc., Vol. 147, No. 697, Sep. 2000.

[9] D. HAN, *Wandering vectors for irrational rotation unitary systems*, Trans. Amer. Math. Soc. 350 (1998), 309–320.
[10] D. HAN, D. LARSON, *Wandering vector multipliers for unitary groups*, Trans. Amer. Math. Soc. 353 (2001), 3347–3370.
[11] E. HERNANDEZ AND G. WEISS, *A First Course on Wavelets*, CRC press, Boca Raton, FL, 1996.
[12] S. MALLAT, *Multiresolution approximations and wavelet orthonormal basis of L2 (R)*, Trans. Amer. Math. Soc. 315 (1989), 69–87.
[13] B. SZ-NAGY, *Expansion Theorems of Paley-Wiener type*, Duke Math. J., 14 (1947), 975–978.

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