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SEMI–SMOOTH POINTS IN SPACE OF OPERATORS ON HILBERT SPACE

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Abstract. The investigations of the smooth points in the operator spaces $\mathscr{K}(\mathscr{H})$ and $\mathscr{L}(\mathscr{H})$ were started in [J. R. Holub, Math. Ann. **201** (1973), 157–163] and [T. J. Abatzoglou, Math. Ann. **239** (1979), 129–135]. The aim of this paper is to present a description of semi-smooth points in the operator spaces $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ and $\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)$.

1. Introduction

It is a result of Holub's [4, Theorem 3.3], that for a compact operator $T\in\mathscr{K}(\mathscr{H})$ on a real Hilbert space $\mathscr{H}$, T is smooth $\Leftrightarrow$ $\|Tx_1\|=\|Tx_2\|=\|T\|$ for some $\|x_1\|=\|x_2\|=1$ implies $x_1=\pm x_2$. Abatzoglou [1, Theorem 3.1] extended this result to $\mathscr{L}(\mathscr{H})$. It is worth mentioning that the program of characterizing smooth operators was recently completed in [8] in the more general setting of Banach spaces.

In this paper, motivated by the results published by Holub, Abatzoglou and Miličić, we study the notion of semi-smooth points in the spaces $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$, where $\mathscr{H}_1,\mathscr{H}_2$ are Hilbert spaces over $\mathbb{R}$.

For a given $A\in\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$, we denote $M(A):=\{x\in\mathscr{H}_1:\ \|x\|=1,\ \|Ax\|=\|A\|\}$. Our new theorem will be related in spirit, though not in proof, to the Holub-Abatzoglou Theorem. More precisely, the Holub-Abatzoglou characterization can be reformulated as:

$$T \text{ is smooth} \Leftrightarrow \dim\operatorname{span} M(A)=1,$$

and we will prove that

$$T \text{ is semi-smooth} \Leftrightarrow \dim\operatorname{span} M(A)\leqslant 2.$$

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2. Preliminaries

Let $(X, \|\cdot\|)$ be a real normed space. X^* denotes the dual space for X. We define *norm derivatives*:

$$\rho'_{\pm}(x,y) := \|x\| \cdot \lim_{\lambda \to 0^{\pm}} \tfrac{\|x+\lambda y\| - \|x\|}{\lambda}, \quad x,y \in X.$$

Convexity of the norm yields that the above definition is meaningful. The following properties can be found, e.g., in [2], [3]:

(n1) $\forall_{\alpha\in\mathbb{R}} \quad \rho'_{\pm}(x, \alpha x + y) = \alpha\|x\|^2 + \rho'_{\pm}(x,y)$;
(n2) $\forall_{\alpha\geqslant 0} \quad \rho'_{\pm}(\alpha x, y) = \alpha\rho'_{\pm}(x,y) = \rho'_{\pm}(x, \alpha y)$;
(n3) $\forall_{\alpha<0} \quad \rho'_{\pm}(\alpha x, y) = \alpha\rho'_{\mp}(x,y) = \rho'_{\pm}(x, \alpha y)$;
(n4) $\forall_{x,y\in X} \quad \rho'_{\pm}(x,x) = \|x\|^2, \ |\rho'_{\pm}(x,y)| \leqslant \|x\| \cdot \|y\|$.

Note, that if $(X, \langle\cdot|\cdot\rangle)$ is an inner product space, then $\langle y|x\rangle = \rho'_{+}(x,y) = \rho'_{-}(x,y)$ for arbitrary $x, y \in X$.

Miličić [5] introduced the following concept. Let us define $\rho' \colon X \times X \to \mathbb{R}$ by

$$\rho'(x,y) := \tfrac{1}{2}\left(\rho'_{-}(x,y) + \rho'_{+}(x,y)\right), \quad x,y \in X.$$

The functional ρ' is also denoted by $\langle y|x\rangle_g$ and called an M-*semi-inner product* – cf. Miličić [7] and [3]. From the properties of the mappings $\rho'_{\pm}$ we get:

(m1) $\forall_{\alpha\in\mathbb{R}} \quad \rho'(x, \alpha x + y) = \alpha\|x\|^2 + \rho'(x,y)$;
(m2) $\forall_{\alpha\in\mathbb{R}} \quad \rho'(\alpha x, y) = \alpha\rho'(x,y) = \rho'(x, \alpha y)$;
(m3) $\forall_{x,y\in X} \quad \rho'(x,x) = \|x\|^2, \ |\rho'(x,y)| \leqslant \|x\| \cdot \|y\|$.

Moreover, the mappings ρ'_{+}, ρ'_{-}, ρ' are continuous with respect to the second variable, but not necessarily with respect to the first one.

We say that X is *smooth at point* x_o if there is a unique functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Now, we consider a set

$$\mathcal{N}_{sm}(X) := \{x \in X : X \text{ is smooth at } x\} \cup \{0\}.$$

If X is a separable real Banach space, then $\mathcal{N}_{sm}(X)$ is dense. We give a characterization of smoothness in terms of the norm derivatives (see, e.g., [3, 2]). Namely, the following statements are equivalent:

(a) X is smooth at x_o, i.e., $x_o \in \mathcal{N}_{sm}(X)$;

(b) $\rho'_{+}(x_o, \cdot) \in X^*$;

(c) $\forall_{y\in X} \quad \rho'_{-}(x_o, y) = \rho'_{+}(x_o, y)$.

It is clear that if X is smooth, then the M-semi-inner product ρ' is linear in the second argument. However, there also exists non-smooth spaces from which the mapping ρ' is linear in the second variable too ([5, Example 8.1], [3, p. 51]).

A real normed space X is called *semi-smooth* (see [6]) if ρ' is additive (or, equivalently, linear; see (m2)) with respect to the second variable, i.e.,

$$\forall_{x,y,z\in X} \quad \rho'(x, y+z) = \rho'(x,y) + \rho'(x,z).$$

So, each smooth space is semi-smooth in the above sense but not conversely (l^1 is a suitable example; see [5, Example 8.1] and [3, p. 51]).

This notion of semi-smooth spaces motivates this paper. We want to introduce the following definition. Namely, we say that a normed space $(X, \|\cdot\|)$ is *semi-smooth at the point* $x_o \in X$ if it satisfies

$$\forall_{y,z\in X} \quad \rho'(x_o, y+z) = \rho'(x_o, y) + \rho'(x_o, z).$$

Similarly as before, we define

$$\mathcal{N}^s_{sm}(X) := \{x \in X \ : \ X \text{ is semi-smooth at } x\}.$$

Thus we can write $X = \mathcal{N}^s_{sm}(X)$ if and only if X is semi-smooth. Observe that

$$\mathcal{N}_{sm}(X) \subseteq \mathcal{N}^s_{sm}(X). \tag{1}$$

We summarize our observations in the following simple result. Let $x_o \in X \setminus \{0\}$. Then the following statements are equivalent:

(a) X is semi-smooth at x_o, i.e., $x_o \in \mathcal{N}^s_{sm}(X)$;

(b) $\rho'(x_o, \cdot) \in X^*$.

It is worth mentioning that in some cases the notion of semi-smoothness may be more convenient than the notion of smoothness. Indeed, recently the concept of semi-smoothness played a significant role in the paper [11].

3. Semi-smoothness in operator spaces

A simple lemma will be useful here. The proof is rather easy, so we omit it.

LEMMA 1. *Let X be a real normed space. Let $x, y \in X$. Suppose that $a^* \in X^*$ is a linear functional such that $\|a^*\| = 1$. If $a^*(x) = \|x\|$, then*

$$\rho'_-(x,y) \leqslant \|x\| \cdot a^*(y) \leqslant \rho'_+(x,y). \tag{2}$$

One should be able to verify with little effort that if $\operatorname{dist}(A, \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)) < \|A\|$, then $M(A) \neq \emptyset$ and $\dim \operatorname{span} M(A) < \infty$. By [10, Theorem 3.2] and [9] we get

THEOREM 1. [10] *Let $A, T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\operatorname{dist}(A, \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)) < \|A\|$. Then the following conditions hold:*

$$\begin{aligned} \rho'_+(A,T) &= \max\{\langle Ax|Tx\rangle : x \in M(A)\}; \\ \rho'_-(A,T) &= \min\{\langle Ax|Tx\rangle : x \in M(A)\}. \end{aligned} \tag{3}$$

The author proved a far more general theorem (see [10]) in that he considered only M-ideals; here we need the special case of [10, Theorem 3.2] (i.e. only (3)) and we did not cite his general version.

Now we are in position to prove the first main result of this paper.

THEOREM 2. *Let $\mathscr{H}_1, \mathscr{H}_2$ be real Hilbert spaces. Suppose that* $\dim \mathscr{H}_1 \geqslant 2$, $\dim \mathscr{H}_2 \geqslant 2$. *Assume* $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ *and* $\operatorname{dist}(A, \mathscr{K}(\mathscr{H}_1, \mathscr{H}_2)) < \|A\|$. *Then the following statements are equivalent:*

(a) *A is semi-smooth, i.e.,* $A \in \mathscr{N}^s_{sm}(\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2))$,

(b) $\dim \operatorname{span} M(A) \leqslant 2$.

Proof. We start with proving (b)$\Rightarrow$(a). If $\dim \operatorname{span} M(A) = 1$, then $M(A) = \{x_1, -x_1\}$. Therefore $\rho'_+(A,T) \overset{(3)}{=} \langle A(\pm x_1)|T(\pm x_1)\rangle \overset{(3)}{=} \rho'_-(A,T)$ for all $T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$. This means that A is a smooth point, and by (1), we have $A \in \mathscr{N}^s_{sm}(\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2))$.

Now, assume that $\dim \operatorname{span} M(A) = 2$. It is not difficult to prove that a restriction

$$A|_{\operatorname{span} M(A)} : \operatorname{span} M(A) \to \mathscr{H}_2$$

has to be a similarity (scalar multiple of an isometry). Namely, $\|Au\| = \|A\| \cdot \|u\|$ for all $u \in \operatorname{span} M(A)$; more precisely, $M(A)$ is a circle in the subspace $\operatorname{span} M(A)$. Since $\operatorname{span} M(A) \subseteq \mathscr{H}_1$ and $A(\operatorname{span} M(A)) \subseteq \mathscr{H}_2$, we have

$$\mathscr{H}_1 = \operatorname{span} M(A) \oplus (\operatorname{span} M(A))^\perp \text{ and}$$
$$\mathscr{H}_2 = A(\operatorname{span} M(A)) \oplus A(\operatorname{span} M(A))^\perp.$$

If $T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$, then T can be written as a 2×2 matrix with operator entries,

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where

$T_1 \in \mathscr{L}(\operatorname{span} M(A), A(\operatorname{span} M(A)))$,
$T_2 \in \mathscr{L}\left(\operatorname{span} M(A)^\perp, A(\operatorname{span} M(A))\right)$,
$T_3 \in \mathscr{L}\left(\operatorname{span} M(A), A(\operatorname{span} M(A))^\perp\right)$, and
$T_4 \in \mathscr{L}\left(\operatorname{span} M(A)^\perp, A(\operatorname{span} M(A))^\perp\right)$.

Let us fix x and y, two orthogonal unit vectors in $M(A)$, and let us define the operators $A_1, B_1, C_1, D_1 \in \mathscr{L}(\operatorname{span} M(A), A(\operatorname{span} M(A)))$ by

$$A_1 x := Ax,\ A_1 y := Ay; \quad C_1 x := Ay,\ C_1 y := 0;$$
$$B_1 x := 0,\quad B_1 y := Ax; \quad D_1 x := Ax,\ D_1 y := -Ay.$$

It is clear that $\mathscr{L}(\operatorname{span} M(A), A(\operatorname{span} M(A))) = \operatorname{span}\{A_1, B_1, C_1, D_1\}$. Moreover, we have also $Ax \perp Ay$, because the operator $A|_{\operatorname{span} M(A)}$ is a similarity. Let us introduce the operators $B, C, D \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$,

$$B := \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C := \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D := \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Take an arbitrary $E \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ such that

$$E := \begin{bmatrix} 0 & E_2 \\ E_3 & E_4 \end{bmatrix}.$$

Fix $b,c,d \in \mathbb{R}$. If we define $F := bB + cC + dD + E$, then we have

$$\begin{aligned} Fx &= dAx + cAy + E_3x, \\ Fy &= bAx - dAy + E_3y. \end{aligned} \tag{4}$$

Now, we define $W := \{\langle Au|Fu\rangle \in \mathbb{R} : u \in M(A)\}$. This yields the conditions $\rho'_+(A,F) = \max W$ (see Theorem 1) and $\rho'_-(A,F) = \min W$. It is helpful to recall that $\|Ax\| = \|Ay\| = \|A\|$, $x \perp y$ and

$$Ax \perp Ay, \quad \operatorname{span} M(A) \perp E_3x, E_3y. \tag{5}$$

Since $M(A)$ is a circle (and $E_3x, E_3y \perp \operatorname{span} M(A)$), it follows from the definition of F that

$$\begin{aligned} W &= \{\langle Au|Fu\rangle \in \mathbb{R} : u \in M(A)\} \\ &= \{\langle A(\alpha x + \beta y)|F(\alpha x + \beta y)\rangle : \alpha x + \beta y \in M(A)\} \\ &= \left\{\langle \alpha Ax + \beta Ay|\alpha Fx + \beta Fy\rangle : \alpha^2 + \beta^2 = 1\right\} \\ &\overset{(4)}{=} \{\langle \alpha Ax + \beta Ay|\alpha(dAx + cAy) + \beta(bAx - dAy)\rangle + \langle \alpha Ax + \beta Ay|\alpha E_3x + \beta E_3y\rangle : \\ &\quad\ \alpha^2 + \beta^2 = 1\} \\ &\overset{(5)}{=} \{(\alpha^2 d + \alpha\beta b)\|Ax\|^2 + (\alpha\beta c - \beta^2 d)\|Ay\|^2 : \\ &\quad\ \alpha^2 + \beta^2 = 1\} \\ &= \|A\|^2 \cdot \left\{(\alpha^2 d + \alpha\beta b) + (\alpha\beta c - \beta^2 d) : \alpha^2 + \beta^2 = 1\right\} \\ &= \|A\|^2 \cdot \left\{(\alpha^2 - \beta^2)d + \alpha\beta(b+c) : \alpha^2 + \beta^2 = 1\right\}. \end{aligned}$$

Now we consider a circle $\mathbb{T} := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1\}$ and a function $\varphi\colon \mathbb{T} \to \mathbb{R}$, $\varphi(\alpha,\beta) := (\alpha^2 - \beta^2)d + \alpha\beta(b+c)$. From this it is very easy to see that $-\varphi(\alpha,\beta) = \varphi(\beta,-\alpha)$, so $\max\varphi(\mathbb{T}) = -\min\varphi(\mathbb{T})$. On the other hand $\|A\|^2 \cdot \varphi(\mathbb{T}) = W$, and we obtain $\max W = -\min W$. Therefore

$$\begin{aligned} \rho'(A,F) &= \frac{1}{2}\left(\rho'_-(A,F) + \rho'_+(A,F)\right) \\ &= \frac{1}{2}(\min W + \max W) = 0. \end{aligned}$$

To summarize, it has been shown that

$$\rho'(A, bB + cC + dD + E) = 0 \ \text{ for all } \ E = \begin{bmatrix} 0 & E_2 \\ E_3 & E_4 \end{bmatrix} \tag{6}$$

and for all b,c,d in $\mathbb{R}$. Finally, we show that $A \in \mathcal{N}^s_{sm}(\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2))$. Fix $S,T \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ and recall that $\mathcal{L}(\operatorname{span} M(A), A(\operatorname{span} M(A))) = \operatorname{span}\{A_1,B_1,C_1,D_1\}$. Thus we obtain

$$\begin{aligned} S &= \alpha_1 A + \beta_1 B + \gamma_1 C + \delta_1 D + E, \\ T &= \alpha_2 A + \beta_2 B + \gamma_2 C + \delta_2 D + \tilde{E} \end{aligned}$$

for some $E = \begin{bmatrix} 0 & E_2 \\ E_3 & E_4 \end{bmatrix}$, $\tilde{E} = \begin{bmatrix} 0 & \tilde{E_2} \\ \tilde{E_3} & \tilde{E_4} \end{bmatrix}$, and for some $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2 \in \mathbb{R}$. Applying (m1) we get

$$\begin{aligned} \rho'(A, S+T) &= \rho'(A, (\alpha_1+\alpha_2)A + (\beta_1+\beta_2)B + (\gamma_1+\gamma_2)C + (\delta_1+\delta_2)D + E_1 + E_2) \\ &\overset{(m1)}{=} (\alpha_1+\alpha_2)\|A\|^2 + \rho'(A, (\beta_1+\beta_2)B + (\gamma_1+\gamma_2)C + (\delta_1+\delta_2)D + E_1 + E_2) \\ &\overset{(6)}{=} (\alpha_1+\alpha_2)\|A\|^2 + 0 = \alpha_1\|A\|^2 + \alpha_2\|A\|^2. \end{aligned}$$

On the other hand, by (m1) and (6) we also have

$$\begin{aligned} \rho'(A, S) &= \rho'(A, \alpha_1 A + \beta_1 B + \gamma_1 C + \delta_1 D + E) \\ &\overset{(m1)}{=} \alpha_1\|A\|^2 + \rho'(A, \beta_1 B + \gamma_1 C + \delta_1 D + E) \\ &\overset{(6)}{=} \alpha_1\|A\|^2 + 0 = \alpha_1\|A\|^2. \end{aligned}$$

In a similar way one can prove $\rho'(A,T) = \alpha_2\|A\|^2$. Thus we obtain

$$\rho'(A, S+T) = \rho'(A,S) + \rho'(A,T)$$

for all $S, T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$, which means that A is a semi-smooth point.

For the proof of the implication (a)$\Rightarrow$(b), we assume that A is a semi-smooth point. Assume, contrary to our claim, that $\dim \mathrm{span} M(A) > 2$. Again, it is easy to see that $M(A)$ is a unit sphere in a subspace $\mathrm{span} M(A)$. Thus there are vectors $x, y, z \in M(A)$ such that $x \perp y$, $x \perp z$, $y \perp z$. Since $A|_{\mathrm{span} M(A)}$ is a similarity, we have also $Ax \perp Ay$, $Ax \perp Az$, $Ay \perp Az$ and $\|Ax\| = \|Ay\| = \|Az\| = \|A\|$. Let $B, C \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ be defined by

$$\begin{aligned} &Bx := Ax,\ By := -Ay,\ Bz := 0, \\ &Bw := 0 \ \text{ for } \ w \in \{x,y,z\}^{\perp}; \end{aligned} \tag{7}$$

and

$$\begin{aligned} &Cx := 0,\ Cy := -Ay,\ Cz := Az, \\ &Cw := 0 \ \text{ for } \ w \in \{x,y,z\}^{\perp}. \end{aligned} \tag{8}$$

A moment's reflection shows that $\|B\| = \|A\| = \|C\|$. If we define linear functionals $f, g \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)^*$ by the formulas

$$f(T) := \left\langle \tfrac{Ax}{\|Ax\|} | Tx \right\rangle \quad \text{and} \quad g(T) := \left\langle \tfrac{Ay}{\|Ay\|} | Ty \right\rangle,$$

then $\|f\| = 1 = \|g\|$, $f(A) = \|A\| = g(A)$ and $f(A) = f(B)$, $g(A) = -g(B)$. Thus

$$\begin{aligned} \|A\|^2 &= \|A\| \cdot f(A) = \|A\| \cdot f(B) \\ &\overset{(2)}{\leqslant} \rho'_+(A,B) \overset{(m3)}{\leqslant} \|A\| \cdot \|B\| = \|A\|^2, \end{aligned}$$

whence $\rho'_+(A,B) = \|A\|^2$. In a similar way, we obtain

$$\begin{aligned} \|A\|^2 = \|A\| \cdot g(A) = \|A\| \cdot (-g(B)) &\overset{(2)}{\leqslant} -\rho'_-(A,B) \\ \leqslant |\rho'_-(A,B)| &\overset{(m3)}{\leqslant} \|A\| \cdot \|B\| = \|A\|^2, \end{aligned}$$

thus $\rho'_-(A,B) = -\|A\|^2$. Therefore $\rho'(A,B) = 0$. We now apply this method again, with B replaced by C, to obtain $\rho'(A,C) = 0$.

From (7) and (8), we have

$$(B+C)x = Ax,\ (B+C)y = -2Ay,$$
$$(B+C)z = Az,\ (B+C)w = 0 \text{ for } w \in \{x,y,z\}^\perp.$$

It is easy to check that $\|B+C\| = 2\|A\|$. Moreover, in a similar way one can prove $\rho'_-(A,B+C) = -2\|A\|^2$. A calculation is identical and we do not repeat this process. But we should show that $\rho'_+(A,B+C) = \|A\|^2$. It follows from Theorem 1 that

$$\begin{aligned}\rho'_+(A,B+C) &= \max\{\langle Au|(B+C)u\rangle :\ u \in M(A)\}\\ &= \max\{\alpha^2\|Ax\|^2 - 2\beta^2\|Ay\|^2 + \gamma^2\|Az\|^2 : \alpha^2+\beta^2+\gamma^2 = 1\}\\ &= \|A\|^2 \cdot \max\{\alpha^2 - 2\beta^2 + \gamma^2 : \alpha^2+\beta^2+\gamma^2 = 1\}\\ &= \|A\|^2 \cdot \max\{1-\beta^2-2\beta^2 : \beta \in [-1,1]\} = \|A\|^2 \cdot 1.\end{aligned}$$

Summarizing, we obtain

$$\begin{aligned}\rho'(A,B+C) &= \frac{1}{2}\left(\rho'_-(A,B+C) + \rho'_+(A,B+C)\right)\\ &= -\frac{1}{2}\|A\|^2 \neq 0 = \rho'(A,B) + \rho'(A,C)\end{aligned}$$

and therefore $A \notin \mathscr{N}^s_{sm}(\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2))$, a contradiction. The proof of Theorem 2 is complete. $\square$

Analysis similar to that in the proof of Theorem 2 shows that we can obtain the following theorem.

THEOREM 3. *Let $\mathscr{H}_1,\mathscr{H}_2$ be Hilbert spaces. Suppose that $2 \leqslant \dim\mathscr{H}_1, \dim\mathscr{H}_2$. Assume $A \in \mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)$ and $A \neq 0$. The following statements are equivalent:*

(a) *A is semi-smooth, i.e., $A \in \mathscr{N}^s_{sm}(\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2))$,*

(b) $\dim \operatorname{span} M(A) \leqslant 2$.

Finally, on account of Theorem 2, as a corollary we get the following result.

THEOREM 4. *Suppose that $2 \leqslant \dim\mathscr{H}_1$, $2 \leqslant \dim\mathscr{H}_2$. The following statements are equivalent:*

(a) *$\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ is semi-smooth;*

(b) $\dim\mathscr{H}_1 = 2$ *or* $\dim\mathscr{H}_2 = 2$.

Proof. Using Theorem 2 and the inequalities $\dim \operatorname{span} M(A) \leqslant \dim\mathscr{H}_1$ and

$$\dim \operatorname{span} M(A) = \dim \operatorname{span} A(\operatorname{span} M(A)) \leqslant \dim\mathscr{H}_2,$$

we have our assertion. $\square$

4. Concluding remark and problem

It is worth mentioning that in order to prove our main result (i.e. Theorem 2) we assumed that $\operatorname{dist}(A,\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)) < \|A\|$. However, for infinite-dimensional spaces we may consider yet another case, i.e. $\operatorname{dist}(A,\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)) = \|A\|$. Let us consider two subsets $\mathscr{L}_a,\mathscr{L}_b \subseteq \mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ defined by $\mathscr{L}_a := \{A\ :\ \operatorname{dist}(A,\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)) < \|A\|\}$ and $\mathscr{L}_b := \{A\ :\ \operatorname{dist}(A,\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)) = \|A\|\}$. Of course, main result, Theorem 2, gave the almost complete description of semi-smoothness. Indeed, $\mathscr{L}_a$ is huge (namely, $\mathscr{L}_a$ is open and dense) and $\mathscr{L}_b$ is very small (more precisely, $\mathscr{L}_b$ is nowhere dense). So, a large class of operators was investigated. However, the case $\operatorname{dist}(A,\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)) = \|A\|$ remains an open problem.

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