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FACTORIZATION AND RANGE INCLUSION OF ADJOINTABLE OPERATORS ON THE WEIGHTED HILBERT C^*–MODULES

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Abstract. The indefinite inner products induced by invertible and self-adjoint weights are introduced for elements in Hilbert C^*-modules. The solvability of the equation $AX = C$ is considered for Hilbert C^*-module operators. Some equivalent conditions concerning two aspects of factorization and range inclusion are refined and generalized to the weighted case.

1. Introduction

For a linear operator T, the range and the null space of T are denoted by $\mathscr{R}(T)$ and $\mathscr{N}(T)$, respectively, and by $\overline{\mathscr{R}(T)}$ the norm closure of $\mathscr{R}(T)$. Douglas [4] (see also [11]) studied the equation $AX = C$ for bounded linear operators on Hilbert spaces, and gave the following so called Douglas theorem.

DOUGLAS THEOREM. [4, Theorem 1] *If A and C are in the C^*-algebra $\mathbb{B}(\mathscr{H})$ of all bounded linear operators on a Hilbert space $\mathscr{H}$, then the following statements are equivalent:*

(i) $\mathscr{R}(C) \subseteq \mathscr{R}(A)$;

(ii) *there exists* $X \in \mathbb{B}(\mathscr{H})$ *such that* $C = AX$;

(iii) $CC^* \leqslant k^2 AA^*$ *for some scalar* $k \geqslant 0$.

Moreover, if (i), (ii), *and* (iii) *are valid, then there exists a unique operator* $D \in \mathbb{B}(\mathscr{H})$ (*known as the reduced* (*or Douglas*) *solution in the literature*) *with* $C = AD$ *so that*

(a) $\|D\|^2 = \inf\{\mu : CC^* \leqslant \mu AA^*\}$;

(b) $\mathscr{N}(D) = \mathscr{N}(C)$;

(c) $\mathscr{R}(D) \subseteq \overline{\mathscr{R}(A^*)}$.

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The equivalence between these established conditions are inspected in more general settings [3, 5, 6, 7, 11, 16]. Šmul'jan [13] was the first mathematician who pointed out that the Douglas theorem does not hold in indefinite inner product spaces, in general, even in the finite dimensional ones. Rodman [12] proved the Douglas theorem for Krein space operators. Under the condition that $\overline{\mathscr{R}(A^*)}$ is orthogonally complemented, the equivalence between the solvability of $AX = C$ and the range inclusion $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ is proved in [7, Theorem 1.1] for adjointable operators on Hilbert C^*-modules. The necessity of the orthogonal complementarity of $\overline{\mathscr{R}(A^*)}$ as well as equivalent conditions for the majorization $CC^* \leqslant \lambda AA^*$ are investigated in a recent paper [6, Theorems 2.4 and 3.2].

In the current paper, we restrict our attention to clarify the relationship between two aspects of factorization and range inclusion for Hilbert C^*-module operators, with emphasis imposed on the indefinite inner products induced by weights, which are invertible and self-adjoint, yet may fail to be positive. It is mentionable that the indefinite inner products considered in this paper are quite different from that in [1].

Some results obtained originally in [6] and [7] are refined and generalized in this paper to the weighted case. More specifically, the proof of (iii)$\Longleftrightarrow$(iv) in [7, Theorem 1.1] is reorganized in the weighted case (see the proof of Theorem 1). Note that Theorem 3.2 of [6] actually deals with the reduced solution of $AX = C$ rather than the general solution, and so a gap is contained in its proof of (iii)$\Longrightarrow$(i). The modified versions of [6, Theorem 3.2] are provided in the weighted case with the restriction of the reduced solution; see Theorems 2 and 3 in the next sections, respectively. A result in the non-weighted case is provided towards covering the gap mentioned above; see our Theorem 4 for the details.

Let us recall briefly some basic knowledge about Hilbert C^*-modules and adjointable operators; more details can be found e.g., in [9].

An inner product module over a C^*-algebra $\mathfrak{A}$ is a (right) $\mathfrak{A}$-module $\mathscr{H}$ equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot,\cdot \rangle$, which is $\mathbb{C}$-linear and $\mathfrak{A}$-linear in the second variable and has the properties $\langle x,y\rangle^* = \langle y,x\rangle$ as well as $\langle x,x\rangle \geqslant 0$ with equality if and only if $x = 0$. The $\mathfrak{A}$-module $\mathscr{H}$ is called a (right) Hilbert $\mathfrak{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$.

Suppose that $\mathscr{H}$ and $\mathscr{K}$ are two Hilbert $\mathfrak{A}$-modules, let $\mathscr{L}(\mathscr{H},\mathscr{K})$ be the set of all operators $T : \mathscr{H} \to \mathscr{K}$ for which there is an operator $T^* : \mathscr{K} \to \mathscr{H}$ such that $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$. It is known that (see, e.g., [9, P., 8]) any element $T \in \mathscr{L}(\mathscr{H},\mathscr{K})$ must be a bounded linear operator, which is also $\mathfrak{A}$-linear. In the case when $\mathscr{H} = \mathscr{K}$, we abbreviate $\mathscr{L}(\mathscr{H},\mathscr{H})$ as $\mathscr{L}(\mathscr{H})$, which is a C^*-algebra.

If both $\mathscr{H}_1$ and $\mathscr{H}_2$ are submodules of $\mathscr{H}$ such that $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$, then $\mathscr{H}_1 \dotplus \mathscr{H}_2$ is defined by

$$\mathscr{H}_1 \dotplus \mathscr{H}_2 = \{h_1 + h_2 : h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2\}.$$

Recall that a closed submodule $\mathscr{M}$ of $\mathscr{H}$ is said to be orthogonally complemented in $\mathscr{H}$ if $\mathscr{H} = \mathscr{M} \dotplus \mathscr{M}^\perp$, where

$$\mathscr{M}^\perp = \big\{x \in \mathscr{H} : \langle x,y\rangle = 0, \text{ for all } y \in \mathscr{M}\big\}.$$

A brief description of the arrangement of this paper is as follows. In the next section, we focus on the study of the equivalent conditions concerning two aspects of factorization and range inclusion. Furthermore, in view of the solvability of the equation $AX = C$, the necessity of the orthogonal complementarity of $\overline{\mathscr{R}(A^*)}$ is clarified.

Throughout the rest of this paper, $\mathbb{C}^{m\times n}$ is the set of all $m\times n$ complex matrices, $\mathfrak{A}$ is a C^*-algebra, $\mathscr{H}$, $\mathscr{K}$, $\mathscr{E}$ are Hilbert $\mathfrak{A}$-modules and $I_{\mathscr{H}}$ (or simply I) is the identity operator on $\mathscr{H}$. In addition, $N\in\mathscr{L}(\mathscr{H})$, $M\in\mathscr{L}(\mathscr{K})$, and $G\in\mathscr{L}(\mathscr{E})$ are three weights (see Definition 1).

2. Main results

We start our work with the following definition playing an essential role in our investigation.

DEFINITION 1. An element M of $\mathscr{L}(\mathscr{H})$ is said to be a weight if $M = M^*$ and M is invertible in $\mathscr{L}(\mathscr{H})$. If, furthermore, M is positive, then M is said to be positive definite.

DEFINITION 2. Let $M\in\mathscr{L}(\mathscr{H})$ be a weight. The indefinite inner product on $\mathscr{H}$ induced by M is defined by

$$\langle x,y\rangle_M := \langle x, My\rangle, \text{ for all } x,y\in\mathscr{H}, \tag{1}$$

and the notation $\mathscr{H}_M$ is used to indicate that $\mathscr{H}$ is endowed with this indefinite inner product.

Recall that a weight J on a Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$ is said to be an involution if $J = J^* = J^{-1}$. In this case, $\mathscr{H}_J$ is called a Krein space. It is clear that $\mathrm{P}_+ = \frac{I+J}{2}$ and $\mathrm{P}_- = \frac{I-J}{2}$ are projections such that $\mathscr{R}(\mathrm{P}_+)\subseteq\mathscr{H}_+$ and $\mathscr{R}(\mathrm{P}_-)\subseteq\mathscr{H}_-$, where $\mathscr{H}_+ = \{x\in\mathscr{H} : \langle x,x\rangle_J \geqslant 0\}$ and $\mathscr{H}_- = \{x\in\mathscr{H} : \langle x,x\rangle_J \leqslant 0\}$. Krein spaces have been applied in many disciplines such as the quantum field theory [8] and noncommutative geometry [14]. The reader is referred to [2, 10] for some basic results on Krein spaces.

EXAMPLE 1. Let $(\mathscr{K},\langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module over a unital C^*-algebra $\mathscr{A}$ with the unit e, and let $J = \begin{pmatrix} I & 0\\ 0 & -e\end{pmatrix}$, where I denotes the identity operator in $\mathscr{L}(\mathscr{K})$. Then

$$\langle (x,a),(y,b)\rangle_J = \langle J(x,a),(y,b)\rangle = \langle x,y\rangle - a^*b$$

gives rise to an indefinite inner product on $\mathscr{K}\oplus\mathscr{A}$. The case when $\mathscr{K}$ is taken to be $\mathbb{C}^{n\times n}$ and $\mathscr{A}=\mathbb{C}$ is called the Minkowski space.

The following lemma is known.

LEMMA 1. (See [15, Remark 1.1]) *Let $M\in\mathscr{L}(\mathscr{K})$ and $N\in\mathscr{L}(\mathscr{H})$ be two weights. Then for every $T\in\mathscr{L}(\mathscr{H},\mathscr{K})$,*

$$\langle Tx,y\rangle_M = \langle x,T^{\#}y\rangle_N, \quad \textit{for every } x\in\mathscr{H} \textit{ and } y\in\mathscr{K},$$

where $T^{\#}$ is called the weighted adjoint operator of T and is given by

$$T^{\#} = N^{-1}T^{*}M \in \mathscr{L}(\mathscr{K},\mathscr{H}). \tag{2}$$

REMARK 1. Let $T \in \mathscr{L}(\mathscr{H},\mathscr{K})$, and let $M \in \mathscr{L}(\mathscr{K})$ and $N \in \mathscr{L}(\mathscr{H})$ be two weights. Since N^{-1}, T^{*} and M are all adjointable, the operator $T^{\#}$ formulated by (2) is in fact an element of $\mathscr{L}(\mathscr{K},\mathscr{H})$. Instead of $T \in \mathscr{L}(\mathscr{H},\mathscr{K})$ and $T^{\#} \in \mathscr{L}(\mathscr{K},\mathscr{H})$, we use the notation $T \in \mathscr{L}(\mathscr{H}_N,\mathscr{K}_M)$ and $T^{\#} \in \mathscr{L}(\mathscr{K}_M,\mathscr{H}_N)$ to indicate that $\mathscr{H}$ and $\mathscr{K}$ are endowed with weights N and M, respectively.

DEFINITION 3. Let $A \in \mathscr{L}(\mathscr{H}_N,\mathscr{K}_M)$ and $C \in \mathscr{L}(\mathscr{E}_G,\mathscr{K}_M)$. An operator $D \in \mathscr{L}(\mathscr{E}_G,\mathscr{H}_N)$ is said to be a reduced solution to the system

$$AX = C \qquad \big(X \in \mathscr{L}(\mathscr{E}_G,\mathscr{H}_N)\big), \tag{3}$$

if $AD = C$ and $\mathscr{R}(D) \subseteq \overline{\mathscr{R}(A^{\#})}$, where $A^{\#} = N^{-1}A^{*}M$.

REMARK 2. Since N is not necessarily positive definite, it may happen that $\overline{\mathscr{R}(A^{\#})} \cap \mathscr{N}(A) \neq \{0\}$. For example, take $\mathscr{H} = \mathscr{K} = \mathbb{C}^2$ and let $M = N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be two weights on $\mathscr{H}$ and $\mathscr{K}$, respectively. Put $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. It is evident that $A^{\#} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, and so

$$\overline{\mathscr{R}(A^{\#})} = \big\{(\alpha,\alpha)^t : \alpha \in \mathbb{C}\big\} = \mathscr{N}(A).$$

In addition, $\overline{\mathscr{R}(A^{*})} = \{(\alpha,-\alpha)^t : \alpha \in \mathbb{C}\} \neq \overline{\mathscr{R}(A^{\#})}$.

If $\overline{\mathscr{R}(A^{\#})} \cap \mathscr{N}(A) = \{0\}$, then whenever it exists, the reduced solution to system (3) is unique.

Our main result reads as follows.

THEOREM 1. (Main) *Let $A \in \mathscr{L}(\mathscr{H}_N,\mathscr{K}_M)$ be such that*

$$\mathscr{H} = \overline{\mathscr{R}(A^{\#})} \dotplus \mathscr{N}(A), \text{ where } A^{\#} = N^{-1}A^{*}M. \tag{4}$$

Then for every $C \in \mathscr{L}(\mathscr{E}_G,\mathscr{K}_M)$, the following three statements are equivalent:

(i) $\mathscr{R}(C) \subseteq \mathscr{R}(A)$;

(ii) *system* (3) *has a reduced solution;*

(iii) *system* (3) *has a solution.*

Proof. The proofs of (ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(i) are obvious. It is enough to prove the implication (i)$\Longrightarrow$(ii). Suppose that $C \in \mathscr{L}(\mathscr{E},\mathscr{K})$ is such that $\mathscr{R}(C) \subseteq \mathscr{R}(A)$. Then for every $x \in \mathscr{E}$, there exists $y \in \mathscr{H}$ such that

$$Cx = Ay = Ay_1, \tag{5}$$

where, by assumption (4), y is decomposed uniquely as $y = y_1 + y_2$ with $y_1 \in \overline{\mathscr{R}(A^\#)}$ and $y_2 \in \mathscr{N}(A)$. If $y_1' \in \overline{\mathscr{R}(A^\#)}$ also satisfies (5), that is, $Cx = Ay_1'$, then

$$y_1 - y_1' \in \overline{\mathscr{R}(A^\#)} \cap \mathscr{N}(A) = \{0\}.$$

Hence $y_1 = y_1'$ by assumption (4). It follows that the operator $D : \mathscr{E} \to \mathscr{H}$, $x \mapsto y_1$ is well-defined and

$$\mathscr{R}(D) \subseteq \overline{\mathscr{R}(A^\#)} \text{ and } Cx = ADx, \text{ for all } x \in \mathscr{E}. \tag{6}$$

We prove that D is bounded by using the closed graph theorem. Let $\{x_n\}$ be in $\mathscr{E}$ such that $x_n \to x$ and $Dx_n \to y$. It is clear that $y \in \overline{\mathscr{R}(A^\#)}$, and $Cx_n \to Cx$, $ADx_n \to Ay$, since C and A are bounded. Moreover, by (6), we have

$$ADx_n = Cx_n \to Cx,$$

whence $Cx = Ay$, which ensures that $Dx = y$. This completes the proof of the boundedness of D.

We show that

$$\|C^\# x\| \leqslant M \|A^\# x\|, \text{ where } M = \|N\| \cdot \|D\| \cdot \|G^{-1}\|. \tag{7}$$

In view of (2) and (6), for each $x \in \mathscr{K}$ we have

$$\begin{aligned}
\langle C^\# x, C^\# x\rangle &= \langle G^{-1}C^*Mx, G^{-1}C^*Mx\rangle \\
&= \langle CG^{-2}C^*Mx,\ Mx\rangle \\
&= \langle ADG^{-2}C^*Mx,\ Mx\rangle \\
&= \langle DG^{-2}C^*Mx,\ A^*Mx\rangle \\
&= \langle NDG^{-1}\cdot G^{-1}C^*Mx,\ N^{-1}A^*Mx\rangle \\
&= \langle NDG^{-1}\cdot C^\# x,\ A^\# x\rangle.
\end{aligned}$$

Hence

$$\|C^\# x\|^2 = \|\langle NDG^{-1}\cdot C^\# x,\ A^\# x\rangle\| \leqslant \|N\|\|D\|\|G^{-1}\|\|C^\# x\|\|A^\# x\|,$$

which clearly leads to (7).

We then define an operator V from $\mathscr{H}$ to $\mathscr{E}$ in three steps. It follows from (7) that the operator

$$V : \mathscr{R}(A^\#) \to \mathscr{R}(C^\#), \quad A^\# x \to C^\# x$$

is well-defined, linear, and $\|Vu\| \leqslant M\|u\|$ for every $u \in \mathscr{R}(A^{\#})$. Therefore, the operator V can be extended to be a bounded linear operator from $\overline{\mathscr{R}(A^{\#})}$ to $\overline{\mathscr{R}(C^{\#})}$. A further linear extension of V can be made from assumption (4) by just letting

$$Vu = 0, \text{ for all } u \in \mathscr{N}(A).$$

From the construction of the operator V above, we observe that

$$V|_{\mathscr{N}(A)} \equiv 0, \quad VA^{\#}x = C^{\#}x \ (x \in \mathscr{K}), \quad \|Vu\| \leqslant M\|u\| \ \big(u \in \overline{\mathscr{R}(A^{\#})}\big). \tag{8}$$

Next, we prove that $D^{\#} = V$ (Hence, the boundedness of V will be deduced). Indeed, given any $x \in \mathscr{E}$ and $y \in \mathscr{H}$ with $y = y_1 + y_2$ such that $y_1 \in \overline{\mathscr{R}(A^{\#})}$ and $y_2 \in \mathscr{N}(A)$, we can choose sequences $\{u_n\}$ and $\{v_n\}$ in $\mathscr{K}$ such that

$$\lim_{n\to\infty} A^{\#}u_n = y_1 \text{ and } \lim_{n\to\infty} A^{\#}v_n = Dx \ \big(\text{see } (6)\big),$$

whence, by (8) and (6), we infer that

$$\begin{aligned}
\langle x, Vy\rangle_G =& \langle x, Vy_1\rangle_G = \lim_{n\to\infty}\langle x, VA^{\#}u_n\rangle_G = \lim_{n\to\infty}\langle x, C^{\#}u_n\rangle_G \\
=& \lim_{n\to\infty}\langle Cx, u_n\rangle_M = \lim_{n\to\infty}\langle ADx, u_n\rangle_M = \lim_{n\to\infty}\langle Dx, A^{\#}u_n\rangle_N \\
=& \langle Dx, y_1\rangle_N = \lim_{n\to\infty}\langle A^{\#}v_n, y_1\rangle_N = \lim_{n\to\infty}\langle v_n, Ay_1\rangle_M \\
=& \lim_{n\to\infty}\langle v_n, Ay\rangle_M = \lim_{n\to\infty}\langle A^{\#}v_n, y\rangle_N = \langle Dx, y\rangle_N.
\end{aligned}$$

This completes the proof of $D^{\#} = V$.

Now, for each $x \in \mathscr{E}$ and $y \in \mathscr{H}$, we have

$$\langle Dx, y\rangle = \langle Dx, N^{-1}y\rangle_N = \langle x, VN^{-1}y\rangle_G = \langle x, GVN^{-1}y\rangle,$$

which implies that D is adjointable and $D^* = GVN^{-1}$. It follows from (6) that D is actually the reduced solution to system (3). □

EXAMPLE 2. Let $\mathscr{E} = \mathscr{H} = \mathscr{K} = \mathbb{C}^2$ and $M = N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be two weights on $\mathscr{H}$ and $\mathscr{K}$, respectively. Let G be any weight on $\mathscr{E}$ and put $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. It is evident that

$$\overline{\mathscr{R}(A^{\#})} = \big\{(\alpha, -\alpha)^t : \alpha \in \mathbb{C}\big\} \text{ and } \mathscr{N}(A) = \mathscr{R}(A) = \big\{(\alpha, \alpha)^t : \alpha \in \mathbb{C}\big\},$$

which lead obviously to (4). Let $C \in \mathscr{L}(\mathscr{E}, \mathscr{K}) = \mathbb{C}^{2\times 2}$ be any operator such that $\mathscr{R}(C) \subseteq \mathscr{R}(A)$. Then clearly,

$$C = \begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix} \text{ for some } \alpha, \beta \in \mathbb{C},$$

and therefore the reduced solution to system (3) turns out to be

$$D = \begin{pmatrix} \frac{\alpha}{2} & \frac{\beta}{2} \\ -\frac{\alpha}{2} & -\frac{\beta}{2} \end{pmatrix}.$$

As a reverse direction of (4), we have the following result.

THEOREM 2. *Let $A \in \mathscr{L}(\mathscr{H}_{I_{\mathscr{H}}}, \mathscr{K}_M)$. Then the following statements are equivalent:*

(i) *$\mathscr{H}$ can be decomposed directly as (4) with $N = I_{\mathscr{H}}$ therein;*

(ii) *given every Hilbert C^*-module $\mathscr{E}$ with a weight G and every $C \in \mathscr{L}(\mathscr{E}_G, \mathscr{K}_M)$ with $\mathscr{R}(C) \subseteq \mathscr{R}(A)$, system (3) with $N = I_{\mathscr{H}}$ therein has a reduced solution;*

(iii) *given every Hilbert C^*-module $\mathscr{E}$ and every $C \in \mathscr{L}(\mathscr{E}_{I_{\mathscr{E}}}, \mathscr{K}_M)$ with $\mathscr{R}(C) \subseteq \mathscr{R}(A)$, system (3) with $N = I_{\mathscr{H}}$ therein has a reduced solution.*

Proof. The implication (i)$\Longrightarrow$(ii) is shown in Theorem 1, and the implication (ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(i): Put $\mathscr{E} = \overline{\mathscr{R}(A^\#)} = \overline{\mathscr{R}(I_{\mathscr{H}}^{-1}A^*M)} = \overline{\mathscr{R}(A^*)} \subseteq \mathscr{H}$, which is endowed with the identity weight. Let $C = A|_{\mathscr{E}}$ be the restriction of A on $\mathscr{E}$. It is obvious that $C \in \mathscr{L}(\mathscr{E}, \mathscr{K})$ such that

$$C^*x = A^*x \quad \text{for every } x \in \mathscr{K},$$

which gives by (2) that

$$C^\#x = I_{\mathscr{E}}^{-1}C^*Mx = A^*Mx = A^\#x, \text{ for every } x \in \mathscr{K}.$$

By our assumption, there exists $D \in \mathscr{L}(\mathscr{E}_{I_{\mathscr{E}}}, \mathscr{H}_N)$ such that

$$AD = C \text{ and } \mathscr{R}(D) \subseteq \overline{\mathscr{R}(A^\#)} = \mathscr{E}.$$

Therefore,

$$D^*A^\#x = C^\#x = A^\#x, \text{ for every } x \in \mathscr{K},$$

which implies that

$$D^*u = u, \text{ for all } u \in \mathscr{E}.$$

The equation above together with $D^* \in \mathscr{L}(\mathscr{H}, \mathscr{E})$ yields

$$\mathscr{R}(D^*) = \mathscr{E} \text{ and } D^*D^*x = D^*x \text{ for all } x \in \mathscr{H}.$$

It follows that

$$x = D^*x + (x - D^*x) \in \mathscr{E} + \mathscr{N}(D^*) \text{ for all } x \in \mathscr{H},$$

which means that $\mathscr{H} = \mathscr{E} + \mathscr{N}(D^*)$ and $\mathscr{E} \cap \mathscr{N}(D^*) = \{0\}$.

It remains only to prove that $\mathscr{N}(D^*) = \mathscr{N}(A)$. By assumption $\mathscr{R}(D) \subseteq \mathscr{E}$, hence $D^*Du = Du$ for every $u \in \mathscr{E}$. Therefore for each $u \in \mathscr{E}$,

$$\begin{aligned}\langle Du-u, Du-u\rangle =& \langle Du, Du\rangle - \langle u, Du\rangle - \langle Du, u\rangle + \langle u, u\rangle \\ =& \langle u, D^*Du\rangle - \langle u, Du\rangle - \langle u, D^*u\rangle + \langle u, u\rangle = 0,\end{aligned}$$

which gives $Du = u$. It follows that

$$\begin{aligned}\mathscr{N}(A) &= \left\{x \in \mathscr{H} : \langle u, Ax\rangle_M = 0, \forall u \in \mathscr{K}\right\} \\ &= \left\{x \in \mathscr{H} : \langle A^{\#}u, x\rangle = 0, \forall u \in \mathscr{K}\right\} \\ &= \left\{x \in \mathscr{H} : \langle v, x\rangle = 0, \forall v \in \mathscr{E}\right\} \\ &= \left\{x \in \mathscr{H} : \langle Dv, x\rangle = 0, \forall v \in \mathscr{E}\right\} \\ &= \left\{x \in \mathscr{H} : \langle v, D^*x\rangle = 0, \forall v \in \mathscr{E}\right\} = \mathscr{N}(D^*). \quad \square\end{aligned}$$

REMARK 3. Replacing "has a reduced solution" with "has a solution", we do not know at this moment whether the implication (iii)$\Longrightarrow$(i) in the preceding theorem is valid or not. A partial answer will be given in Theorem 4.

THEOREM 3. *Let $A \in \mathscr{L}(\mathscr{H}_N, \mathscr{H}_M)$ be such that $NA = AN$ and*

$$\langle x, x\rangle_N = 0 \text{ implies } x = 0 \text{ whenever } x \in \overline{\mathscr{R}(A^{\#})}. \tag{9}$$

Then the following statements are equivalent:

(i) *$\mathscr{H}$ can be decomposed directly as (4);*

(ii) *given every Hilbert C^*-module $\mathscr{E}$ with a weight G and every $C \in \mathscr{L}(\mathscr{E}_G, \mathscr{H}_M)$ with $\mathscr{R}(C) \subseteq \mathscr{R}(A)$, system (3) with $\mathscr{K}_M = \mathscr{H}_M$ therein has a reduced solution.*

Proof. The implication (i)$\Longrightarrow$(ii) is shown in Theorem 1.

(ii)$\Longrightarrow$(i): Let $\mathscr{E} = \overline{\mathscr{R}(A^{\#})}$ and $G = N|_{\mathscr{E}}$ be the restriction of N on $\mathscr{E}$. Since $NA = AN$, we have $NA^* = A^*N$ and $N^{-1}A^* = A^*N^{-1}$, so

$$\mathscr{E} = \overline{\mathscr{R}(N^{-1}A^*M)} = \overline{\mathscr{R}(N^{-1}A^*)} = \overline{\mathscr{R}(A^*N^{-1})} = \overline{\mathscr{R}(A^*)}. \tag{10}$$

Hence

$$G\mathscr{E} = \overline{\mathscr{R}(NA^*)} = \overline{\mathscr{R}(A^*N)} = \overline{\mathscr{R}(A^*)} = \mathscr{E}.$$

It follows that G is a weight on $\mathscr{E}$. Put $C = A|_{\mathscr{E}} : \mathscr{E}_G \to \mathscr{H}_M$ as before. Then (10) yields $C \in \mathscr{L}(\mathscr{E}_G, \mathscr{H}_M)$, which satisfies

$$C^*x = A^*x \text{ and } C^{\#}x = N^{-1}C^*Mx = A^{\#}x \text{ for every } x \in \mathscr{H}_M.$$

By the assumption, there exists $D \in \mathscr{L}(\mathscr{E}_G, \mathscr{H}_N)$ such that

$$AD = C \text{ and } \mathscr{R}(D) \subseteq \mathscr{E}.$$

Therefore,

$$D^{\#}A^{\#}x = C^{\#}x = A^{\#}x \text{ for all } x \in \mathscr{H}_M,$$

which gives

$$D^{\#}u = u \text{ and } D^{\#}D^{\#}x = D^{\#}x, \text{ for all } u \in \mathscr{E} \subseteq \mathscr{H}_N \text{ and } x \in \mathscr{H}_N.$$

As in the proof of Theorem 2, we can obtain

$$\mathscr{R}(D^{\#}) = \mathscr{E} \text{ and } \mathscr{H}_N = \mathscr{E} \dot{+} \mathscr{N}(D^{\#}).$$

For each $u \in \mathscr{E}$, since $D^{\#}Du = Du$, we have

$$\begin{aligned}\langle Du-u, Du-u\rangle_N =& \langle Du, Du\rangle_N - \langle u, Du\rangle_N - \langle Du, u\rangle_N + \langle u, u\rangle_N \\ =& \langle u, D^{\#}Du\rangle_G - \langle u, Du\rangle_N - \langle u, D^{\#}u\rangle_G + \langle u, u\rangle_N = 0,\end{aligned}$$

which gives $Du = u$ by (9). From (9) it follows that

$$\begin{aligned}\mathscr{N}(D^{\#}) &= \big\{x \in \mathscr{H}_N : \langle u, D^{\#}x\rangle_G = 0, \forall u \in \mathscr{E}_G\big\} \\ &= \big\{x \in \mathscr{H}_N : \langle Du, x\rangle_N = 0, \forall u \in \mathscr{E}_G\big\} \\ &= \big\{x \in \mathscr{H}_N : \langle v, x\rangle_N = 0, \forall v \in \mathscr{E} \subseteq \mathscr{H}_N\big\} \\ &= \big\{x \in \mathscr{H}_N : \langle A^{\#}z, x\rangle_N = 0, \forall z \in \mathscr{H}_M\big\} \\ &= \big\{x \in \mathscr{H}_N : \langle z, Ax\rangle_M = 0, \forall z \in \mathscr{H}_M\big\} = \mathscr{N}(A).\end{aligned}$$

This completes the proof that $\mathscr{H}$ can be decomposed directly as (4). □

REMARK 4. For a Krein C^*-module $\mathscr{H}_J$, the set

$$\mathscr{H}_0 = \{x \in \mathscr{H} : \langle x, x\rangle_J = 0\}$$

is called the neutral part of $\mathscr{H}$. In this case, (9) can be figured out as $\overline{\mathscr{R}(A^{\#})} \cap \mathscr{H}_0 = \{0\}$.

We finish this paper with a result on the orthogonal complementarity of $\overline{\mathscr{R}(A^*)}$. In what follows we consider only the non-weighted case. Recall that every C^*-algebra $\mathfrak{A}$ can be regarded as a Hilbert $\mathfrak{A}$-module via the inner product given by

$$\langle x, y\rangle := x^*y, \text{ for all } x, y \in \mathfrak{A}.$$

Assume that $\mathfrak{A}$ has a unit e. Then for each $T \in \mathscr{L}(\mathfrak{A}, \mathscr{K})$,

$$T(x) = T(ex) = \big(T(e)\big)x = ax = L_a(x),$$

where $a = T(e)$. If, in addition, $\mathscr{K} = \mathfrak{A}$, then $L_a \in \mathscr{L}(\mathfrak{A})$ and $(L_a)^* = L_{a^*}$. In particular, L_e is the identity operator $I_{\mathfrak{A}}$ on $\mathfrak{A}$.

THEOREM 4. *Let $\mathfrak{A}$ be a unital C^*-algebra and let $A \in \mathscr{L}(\mathfrak{A}, \mathscr{K})$. If for every $C \in \mathscr{L}(\mathscr{E}, \mathscr{K})$ with $\mathscr{R}(C) \subseteq \mathscr{R}(A)$, system (3) with $G = I_{\mathscr{E}}, \mathscr{H} = \mathfrak{A}$ and $N = I_{\mathscr{H}}$ has a solution, then $\mathfrak{A}$ can be decomposed orthogonally as*

$$\mathfrak{A} = \overline{\mathscr{R}(A^*)} \dotplus \mathscr{N}(A).$$

Proof. Let $\mathscr{H} = \mathfrak{A}$, $\mathscr{E} = \overline{\mathscr{R}(A^*)}$ and $C = A|_{\mathscr{E}}$. By the proof of Theorem 2, we have $C \in \mathscr{L}(\mathscr{E}, \mathscr{K})$ such that

$$C^*x = A^*x, \text{ for all } x \in \mathscr{K}.$$

By our assumption, there exists $X \in \mathscr{L}(\mathscr{E}, \mathscr{H})$ such that $AX = C$. Then

$$X^*A^*x = C^*x = A^*x, \text{ for all } x \in \mathscr{K}.$$

It follows that $X^*u = u$ for all $u \in \mathscr{E}$, therefore $\mathscr{R}(X^*) \supseteq \mathscr{E}$, which leads obviously by $X^* \in \mathscr{L}(\mathscr{H}, \mathscr{E})$ to $\mathscr{R}(X^*) = \mathscr{E}$.

Now let $a = X^*(e) \in \mathscr{E}$, where e is the unit of $\mathfrak{A}$. Then

$$X^*(x) = L_a(x), \text{ for all } x \in \mathscr{H},$$

where L_a is an element of $\mathscr{L}(\mathscr{H})$ rather than an element of $\mathscr{L}(\mathscr{H}, \mathscr{E})$. Since $\mathscr{R}(L_a) = \mathscr{R}(X^*) = \mathscr{E}$ is closed in $\mathscr{H}$, by [9, Theorem 3.2], we infer that $\mathscr{H}$ can be decomposed orthogonally as $\mathscr{H} = \mathscr{E} \dotplus \mathscr{E}^{\perp}$. The conclusion then follows from the well-known equation $\overline{\mathscr{R}(A^*)}^{\perp} = \mathscr{N}(A)$. □

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