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ON A FUNCTIONAL CALCULUS FOR UNITARY OPERATORS IN PONTRYAGIN SPACES

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Abstract. This work is devoted to a functional description of the weakly closed algebra generated by a Pontryagin Space unitary operator under the condition that this operator contains a part like the two-side shift operator. The latter means that the inverse operator does not belong to the algebra.

Introduction

This paper is related with the theme treated initially in the papers [15], [16] and is a direct continuation of the paper [19]. For the terminology and the history of the subject, see the monograph [1] (or [2]), more specific concepts and facts can be found in [19]. The setting of the main problem is as follows. Consider a unitary operator U acting in a Pontryagin space, calculate all operator values of polynomials, where the value of the independent variable is taken to be equal to U. Next, close this operator algebra in weak topology denoting the new algebra by $\mathrm{Alg}\,U$. Our aim is to describe a structure of $\mathrm{Alg}\,U$. Depending on some properties of U, the algebra $\mathrm{Alg}\,U$ can (or not) contain the operator U^{-1}. The case $U^{-1} \in \mathrm{Alg}\,U$ was studied in [19], so in the present paper the case $U^{-1} \notin \mathrm{Alg}\,U$ will be analyzed.

Section 1 provides an introduction to known notions and results used in the course of the paper. Section 2 presents a slightly modified version of F. and M. Riesz Lemma on peak sets for the disc algebra, in Section 3 some function spaces related with functional calculus for π-unitary operators are introduced. The role of these spaces is sketched by Proposition 3.1. Section 4 gives some preliminary results concerning the functional calculus for any π-unitary operator with the unbounded spectral function. Section 5 deals with a behavior on some unbounded functionals. Section 6 is devoted to the case of a π-unitary operator U with unbounded spectral function such that $U^{-1} \notin \mathrm{Alg}\,U$. The main result is given by Theorem 6.2.

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1. Preliminaries

In what follows the term "Pontryagin space" means a separable Hilbert space $\mathfrak{H}$ with an ordinary scalar product $(\cdot,\cdot)$ and a Hermitian sesquilinear indefinite form (inner product) $[\cdot,\cdot]$: $(\forall x,y\in\mathfrak{H})\ [x,y]=(Jx,y)$, $J=J^{-1}$, $\dim\operatorname{Ker}(I+J)=\kappa<\infty$. Operator J is called the fundamental symmetry (of $[\cdot,\cdot]$ with respect to $(\cdot,\cdot)$). As usual the symbol $[\perp]$ means the π-orthogonality, i.e. the orthogonality with respect to $[\cdot,\cdot]$ and the symbol $\perp$ means the orthogonality with respect to the Hilbert scalar product. In what follows U means a π-unitary operator, i.e., in particular, $U\mathfrak{H}=\mathfrak{H}$. Next, the symbols $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{T}$ mean complex plane, open unit disk and its border respectively.

Let us present $\mathfrak{H}$ in the form

$$\mathfrak{H}=\mathfrak{H}_{\mathbb{T}}[+]\mathfrak{H}_{\mathbb{T}'},\quad U\mathfrak{H}_{\mathbb{T}}\subset\mathfrak{H}_{\mathbb{T}},\quad U\mathfrak{H}_{\mathbb{T}'}\subset\mathfrak{H}_{\mathbb{T}'},\tag{1.1}$$

where $\sigma(U|_{\mathfrak{H}_{\mathbb{T}}})\subset\mathbb{T}$, $\sigma(U|_{\mathfrak{H}_{\mathbb{T}'}})\subset\mathbb{C}/\mathbb{T}$. The subspace $\mathfrak{H}_{\mathbb{T}'}$ has a finite dimension. Let E_λ be the π-orthogonal spectral function (spectral resolution) generated by $U|_{\mathfrak{H}_{\mathbb{T}}}$ and let $\{\lambda_j\}$ be the set of its critical points. Denote $\widetilde{\mathfrak{H}}:\ =\operatorname{CLin}\{E(\Delta)\mathfrak{H}\}$, where Δ runs the set of all closed arcs $\Delta\subset\mathbb{T}$, such that $\Delta\cap\{\lambda_j\}=\emptyset$, so

$$\text{the subspace } E(\Delta)\mathfrak{H} \text{ is positive.}\tag{1.2}$$

The subspace $\widetilde{\mathfrak{H}}$ is non-negative and generally speaking has a non-trivial isotropic part. We shall suppose

$$\widetilde{\mathfrak{H}}\cap\widetilde{\mathfrak{H}}^{[\perp]}\neq\{0\}.\tag{1.3}$$

Note that Condition (1.3) is fulfilled if and only if the spectral function E_λ is unbounded. Let $\mathfrak{G}_1=\widetilde{\mathfrak{H}}\cap\widetilde{\mathfrak{H}}^{[\perp]}$, $\mathfrak{G}_0=J\mathfrak{G}_1$, $\mathfrak{G}_2=\widetilde{\mathfrak{H}}\cap\mathfrak{G}_1^{\perp}$, $\mathfrak{G}_3=(\widetilde{\mathfrak{H}}\oplus\mathfrak{G}_0)^{[\perp]}$. All considerations would be only simpler if $\mathfrak{G}_3$ is trivial so we will suppose $\mathfrak{G}_3\neq\{0\}$. Without loss of generality one can suppose that $\mathfrak{G}_3=(\widetilde{\mathfrak{H}}\oplus\mathfrak{G}_0)^{\perp}$ and that the Hilbert scalar product on $\mathfrak{G}_2$ coincides with $[\cdot,\cdot]$. Thus,

$$\left.\begin{array}{l}\mathfrak{H}_{\mathbb{T}}=\mathfrak{G}_0\oplus\mathfrak{G}_1\oplus\mathfrak{G}_2\oplus\mathfrak{G}_3\\[2mm] J|_{\mathfrak{H}_{\mathbb{T}}}=\begin{pmatrix}0&V^{-1}&0&0\\V&0&0&0\\0&0&I_2&0\\0&0&0&J_3\end{pmatrix},U|_{\mathfrak{H}_{\mathbb{T}}}=\begin{pmatrix}U_{00}&0&0&0\\U_{10}&U_{11}&U_{12}&U_{13}\\U_{20}&0&U_{22}&0\\U_{30}&0&0&U_{33}\end{pmatrix}.\end{array}\right\}\tag{1.4}$$

In this representation V is an isometry, I_2 is the identity in $\mathfrak{G}_2$, J_3 is a fundamental symmetry in $\mathfrak{G}_3$ and the elements of the matrix representation for $U|_{\mathfrak{H}_{\mathbb{T}}}$ have the following relations

$$\left.\begin{array}{rl}(U_{00})^{-1}=&V^{-1}(U_{11})^*V,\\ U_{10}=&-\frac{1}{2}U_{11}V\left((U_{20})^*U_{20}+(U_{30})^*J_3U_{30}+iA\right),\\ U_{20}=&-U_{22}(U_{12})^*VU_{00},\\ U_{30}=&-U_{33}J_3(U_{13})^*VU_{00},\end{array}\right\}\tag{1.5}$$

where A is a self-adjoin operator. Moreover, the operators U_{22} and U_{33} are, respectively, unitary and J_3-unitary in the corresponding subspaces. Let us put

$$\widetilde{U}: = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}, \widetilde{U}^{\uparrow}: = \begin{pmatrix} U_{00} & 0 \\ U_{20} & U_{22} \end{pmatrix}. \tag{1.6}$$

Operators $\widetilde{U}$ and $\widetilde{U}^{\uparrow}$ act in the spaces $\widetilde{\mathfrak{H}} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$ and $\widetilde{\mathfrak{H}}^{\uparrow}: = \mathfrak{G}_0 \oplus \mathfrak{G}_2$ respectively. Since U_{22} is a unitary operator, its model is e^{iT}, i.e. the multiplication operator by e^{it} in a suitable space $L^2_{\vec{\sigma}}(\mathfrak{E})$, where $\mathfrak{E}$ is a Hilbert space (maybe finite-dimensional), a vector-valued measure is defined as usually in the theory of multiplicity of self-adjoint operators (see [10], §41; [4], Chapter 7; [5], Chapter 4.4; [11], Chapter VII). So, assume that $\sigma(t)$ is a non-decreasing function defined on the segment $[0,2\pi]$, continuous (at least) from the left in all points of the segment and having an infinite number of growth points. The mentioned function generates on the segment the scalar Lebesgue-Stieltjes measure μ_σ. Let $t \mapsto \mathscr{E}_t$, $t \in [0,2\pi]$ be a map such that

- $\mathscr{E}_t \subset \mathscr{E}$,
- $\dim(\mathscr{E}_t)$ is a μ_σ-measurable (but not necessarily finite a.e.) function,
- if $\dim(\mathscr{E}_{t_1}) = \dim(\mathscr{E}_{t_2})$, then $\mathscr{E}_{t_1} = \mathscr{E}_{t_2}$,
- if $\dim(\mathscr{E}_{t_1}) < \dim(\mathscr{E}_{t_2})$, then $\mathscr{E}_{t_1} \subset \mathscr{E}_{t_2}$.

Denote by $M_{\vec{\sigma}}(\mathscr{E})$ the space of the vector-valued functions $f(t)\colon\ t \mapsto \mathscr{E}_t$ μ_σ-measurable in the weak sense, defined a.e. and finite a.e. on the segment $[0,2\pi]$. Accordingly, the symbol $L^2_{\vec{\sigma}}(\mathscr{E})$ means here a Hilbert space of functions $f(t) \in M_{\vec{\sigma}}(\mathscr{E})$, such that $\int_0^{2\pi} \|f(t)\|^2_{\mathscr{E}} d\sigma(t) < \infty$, $(f(t),g(t))_{L^2_{\vec{\sigma}}} = \frac{1}{2\pi}\int_0^{2\pi}(f(t),g(t))_{\mathscr{E}} d\sigma(t)$. Due to these conditions there is some vector $d \in \mathscr{E}$ such that

$$d(t) \equiv d \in L^2_{\vec{\sigma}}(\mathscr{E}), \ \sigma(t) = \int_0^t \|d(t)\|^2_{\mathscr{E}} d\sigma(t). \tag{1.7}$$

Let $\{\widetilde{g}_j(t)\}_{j=1}^k \subset M_{\vec{\sigma}}(\mathfrak{E})$ be a finite set of functions with the following properties

$$\left.\begin{array}{l} \text{a) the system } \{\widetilde{g}_j(t)\}_{j=1}^k \text{ is linear independent modulo } L^2_{\vec{\sigma}}(\mathfrak{E}); \\ \text{b) for all } j = 1,2,\dots,k \text{ the function } e^{it}\widetilde{g}_j(t) \text{ has the representation} \\ \qquad e^{it}\widetilde{g}_j(t) = \eta_j(t) + \sum_{l=1}^k c_{jl}\widetilde{g}_l(t), \text{ where } \eta_j(t) \in L^2_{\vec{\sigma}}(\mathfrak{E}); \\ \text{c) eigenvalues of the matrix } C = \left(c_{jl}\right)_{k\times k} \text{ have unit module.} \end{array}\right\} \tag{1.8}$$

Denote by $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ the Hilbert functional space generated by the linear span of $L^2_{\vec{\sigma}}(\mathfrak{E})$ and $\{\widetilde{g}_j(t)\}_{j=1}^k$, where functions from the set $\{\widetilde{g}_j(t)\}_{j=1}^k$ are supposed by definition to be normalized, pairwise orthogonal and orthogonal to $L^2_{\vec{\sigma}}(\mathfrak{E})$.

Conditions (1.8) allow us to define on $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ the operator of multiplication by e^{it}: $f(t) \mapsto e^{it} f(t)$. This operator will be denote by e^{iT}. Conditions (1.8) also allow us to define on $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ the operator of multiplication by e^{-it} (denoted by e^{-iT}). It is easy to check that e^{-iT} is inverse to e^{iT}.

THEOREM 1.1. ([19]) *For the operator U the same as above there are a Hilbert space $L^2_{\vec{\sigma}}(\mathfrak{E})$, a function set $\{\widetilde{g}_j(t)\}_{j=1}^k$ satisfying Conditions (1.8) and an isometric operator $W: \widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E}) \mapsto \widetilde{\mathfrak{H}}$, $WL^2_{\vec{\sigma}}(\mathfrak{E}) = \mathfrak{G}_2$, such that*

$$\widetilde{U} = W(e^{-iT})^* W^{-1}, \qquad \widetilde{U}^{\uparrow} = W^{\uparrow} e^{iT} (W^{\uparrow})^{-1}, \tag{1.9}$$

where $W^{\uparrow} = (I_2 \oplus V^{-1})W$, I_2 is the identical operator on $\mathfrak{G}_2$, V is an isometric operator mapping $\mathfrak{G}_0$ onto $\mathfrak{G}_1$: $Vx = Jx$ for every $x \in \mathfrak{G}_0$, $\sigma(t)$ is continuous for every t such that $e^{it} \in \sigma(U_{11}) \cup \sigma(U_{33})$. $L^2_{\vec{\sigma}}(\mathfrak{E})$ and $\{\widetilde{g}_j(t)\}_{j=1}^k$ can be chosen by a such way that for all $j = 1, 2, \ldots, k$ and at a.a. $t \in [0, 2\pi]$ the condition $\widetilde{g}_j(t) \in \mathfrak{E}_1$ holds, where $\mathfrak{E}_1$ is some subspace of $\mathfrak{E}$ with the dimension no greater than k.

DEFINITION 1.1. The space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ in Theorem 1.1 is called a *basic model space* for the operator U and W is called an *operator of similarity* (generated by $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$).

Let us note that a basic model space and a corresponding operator of similarity are not uniquely determined (see [18], Subsection 6.2 for details).

Let us introduce some notions and notations from [17].

DEFINITION 1.2. Every operator $B \in \operatorname{Alg} U$ can be associated with a scalar function $\phi_B(t)$ (the *portrait* of B) such that

$$BE(\Delta) = \int_{\Delta} \phi_B(t) dE_t,$$

where Δ runs over the set of all closed subintervals of $[0, 2\pi]$ disjoint from Λ. If $\phi_B(t)$ is the portrait of B then B is called an *original* for $\phi_B(t)$ (the same function can have different originals).

An alternative approach that establishes the connection between the operator B and its portrait can be presented (see (1.9)) as follows:

$$P^{\uparrow} B|_{\widetilde{\mathfrak{H}}^{\uparrow}} = W^{\uparrow} \Phi_B (W^{\uparrow})^{-1} \tag{1.10}$$

where $P^{\uparrow}$ is the orthogonal projection onto $\widetilde{\mathfrak{H}}^{\uparrow}$ and Φ_B is the operator of multiplication by $\phi_B(t)$.

2. On peak sets for holomorphic functions.

Before proceeding further we need a technical result.

PROPOSITION 2.1. *For an arbitrary finite set of pairwise different points* $\{\nu_j\}_1^m \subset (0,2\pi]$ *there is a system of* $2m$ *real functions* $\{\eta_j^\pm(t)\}_1^m$ *given on* $[0,2\pi]$, *such that*

$$\left.\begin{array}{l}
\text{a) for all } j=1,2,\dots,m \text{ the functions } \eta_j^+(t) \text{ and } \eta_j^-(t) \text{ are continuously differentiable and } \eta_j^\pm(0)=\eta_j^\pm(2\pi);\\
\text{b) the inequality } \eta_j^\pm(t)\geqslant 0 \text{ holds for all } j=1,2,\dots,m \text{ and } t\in[0,2\pi];\\
\text{c) for every } j=1,2,\dots,m \text{ there is a neigborhood } \delta(\nu_j) \text{ of } \nu_j, \text{ such that the equality } \eta_q^\pm(t)=0 \text{ holds for all } t\in\delta(\nu_j) \text{ and } q=1,2,\dots,m;\\
\text{d) for all } j,q=1,2,\dots,m \text{ the equalities } \frac{1}{2\pi}\int_0^{2\pi}\eta_j^+(t)\operatorname{ctg}\frac{t-\nu_q}{2}dt=\delta_{jq},\ \frac{1}{2\pi}\int_0^{2\pi}\eta_j^-(t)\operatorname{ctg}\frac{t-\nu_q}{2}dt=-\delta_{jq}\\
\quad\text{(briefly } \frac{1}{2\pi}\int_0^{2\pi}\eta_j^\pm(t)\operatorname{ctg}\frac{t-\nu_q}{2}dt=\pm\delta_{jq}\text{) hold,}
\end{array}\right\}\quad(2.11)$$

where δ_{jq} *is the Kronecker symbol.*

Proof. First, let us compose a set $\{\mu_j^\pm(t)\}_1^m$ of continuously differentiable functions under the following conditions

$$\left.\begin{array}{l}
\text{a) } \mu_j^\pm(t)\geqslant 0 \text{ for } t\in[0,2\pi],\ \mu_j^\pm(t)=0 \text{ if } t\notin(\alpha_j^\pm,\beta_j^\pm), \text{ where the interval } (\alpha_j^\pm,\beta_j^\pm)\subset(0,2\pi] \text{ is chosen such that } \nu_q\notin(\alpha_j^\pm,\beta_j^\pm),\\
\quad j,q=1,2,\dots,m,\ \mu_j^\pm(0)=\mu_j^\pm(2\pi);\\
\text{b) } \frac{1}{2\pi}\int_0^{2\pi}\mu_j^\pm(t)\operatorname{ctg}\frac{t-\nu_j}{2}dt=\pm1,\ j=1,2,\dots,m;\\
\text{c) } \frac{1}{2\pi}\left|\int_0^{2\pi}\mu_j^\pm(t)\operatorname{ctg}\frac{t-\nu_q}{2}dt\right|\leqslant 4^{-m},\ j,q=1,2,\dots,m,\ j\neq q.
\end{array}\right\}$$

The mentioned above system $\{\mu_j^\pm(t)\}_1^m$ it is easy to construct taking, for instance, the function

$$\tau(t)=\begin{cases}(15/16)(t^2-1)^2, & t\in[-1;1];\\ 0, & t\in(-\infty;-1]\cup[1;+\infty);\end{cases}$$

and using the relations ($|\alpha|$ and $|\beta|$ are sufficiently small)

$$\lim_{\alpha,\beta\to+0,\alpha<\beta}\frac{1}{\beta-\alpha}\int_{-\pi}^{\pi}\tau\left(\frac{2}{\beta-\alpha}t-\frac{\beta+\alpha}{\beta-\alpha}\right)\operatorname{ctg}\frac{t}{2}dt=+\infty,$$

$$\lim_{\alpha,\beta\to-0,\alpha<\beta}\frac{1}{\beta-\alpha}\int_{-\pi}^{\pi}\tau\left(\frac{2}{\beta-\alpha}t-\frac{\beta+\alpha}{\beta-\alpha}\right)\operatorname{ctg}\frac{t}{2}dt=-\infty,$$

$$\lim_{\alpha,\beta\to\pm 0,\alpha<\beta} \frac{2}{\beta-\alpha}\int_{-\pi}^{\pi}\tau\left(\frac{2}{\beta-\alpha}t-\frac{\beta+\alpha}{\beta-\alpha}\right)\operatorname{ctg}\frac{t-\nu}{2}dt=-ctg(\nu/2),$$

where $\nu\in(-\pi,0)\cup(0,\pi)$.

Functions $\eta_j^{\pm}(t)$ we will construct stage by stage and (within the same stage) step by step. The process slightly remains a construction of biorthogonal families (see, for instance, [13]) but has some details related principally with the item b) from (2.11).

First stage. Put $\gamma_{jq}^{\pm}=\frac{1}{2\pi}\int_0^{2\pi}\mu_j^{\pm}(t)\operatorname{ctg}\frac{t-\nu_q}{2}dt$, $j,q=1,2,\dots,m$. Let q=1. During this stage we will construct a system of functions "orthogonal" to $\operatorname{ctg}\frac{t-\nu_1}{2}$.

First step. Let us consider γ_{21}^{+}. If $\gamma_{21}^{+}=0$, then the function $\mu_2^{+}(t)$ is not changed, if $\gamma_{21}^{+}>0$, then we set $\widetilde{\mu}_2^{+}(t)=\mu_2^{+}(t)+\gamma_{21}^{+}\mu_1^{-}(t)$, if $\gamma_{21}^{+}<0$, then we set $\widetilde{\mu}_2^{+}(t)=\mu_2^{+}(t)-\gamma_{21}^{+}\mu_1^{+}(t)$. Thus, in any case we have $\widetilde{\mu}_2^{+}(t)\geqslant 0$,

$$\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mu}_2^{+}(t)\operatorname{ctg}\frac{t-\nu_1}{2}dt=0\,,\quad \frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mu}_2^{+}(t)\operatorname{ctg}\frac{t-\nu_2}{2}dt=1+|\gamma_{21}^{+}|\gamma_{12}^{\mp},$$

$$\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mu}_2^{+}(t)\operatorname{ctg}\frac{t-\nu_j}{2}dt=\gamma_{2j}^{+}+|\gamma_{21}^{+}|\gamma_{1j}^{\mp},\quad j=3,4,\dots,m,$$

where the choice of the signs "$-$" or "$+$" for the multiple $\gamma_{1j}^{\mp}$ corresponds to the sign of $-\gamma_{21}^{+}$. Let us redefine $\mu_2^{+}(t)$ by the following way: $\mu_2^{+}(t):=\widetilde{\mu}_2^{+}(t)/(1+|\gamma_{21}^{+}|\gamma_{12}^{\mp})$. After that we can write

$$\frac{1}{2\pi}\int_0^{2\pi}\mu_2^{+}(t)\operatorname{ctg}\frac{t-\nu_1}{2}dt=0,\ \frac{1}{2\pi}\int_0^{2\pi}\mu_2^{+}(t)\operatorname{ctg}\frac{t-\nu_2}{2}dt=1,$$

$$\frac{1}{2\pi}\left|\int_0^{2\pi}\mu_2^{+}(t)\operatorname{ctg}\frac{t-\nu_j}{2}dt\right|<\frac{\frac{1}{4^m}+\frac{1}{4^{2m}}}{1-\frac{1}{4^{2m}}}<1/2^{2m-1},\quad j=3,4,\dots,m.$$

By these formulae the first step is finished. Note that during this we we did not use the functions $\mu_2^{-}(t)$, $\mu_j^{\pm}(t)$, $j=3,4,\dots,m$ and only the function $\mu_2^{+}(t)$ was redefined.

Second step. During this step we will redefined the function $\mu_2^{-}(t)$. Let us consider γ_{21}^{-}. If $\gamma_{21}^{-}=0$, then the function $\mu_2^{+}(t)$ is not changed, if $\gamma_{21}^{-}>0$, then we set $\widetilde{\mu}_2^{-}(t)=\mu_2^{-}(t)+\gamma_{21}^{-}\mu_1^{-}(t)$, if $\gamma_{21}^{-}<0$, then we set $\widetilde{\mu}_2^{-}(t)=\mu_2^{-}(t)-\gamma_{21}^{-}\mu_1^{+}(t)$. Thus, in any case we have $\widetilde{\mu}_2^{-}(t)\geqslant 0$,

$$\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mu}_2^{-}(t)\operatorname{ctg}\frac{t-\nu_1}{2}dt=0,\ \frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mu}_2^{-}(t)\operatorname{ctg}\frac{t-\nu_2}{2}dt=-1+|\gamma_{21}^{-}|\gamma_{12}^{\mp},$$

$$\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mu}_2^{-}(t)\operatorname{ctg}\frac{t-\nu_j}{2}dt=\gamma_{2j}^{-}+|\gamma_{21}^{-}|\gamma_{1j}^{\mp},\quad j=3,4,\dots,m,$$

where the choice of the signs "$-$" or "$+$" for the multiple $\gamma_{1j}^{\mp}$ corresponds to the sign of $-\gamma_{21}^{-}$. Let us redefine $\mu_2^{-}(t)$ by the following way: $\mu_2^{-}(t):=\widetilde{\mu}_2^{-}(t)/(1-|\gamma_{21}^{-}|\gamma_{12}^{\mp})$. After that we can write

$$\frac{1}{2\pi}\int_0^{2\pi}\mu_2^{-}(t)\operatorname{ctg}\frac{t-\nu_1}{2}dt=0,\ \frac{1}{2\pi}\int_0^{2\pi}\mu_2^{-}(t)\operatorname{ctg}\frac{t-\nu_2}{2}dt=-1,$$

$$\frac{1}{2\pi}\left|\int_0^{2\pi}\mu_2^+(t)\operatorname{ctg}\frac{t-\nu_j}{2}dt\right| < \frac{\frac{1}{4^m}+\frac{1}{4^{2m}}}{1-\frac{1}{4^{2m}}} < 1/2^{2m-1}, \quad j=3,4,\dots,m.$$

So, the second step is finished.

Third step. The aim of this step is a redefinition of $\mu_3^+(t)$ making it "orthogonal" to $\operatorname{ctg}\frac{t-\nu_1}{2}$. Consider γ_{31}^+. If $\gamma_{31}^- = 0$, then the function $\mu_3^+(t)$ is not changed, if $\gamma_{31}^+ > 0$, then we set $\widetilde{\mu}_3^+(t) = \mu_3^+(t) + \gamma_{31}^+\mu_1^-(t)$, if $\gamma_{31}^+ < 0$, then we set $\widetilde{\mu}_3^+(t) = \mu_3^+(t) - \gamma_{31}^+\mu_1^+(t)$. Thus, we have $\widetilde{\mu}_2^-(t) \geqslant 0$. A final part of this step is similar to the corresponding one of the first step.

The subsequent steps are devoted to a redefinition of $\mu_3^-(t)$ and $\mu_j^{\pm}(t)$, $j = 4,\dots,m$ with the aim to obtain the conditions of the type (2.11), where $\eta_j^{\pm}(t)$ is substituted by $\mu_j^{\pm}(t)$, $q=1$.

Thus, in the first stage we obtained the system that satisfies the following conditions

$$\left.\begin{array}{ll}
\text{a)} & \text{for all } j=1,2,\dots,m \text{ the functions } \mu_j^+(t) \text{ and } \mu_j^-(t) \text{ are continuously differentiable and } \mu_j^{\pm}(0)=\mu_j^{\pm}(2\pi);\\
\text{b)} & \text{the inequality } \mu_j^{\pm}(t)\geqslant 0 \text{ holds for all } j=1,2,\dots,m \text{ and } t\in[0,2\pi];\\
\text{c)} & \text{for any } j=1,2,\dots,m \text{ there is a neigborhood } \delta(\nu_j) \text{ of } \nu_j, \text{ such that the equality } \mu_q^{\pm}(t)=0 \text{ holds for all } t\in\delta(\nu_j) \text{ and } q=1,2,\dots,m;\\
\text{d)} & \text{for all } j=1,2,\dots,m \text{ the equalities } \frac{1}{2\pi}\int_0^{2\pi}\mu_j^{\pm}(t)\operatorname{ctg}\frac{t-\nu_j}{2}dt=\pm 1 \text{ hold;}\\
\text{e)} & \text{for all } j=2,3,\dots,m \text{ the equalities } \int_0^{2\pi}\mu_j^{\pm}(t)\operatorname{ctg}\frac{t-\nu_1}{2}dt=0 \text{ hold;}\\
\text{f)} & \text{for all } j=1,2,\dots,m \text{ and } q=2,3,\dots,m,\ j\neq q \text{ the inequalities } \frac{1}{2\pi}|\int_0^{2\pi}\mu_j^{\pm}(t)\operatorname{ctg}\frac{t-\nu_q}{2}dt|\leqslant 1/2^{2m-1}) \text{ hold.}
\end{array}\right\} \quad (2.12)$$

The system $\{\mu_j^{\pm}(t)\}_{j=1}^{\infty}$ under Conditions (2.12) represents a base for the next stage.

Second stage. Put $\gamma_{jq}^{\pm} = \frac{1}{2\pi}\int_0^{2\pi}\mu_j^{\pm}(t)\operatorname{ctg}\frac{t-\nu_q}{2}dt$, $j,q=1,2,\dots,m$. During this stage we will construct a system of functions "orthogonal" to $\operatorname{ctg}\frac{t-\nu_2}{2}$. A way to do it is similar to the process used during the first stage. In particular, let us describe the first step. If $\gamma_{12}^+ = 0$ then we do not change $\mu_1^+(t)$, in the opposite case we put $\widetilde{\mu}_1^+(t) = \mu_1^+(t) + |\gamma_{12}^+|\mu_2^{\mp}(t)$, where the choice of the signs "$-$" or "$+$" for the multiple $\mu_2^{\mp}(t)$ corresponds to the sign of $-\gamma_{12}^+$. Note that the latter formula can be considered as valid in the case $\gamma_{12}^+ = 0$ too. Next, we redefine $\mu_2^+(t) := \widetilde{\mu}_2^+(t)$ (note that $\gamma_{21}^{\mp} = 0$). By the same manner we, first, put $\widetilde{\mu}_j^+(t) = \mu_j^+(t) + |\gamma_{j2}^+|\mu_2^{\mp}(t)$ and, second, redefine $\mu_j^+(t) := \widetilde{\mu}_j^+(t)/(1+|\gamma_{j2}^+|\gamma_{2j}^{\mp})$, $j=3,4,\dots m$, etc.

As a next stage (for q=3) we redefine by the same manner the functions $\mu_1^{\pm}(t)$, $\mu_2^{\pm}(t)$, $\mu_4^{\pm}(t)$, $\dots$, $\mu_m^{\pm}(t)$, etc. Finally, concluding the last stage (for q=m) we put $\eta_j^{\pm}(t) = \mu_j^{\pm}(t)$, $j=1,2,\dots,m$. $\square$

The following lemma represents a slightly modified version of F. and M. Riesz Lemma on peak sets for the disc algebra (see, for instance, Garnett [7], Part III, Page 125).

LEMMA 2.1. *Let* $\mathscr{E} \subset \mathbb{T}$ *be a compact of Lebesgue measure equal to zero and let* $\{\nu_j\}_1^m \subset \mathbb{T}$ *be a some some finite set of pairwise different points and* $\mathscr{E} \cap \{\nu_j\}_1^m = \emptyset$. *Then there is a holomorphic on* $\mathbb{D}$ *and continuous up to the border* $\mathbb{T}$ *numerical function* $\varphi(\xi)$ *such that*

$$\left.\begin{array}{l} a)\ \varphi(\xi) = 1 \text{ for } \xi \in \mathscr{E}; \\ b)\ |\varphi(\xi)| < 1 \text{ for } \xi \in \mathbb{T}/\mathscr{E}; \\ c)\ \text{the function } \varphi(\xi)/\Pi_{j=1}^m(\xi - \nu_j) \text{ is uniformly bounded on } \mathbb{D}. \end{array}\right\} \tag{2.13}$$

Proof. Without loss of generality one can assume that $-1 \notin \mathscr{E} \cup \{\nu_j\}_1^m$, so the function $\frac{1}{i}\ln\xi$ maps the set $\mathscr{E} \cup \{\nu_j\}_1^m$ to an interior part of the interval $(0, 2\pi]$. By the Fatou's theorem (see Garnett [1], Ch.III, p. 125) there is a function $\omega\colon \mathbb{T} \mapsto [-\infty; 0)$, such that the pre-image of the set $\{-\infty\}$ coincides with $\mathscr{E}$, $\omega(e^{it}) \in L^1(0, 2\pi)$, $\omega(\xi)$ is continuous (taking into account the natural topology of semi-closed interval $[-\infty; 0)$) on $\mathbb{T}$ and the function $\omega(\xi)$ has a continuous derivative on the set $\mathbb{T}/\mathscr{E}$. Next, let a real-valued function $\psi(\xi)$ is defined and has the continuous derivative on the set $\mathbb{T}$, $\psi(\xi) \geqslant 0$ for all $\xi \in \mathbb{T}$, $\psi(\xi)$ is equal to zero in some neighborhood of the set $\{\nu_j\}_1^m$, but the pre-image of zero for $\psi(\xi)$ has no intersection with $\mathscr{E}$. Let $\omega_1(t) = \omega(e^{it}) \times \psi(e^{it})$, $t \in [0; 2\pi]$, and let $\widetilde{\omega}_1(t)$ be the conjugate of $\omega_1(t)$ function, i.e.

$$\widetilde{\omega}_1(t) = \frac{1}{2\pi}\int_0^{2\pi} \omega_1(\tau)\,\mathrm{ctg}\,\frac{t-\tau}{2} d\tau$$

(the integral is treated as a principal value integral). Due to the conditions imposed on $\omega_1(t)$ the function $\widetilde{\omega}_1(t)$ is continuous on $\frac{1}{i}\ln(\mathbb{T}\backslash\mathscr{E})$ and is analytic as a real-valued function in some neighborhood of the points $\frac{1}{i}\ln\nu_j$, $j = 1, 2, \ldots, m$. Put $\omega_2(t) = \omega_1(t) - \sum\limits_{j=1}^m |\widetilde{\omega}_1(\nu_j)|\eta_j^{\pm}(t)$, the choice of the sign for $\eta_j^{\pm}(t)$ is the following: it is "+" if $\widetilde{\omega}_1(\nu_j) \geqslant 0$, and it is "−" in the opposite case, the functions $\eta_j^{\pm}(t)$ are the same as in Proposition 2.1. Due to the construction of $\omega_2(t)$ the conjugate function $\widetilde{\omega}_2(t)$ is analytic as a real-valued function at points $\frac{1}{i}\ln\nu_j$,

$$\widetilde{\omega}_2\left(\frac{1}{i}\ln\nu_j\right) = 0, \qquad j = 1, 2, \ldots, m, \tag{2.14}$$

and for the function $\omega_2(t)$ the inequality $\omega_2(t) \leqslant 0$ is fulfilled everywhere. Let us put $\zeta(\xi) = \zeta(\rho e^{i\tau}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\xi}{e^{it}-\xi}\omega_2(t)dt$. The function $\zeta(\xi)$ is holomorphic in $\mathbb{D}$ and continuous in $\mathbb{D} \cup (\mathbb{T}\backslash\mathscr{E})$ and its range is a part of the closed left complex semi-plane, moreover due to (2.14) the equalities $\zeta(\nu_j) = 0$, $j = 1, 2, \ldots, m$ are fulfilled and the function $\zeta(\xi)/\Pi_{j=1}^{m}(\xi-\nu_j)$ is bounded on the intersection $\mathbb{T}$ with some neighborhood of the set $\{\nu_j\}_1^m$. Let us put $\varphi(\xi)=\zeta(\xi)/(\zeta(\xi)-1)$. It is clear that the function $\varphi(\xi)$ has the required properties. □

3. Some function spaces

Let us recall that U is a π-unitary operator satisfying Condition (1.3). Let us choose and fix a basic model space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ for U, where $\sigma(t)$ is a non decreasing continuous from the left function given on the segment $[0,2\pi]$. Then $\sigma(t)$

$$\sigma(t)=\sigma_c(t)+\sigma_s(t), \tag{3.15}$$

where $\sigma_c(t)$ and $\sigma_s(t)$ are uniquely determined non-decreasing functions such that $\sigma_c(t)$ generates absolutely continuous and $\sigma_s(t)$ generates singular (including the atomic component) measures with respect to the standard Lebesgue measure.

Due to Theorem 1.1 $\mu_\sigma(\ln(\sigma(U_{11})\cup\sigma(U_{33})))=0$, so

$$\mu_{\sigma_s}(\ln(\sigma(U_{11})\cup\sigma(U_{33})))=0\,. \tag{3.16}$$

Next, let $\{\widetilde{g}_j(t)\}_{j=1}^k$ be a set of unbounded elements generating the space $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$. In concordance with the notations given in [17] (formula (4.4)) (compare also with [18], formula (6.79)) we define

$$\iota_{jq}(t)=(\widetilde{g}_j(t),\widetilde{g}_q(t))_{\mathfrak{E}},\quad j,q=1,2,\dots,k,\quad G(t)=1+\sum_{j=1}^{k}\iota_{jj}(t). \tag{3.17}$$

For simplicity we assume that

$$1\notin\sigma(U_{11})\cup\sigma(U_{33}). \tag{3.18}$$

Condition (3.18) means that the operator $E_{2\pi}$ is correctly defined and projects all the space on $\mathfrak{H}_{\mathbb{T}}$.

Next, let numbers $0=\alpha_0<\alpha_1<\alpha_2<\dots<\alpha_m<\alpha_{m+1}=2\pi$ be such that $\{e^{i\alpha_j}\}_{j=1}^m=\sigma(U_{11})\cup\sigma(U_{33})$, let $\beta_j\in(\alpha_j,\alpha_{j+1})$ be some fixed numbers, $j=0,1,2,\dots,m$. Then for $t\in(\alpha_j,\alpha_{j+1})$ we put

$$\nu_c(t)=\int_{\beta_j}^{t}G(t)d\sigma_c(t),\ \nu_s(t)=\int_{\beta_j}^{t}G(t)d\sigma_s(t),\ \nu(t)=\nu_c(t)+\nu_s(t). \tag{3.19}$$

By the same way we introduce functions

$$\eta_c(t)=\int_0^t(1/G(t))d\sigma_c(t),\ \eta_s(t)=\int_0^t(1/G(t))d\sigma_s(t),\ \eta(t)=\eta_c(t)+\eta_s(t), \tag{3.20}$$

where $t\in[0,2\pi]$.

The space L^2_ν is by definition the natural closure of the set of all continuous functions given on $[0,2\pi]$ and vanishing on some neighborhood of the set $\{\alpha_j\}_{j=1}^m$. By the similar way the spaces $L^2_{\nu_c}$ and $L^2_{\nu_s}$ are introduced, the spaces L^2_η, $L^2_{\eta_c}$, $L^2_{\eta_s}$, etc, are defined as usual.

Note that the spaces L^∞_σ and L^2_ν, as well as the spaces L^1_σ and L^2_η, form compatible pairs or, the so called Banach pairs (for details see [3] or [9]). Therefore, the spaces $L^1_\sigma + L^2_\eta$ and $L^\infty_\sigma \cap L^2_\nu$ are well defined. In particular, the standard norm on $L^1_\sigma + L^2_\eta$ is given by the formula

$$\|f\| = \inf_{f_1+f_2=f} \{\|f_1\|_{L^1_\sigma} + \|f_2\|_{L^2_\eta}\}.$$

The space $L^\infty_\sigma \cap L^2_\nu$ can be considered as the adjoin one to the space $L^1_\sigma + L^2_\eta$ if the duality between these two spaces is given by the formula $(f(t), g(t)) = \int_0^{2\pi} f(t)\overline{g(t)}d\sigma(t)$, where $f(t) \in L^1_\sigma + L^2_\eta$ and $g(t) \in L^\infty_\sigma \cap L^2_\nu$.

An importance of above function spaces for our aims can be illustrated by the following manner (see [17], Proof of Proposition 4.1 for more details).

PROPOSITION 3.1. *Let the spaces* $\widetilde{L}^2_{\tilde{\sigma}}(\mathfrak{E})$ *and* $L^1_\sigma + L^2_\eta$ *be related by Definition 1.1 and Formulae (3.20). Then for any function* $\psi(t) \in L^1_\sigma + L^2_\eta$ *there are vector-functions* $g_{-1}(t), g_0(t), f_{-1}(t), f_0(t), \dots, f_k(t) \in L^2_{\tilde{\sigma}}(\mathfrak{E})$, *such that*

$$\psi(t) = (g_{-1}(t), f_{-1}(t))_{\mathfrak{E}} + (g_0(t), f_0(t))_{\mathfrak{E}} + \sum_{j=1}^{k} (\widetilde{g}_j(t), f_j(t))_{\mathfrak{E}}.$$

Below we will also use the symbol μX for the standard Lebesgue measure of a set X and the symbol L^1 for for the Lebesgue space of scalar complex valued absolutely integrable functions, given on the interval $[0, 2\pi]$, etc.

4. Constructing $\mathrm{Alg}_M U$: first steps

We study the structure of $\mathrm{Alg}\, U$ in two stages. In this section we consider the set of operators $M(U)$, where $M(\xi)$ is a polynomial such that $M(U)$ is completely defined by the basic model space of the operator U, and, next, some other functions with similar properties. The second stage starts in Section 5. For the beginning let us note that the spectrum of the operator U_{33} is a subset of the set Λ and therefore forms a finite set. Therefore $\mathfrak{G}_3$ is a Pontryagin space, it is a finite-dimensional space or can be presented as a π-orthogonal sum of two subspaces invariant with respect to U_{33} such that the first one is finite-dimensional and the other one is positive. Thus, U_{33} has the minimal polynomial $M_3(\xi)$: $M_3(U_{33}) = 0$. Next, let $M_0(\xi)$ $(\equiv M_1(\xi))$ be the minimal polynomial for the finite-dimensional operator U_{00} (or U_{11}), and $M_{\mathbb{T}'}(\xi)$ be the minimal polynomial for the operator $U|_{\mathfrak{H}_{\mathbb{T}'}}$. In this case for any polynomial $M(\xi)$ of the form

$$M(\xi) = M_0^2(\xi) M_3^2(\xi) M_{\mathbb{T}'}(\xi) Q(\xi), \tag{4.21}$$

where $Q(\xi)$ is an arbitrary polynomial, all matrix elements of $M(U)$ with respect to the decomposition

$$\mathfrak{H} = \mathfrak{G}_0 \oplus \mathfrak{G}_1 \oplus \mathfrak{G}_2 \oplus \mathfrak{G}_3 \oplus \mathfrak{H}_{\mathbb{T}'} \tag{4.22}$$

are equal to zero with exception, maybe, for ones corresponding to the mappings $\mathfrak{G}_0 \to \mathfrak{G}_1$, $\mathfrak{G}_0 \to \mathfrak{G}_2$, $\mathfrak{G}_2 \to \mathfrak{G}_1$, $\mathfrak{G}_2 \to \mathfrak{G}_2$. The exceptional elements $\mathfrak{G}_0 \to \mathfrak{G}_2$, $\mathfrak{G}_2 \to \mathfrak{G}_1$ and $\mathfrak{G}_2 \to \mathfrak{G}_2$ can be calculated via the basic model space of U, but this way does not work for $\mathfrak{G}_0 \to \mathfrak{G}_1$. Indeed, let W be an operator of similarity generated by $L^2_{\vec{\sigma}}(\mathfrak{E})$. Let us define (see (1.9))

$$h_j = W^{\uparrow}\widetilde{g}_j(\xi), \quad e_j = W\widetilde{g}_j(\xi), \quad j = 1,2,\ldots,k. \tag{4.23}$$

The equality $M_0(U_{00}) = 0$ yields $M_0(e^{it})\widetilde{g}(t) \in L^2_{\vec{\sigma}}(\mathfrak{E})$. Due to (1.1) $\overline{M_0(e^{it})M_3(e^{it})} = c \cdot e^{-ipt} \cdot M_0(e^{it})M_3(e^{it})$, where $|c| = 1$ and p is the degree of $M_0(\xi)M_3(\xi)$, hence (see (3.17) and (4.21)) the integrals $\int_0^{2\pi} M(e^{it})\iota_{jq}(t)d\sigma(t)$, $j,q = 1,2,\ldots,k$, converge but, generally speaking (see Example 1 from [19]),

$$(M(U)h_j, e_q) \neq \int_0^{2\pi} M(e^{it})\iota_{jq}(t)d\sigma(t), \quad j,q = 1,2,\ldots,k. \tag{4.24}$$

The latter problem can be resolved by the following way. Let us redefine (4.21)

$$M(\xi) = M_0^4(\xi)M_3^4(\xi)M_{\mathbb{T}'}(\xi)Q(\xi). \tag{4.25}$$

Then (see the justification of Formula (9) in [19])

$$(M(U)h_j, e_q) = \int_0^{2\pi} M(e^{it})\iota_{jq}(t)d\sigma(t), \quad j,q = 1,2,\ldots,k. \tag{4.26}$$

Everywhere below we put

$$\mathfrak{M} = \{M_0^4(\xi)Q(\xi)\},\ \mathfrak{M}' = \{M_0^4(\xi)M_3^4(\xi)M_{\mathbb{T}'}(\xi)Q(\xi)\}, \tag{4.27}$$

where $Q(\xi)$ runs throw the set of all polynomials. Using the equalities like (4.26) let us introduce values $\varphi(U)$ for some class of functions $\varphi(\xi)$. Note that we do not make the assumption that the condition $\varphi(U) \in \operatorname{Alg} U$ is satisfied. Thus, let the function $\varphi(\xi)$ be such that the function $\varphi(e^{it})$ is measurable, defined a.e., uniformly bounded on $[0,2\pi]$ and

$$\int_0^{2\pi} |\varphi(e^{it})|G(t)d\sigma(t) < \infty. \tag{4.28}$$

Let us introduce the operator $\overset{\diamond}{\varphi}(U)$ as it is described (see (1.9) and (4.26)):

$$\left.\begin{array}{l}
\bullet\ P^{\uparrow}\overset{\diamond}{\varphi}(U)|_{\widetilde{\mathfrak{H}}^{\uparrow}} = W^{\uparrow}\Phi(W^{\uparrow})^{-1}\text{, where } P^{\uparrow} \text{ is the ortoprojection on } \widetilde{\mathfrak{H}}^{\uparrow}\text{, and } \Phi \text{ is the multiplication operator by function } \varphi(e^{it}) \text{ acting on } \widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E});\\
\bullet\ \text{the subspace } \widetilde{\widetilde{\mathfrak{H}}} \text{ is invariant with respect to } \overset{\diamond}{\varphi}(U) \text{ and } \overset{\diamond}{\varphi}(U)|\widetilde{\widetilde{\mathfrak{H}}} = W\bar{\Phi}^{*}(W)^{-1}\text{, where } \bar{\Phi} \text{ is the multiplication operator by } \bar{\varphi}(e^{it});\\
\bullet\ (\overset{\diamond}{\varphi}(U)h_j, e_q) = \int_0^{2\pi} \varphi(e^{it})\iota_{jq}(t)d\sigma(t),\ j,q = 1,2,\ldots,k;\\
\bullet\ \text{all other elements of the matrix representation of the operator } \overset{\diamond}{\varphi}(U) \text{ with respect to the decomposition (4.22) are equal to zero.}
\end{array}\right\} \tag{4.29}$$

Due to (4.28) the operator $\overset{\diamond}{\varphi}(U)$ is well defined by Conditions (4.29). The answer to the natural question concerning the relation between $\overset{\diamond}{\varphi}(U)$ and $\mathrm{Alg}\, U$ is directly connected (see below) with some properties of σ_c.

Condition (4.28) is satisfied if $\varphi(\xi)$ is such that

$$\varphi(\xi) = \psi(\xi) M_0^4(\xi) M_3^4(\xi) M_{\mathbb{T}'}(\xi), \tag{4.30}$$

where $\psi(\xi)$ is a holomorphic on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \mathbb{T}$ function. The next proposition follows directly from Theorem 1.1 and comparison with Formulae (4.26) and (4.29).

PROPOSITION 4.1. *If a function $\varphi(\xi)$ has Form (4.30), then the corresponding operator $\overset{\diamond}{\varphi}(U)$ belongs to the closure in norm topology of the set of elements $M(U)$, where $M(\xi) \in \mathfrak{M}'$.*

Let $\mathrm{Alg}_M U$ be the weak closure of the operator set $\{M(U)\}$, where $M(\xi)$ runs throw the set $\mathfrak{M}'$. Due to Proposition 4.1 the relation $\overset{\diamond}{\varphi}(U) \in \mathrm{Alg}_M U$ is fulfilled for any function $\varphi(\xi)$ that have the representation (4.30).

THEOREM 4.1. *Let $X \subset \mathbb{T}$ be a set such that its pre-image $\ln^{-1}(X)$ is measurable both with respect to Lebesgue measure and μ_σ, moreover $\mu_\sigma(\ln^{-1}(X)) > 0$, $\mu(\ln^{-1}(X)) = 0$ and for the closure $\bar{X}$ the equality $\bar{X} \cap (\sigma(U_{11}) \cup \sigma(U_{33})) = \emptyset$ holds. Then $E(X) \in \mathrm{Alg}_M U$.*

Proof. As it is well known (see, for instance, [6], Theorem 1.18) for the measurable set X there is a subset X_c such that X_c is a finite or countable union of closed sets and $\mu_\sigma(X \setminus X_c) = 0$. Moreover, due to (1.2) $E(X)(\mathfrak{H})$ is a positive subspace and the inner product $[\cdot,\cdot]$ defines on $E(X)(\mathfrak{H})$ a structure of Hilbert space. Thus, $E(X) = E(X_c)$ and $E(X_c)$ is a strong limit of projections, where each projection corresponds to a closed subset. The latter means that it is enough to prove the theorem for the case of closed X. So, let us assume that X is closed. Then due to Lemma 2.1 there is a function $\varphi(\xi)$, such that for $\varphi(\xi)$ Conditions ((2.13)) hold with $\mathscr{E} = X$ and $\{\nu_j\}_1^m = \sigma(U_{11}) \cup \sigma(U_{33})$. Next, there is a natural number q_0 such that all functions from the sequence $\theta_l(\xi) = M_{\mathbb{T}'}(\xi)(\varphi(\xi))^{q_0+l}$, $l = 0, 1, 2, \ldots$ have Representation (4.30), hence the sequence $\{\overset{\diamond}{\theta}_l(U)\}_0^\infty$ is well defined. Moreover, the choice of the function $\varphi(\xi)$ is such that the sequence $\{\theta_l(\xi)\}_0^\infty$ is uniformly bounded on $\mathbb{T}$, everywhere on $\mathbb{T}$ converges to the function $\theta_X(\xi) = M_{\mathbb{T}'}(\xi)\chi_X(\xi)$, where $\chi_X(\xi)$ is the indicator of the set X. Moreover, if $\delta(X)$ is an arbitrary neighborhood X, then the sequence $\{(\varphi(\xi))^l\}_0^\infty$ uniformly converges to zero on $\mathbb{T} \setminus \delta(X)$. Thus, $\mathrm{w} - \lim_{l \to \infty} \overset{\diamond}{\theta}_l(U) = \overset{\diamond}{\theta}_X(U)$, where the operator $\overset{\diamond}{\theta}_X(U)$ is defined by $\tilde{n}$ (4.29). If X is a single-point set, then the theorem has been already because in this case the operator $\overset{\diamond}{\theta}_X(U)$ is up to a non-zero multiple a projection. In the general case the operator $E(X)$ can be obtained as the Riesz projection for the non-zero spectrum of the operator

$\overset{\diamond}{\theta}_X(U)$. Indeed, $X \subset \mathbb{T}$ is a closed set and its Lebesgue measure is equal to zero, so it is nowhere dense on $\mathbb{T}$ and the non-zero spectrum of $\overset{\diamond}{\theta}_X(U)$ can be surrounded by a simple contour. □

PROPOSITION 4.2. *If a polynomial $M(\xi)$ has Form (4.21) then $\overset{\diamond}{M}(U) \in Alg_M U$.*

Proof. If $\xi \in \mathbb{C}$ then the expression

$$\mathscr{L}_m(\xi) = \frac{1 - (\frac{m\xi}{m+1})^{m^2}}{1 - \frac{m\xi}{m+1}} \tag{4.31}$$

represents in fact a polynomial. If $\xi = e^{it}$, $t \in (0, 2\pi)$ then $\mathscr{L}_m(e^{it}) \to \frac{1}{1-e^{it}}$ for $m \to \infty$. Next,

$$|(1-e^{it})\mathscr{L}_m(e^{it})| = \left|1 - \left(\frac{me^{it}}{m+1}\right)^{m^2}\right| \left|1 - \frac{e^{it}}{(m+1-me^{it})}\right| \tag{4.32}$$

the expression $(1-e^{it})\mathscr{L}_m(e^{it})$ is bounded on $[0, 2\pi]$ y trends to 1 for all $t \neq 0, 2\pi$. Using this construction one can prove the existence of a sequence of polynomials $Q_m(\xi)$ such that

- $\lim_{m\to\infty} Q_m(e^{it})M_0^2(e^{it})M_3^2(e^{it}) = 1$ if $t \notin \sigma(U_{00}) \cup \sigma(U_{33})$;
- $\lim_{m\to\infty} Q_m(e^{it})M_0^2(e^{it})M_3^2(e^{it}) = 0$ if $t \in \sigma(U_{00}) \cup \sigma(U_{33})$;
- There is a constant $c > 0$ such that $|Q_m(e^{it})M_0^2(e^{it})M_3^2(e^{it})| \leqslant c$ for all $t \in [0, 2\pi]$ and $m = 1, 2, \ldots$.

Latter Conditions jointly with the basic model representation for U and Equality (3.16) show that

$$\mathrm{w} - \lim_{m\to\infty} M_0^4(U)M_3^4(U)Q_m(U) = M_0^2(U)M_3^2(U),$$

where $\mathrm{w} - \lim$ is a symbol of limit in the week operator topology. The rest is straightforward. □

Let us present $M_{\mathbb{T}'}(\xi)$ from (4.21) and (4.25) in the form

$$M_{\mathbb{T}'}(\xi) = M_{\underset{\mathbb{T}'}{\vartriangle}}(\xi) \cdot M_{\underset{\mathbb{T}'}{\triangledown}}(\xi), \tag{4.33}$$

where $M_{\underset{\mathbb{T}'}{\vartriangle}}(\xi)$ and $M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)$ are two polynomials with zero sets out of $\mathbb{D} \cup \mathbb{T}$ and within $\mathbb{D}$ respectively.

PROPOSITION 4.3. *If a polynomial $M(\xi)$ has the form*

$$M(\xi) = M_0^2(\xi)M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)Q(\xi), \tag{4.34}$$

then $\overset{\diamond}{M}(U) \in Alg_M U$.

Proof. One can removed first $M_3^2(U)$ by the same way as in the proof of Proposition 4.2 and next $M_{\underset{\mathbb{T}'}{\triangle}}(\xi)$ using the idea of Proposition 4.1 replacing $\psi(\xi)$ by $\frac{Q(\xi)}{M_{\underset{\mathbb{T}'}{\triangle}}(\xi)}$. □

In the next discursion the polynomials (4.34) will play a special role, so we denote

$$\mathfrak{K}=\{M(\xi):\,M(\xi)=M_0^2(\xi)Q(\xi)\},\,\mathfrak{L}=\{M(\xi):\,M(\xi)=M_0^2(\xi)M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)Q(\xi)\},\tag{4.35}$$

where $Q(\xi)$ runs throw the set of all polynomials. Now let us discuss on a functional space in which the the functional set corresponding to $\mathrm{Alg}_M(U)$ could be embedded. It does not mean that there is one-to-one correspondence between this set and $\mathrm{Alg}_M(U)$, but it is clear that for any operator from $\mathrm{Alg}_M(U)$ there exists the corresponding function. Thus, let $M(\xi)$ is defined as in (4.34) and the corresponding operator $\overset{\diamond}{M}(U)$ is described as in (4.29). Then, in particular, for $x,y\in\mathfrak{G}_2$ we have

$$[\overset{\diamond}{M}(U)x,y]=\int_0^{2\pi}M(e^{it})(f(t),g(t))_{\mathfrak{E}}d\sigma(t),\tag{4.36}$$

where (see Theorem 1.1) $f(t)=W^{-1}x,\,g(t)=W^{-1}y$. Then $(f(t),g(t))_{\mathfrak{E}}$ runs throw the whole space L^1_σ. So, one can consider Expression (4.36) as a continuous linear functional related with M and defined on L^1_σ. If we have a sequence $\{N_l(\xi)=M_0^2(\xi)P_{\underset{\mathbb{T}'}{\triangledown}}(\xi)Q_l(\xi)\}$ such that $\{[\overset{\diamond}{N_l}(U)x,y]\}$ converges for every $x,y\in\mathfrak{G}_2$ then due to Banach-Steinhaus Theorem the sequence $\{N_j(e^{it})\}$ converges in w*-topology to a function $\phi(t)\in L^\infty_\sigma$. On the other hand, for (see (4.22) and (4.23)) $h_j\in\mathfrak{G}_0$ and the same x the expression

$$[\overset{\diamond}{M}(U)x,h_j]=\int_0^{2\pi}M(e^{it})(f(t),\widetilde{g}_j(t))_{\mathfrak{E}}d\sigma(t),$$

can be consider as a linear continuous linear functional acting on $L^2_{\vec{\sigma}}(\mathfrak{E})$, so, taking the same polynomial sequence we get that $\overline{\phi(t)}\widetilde{g}_j(t)\in L^2_{\vec{\sigma}}(\mathfrak{E})$. Putting $j=1,2,\dots,k$ and taking into account that $\phi(t)\in L^\infty_\sigma$ we obtain (see (3.18) and(3.19)) $\phi(t)\in L^2_\nu$. So, for every operator $B\in\mathrm{Alg}_M U$ there is a continuous linear functional defined on $L^1_\sigma+L^2_\eta$, i.e. there is a function $\phi(t)\in L^2_\nu\cap L^\infty_\sigma$ that corresponds to the model representation of B or, in the terms of Definition 1.2, it is the portrait of B.

Note that, generally speaking, Condition (4.28) is not fulfilled for $\phi(t)\in L^2_\nu\cap L^\infty_\sigma$, because $G(t)\notin L^1_\sigma+L^2_\eta$ therefore (compare with (4.29)) we must explain a way to define the set of numbers $\{[Bh_j,e_q]\}_{j,q=1}^k$. Another important question is related with a description of the closure of the polynomial set $\mathfrak{L}$ as a linear manifold in the space $L^2_\nu\cap L^\infty_\sigma$. Let us denote this closure as $\mathfrak{B}$. Note that $\mathfrak{B}$ coincides with the set of portraits for the operators from $\mathrm{Alg}_M U$. The structure of $\mathfrak{B}$ strongly depends of the corresponding measure. Let us illustrate this statement.

Due to the definition of $\sigma_c(t)$ (see (3.15)) there is a Lebesgue integrable functions $\omega(t)\geqslant 0$ given e.e. in segment $[0,2\pi]$ and such that for every $t\in[0,2\pi]$

$$\sigma_c(t)=\int_0^t\omega(t)dt.\tag{4.37}$$

Below the symbol μX means the standard Lebesgue measure of a set X.

In [19] it was explicitly proved (see Theorem 5) that if

$$\mu\{t:\,\omega(t)=0\}>0, \tag{4.38}$$

then $\mathfrak{B}=L^2_\nu\cap L^\infty_\sigma$, and implicitly it was shown (see Theorem 2) that $\mathfrak{B}\neq L^2_\nu\cap L^\infty_\sigma$ if

$$\mu\{t:\,\omega(t)=0\}=0. \tag{4.39}$$

In the next section we present a complete description of $\mathfrak{B}$ for the case (4.39).

As a previous analysis let us consider the expression

$$\vartheta_{jq}(M)=\int_0^{2\pi}M(e^{it})\iota_{jq}(t)d\sigma(t) \tag{4.40}$$

with $M(e^{it})$ from (4.34). It can be treated as a linear functional well defined on $\mathfrak{L}$ and (maybe) unbounded with a dense domain in $\mathfrak{B}$. The unboundedness of at least one of these functionals takes place if $\mathfrak{B}=L^2_\nu\cap L^\infty_\sigma$, but if $\mathfrak{B}\neq L^2_\nu\cap L^\infty_\sigma$ the situation can be different. Let us given two examples.

EXAMPLE 4.1. Put $\widetilde{g}(t)=\frac{1}{e^{it}-1}$ and take the linear span of $\widetilde{g}(t)$, L^2 and a formal vector h as a Hilbert space $\mathfrak{H}$. In the latter space put $\|\widetilde{g}(t)\|=\|h\|=1$, $\widetilde{g}(t)\perp h\perp L^2$ and define a Hilbert structure on L^2 as usual. Next, introduce on $\mathfrak{H}$ a Pontryagin space structure using a fundamental symmetry J as follows: $J\widetilde{g}(t)=h$, $Jh=\widetilde{g}(t)$. Finally, put $U\widetilde{g}(t)=e^{it}\widetilde{g}(t)=1+\widetilde{g}(t)-\frac{1}{2}h$, $Uh=h$, $Ue^{-it}=1-h$, $Ue^{int}=e^{i(n+1)t}$ for $n=0,1,\pm2,\pm3,\ldots$. It is easy to check that U is a π-unitary operator, $\mathfrak{G}_1$ is a one-dimensional subspace spanned by h, $M_0(\xi)=\xi-1$ and $\iota_{11}(t)=\frac{-e^{it}}{(e^{it}-1)^2}$. Thus, in this case there is only one functional (4.40). The direct calculation brings $\int_0^{2\pi}e^{imt}(e^{it}-1)^2\iota_{11}(t)dt=0$ for all $m=0,1,\ldots$, therefore $\vartheta_{11}(M)\equiv 0$.

EXAMPLE 4.2. Here the construction is similar to one in Example 4.1, J acts under the same rules, but $\widetilde{g}(t)=\frac{1}{e^{it}-1}(-ln(2\sin\frac{t}{2})+i(\frac{t-\pi}{2}))$, $t\in(0,2\pi)$. Then $e^{it}\widetilde{g}(t)=\widetilde{g}(t)+\upsilon(t)$, where $\upsilon(t)=(-ln(2\sin\frac{t}{2})+i(\frac{t-\pi}{2}))$. Note that in L^2

$$\upsilon(t)=\sum_{p=1}^{\infty}\frac{e^{-ipt}}{p}, \tag{4.41}$$

so due to (1.5) $Ue^{int}=e^{i(n+1)t}$ for $n=-1,0,1,2,\ldots$ and $Ue^{-int}=e^{-i(n-1)t}-\frac{1}{n-1}h$ for $n=2,3\ldots$, $Uh=h$, moreover taking $A=0$ in (1.5)) we can put $U\widetilde{g}(t)=1+\widetilde{g}(t)-\frac{1}{2}\int_0^{2\pi}|\upsilon(t)|^2dt\cdot h$. Note that due to (4.41) we have the following representation

$$|\upsilon(t)|^2=\sum_{-\infty}^{+\infty}\alpha_l e^{ilt},\ \text{where } \alpha_{-l}=\alpha_l=\sum_{p=1}^{\infty}\frac{1}{p(p+l)},\ l=0,1,\ldots,$$

or, after a transformation,

$$\alpha_1=1,\quad \alpha_l=\frac{1}{l}\sum_{p=1}^{l}\frac{1}{p},\quad l=2,3,\ldots,\ \text{so}\ \sum_{l=0}^{+\infty}\alpha_l=\infty.$$

So, $\iota_{11}(t) = |\widetilde{g}(t)|^2 = \frac{-e^{it}}{(e^{it}-1)^2}|\upsilon(t)|^2$, so $G(T) = 1 + \frac{-e^{it}}{(e^{it}-1)^2}|\upsilon(t)|^2$. Let us consider the sequence $\{Q_m(e^{it}) = (e^{it}-1)^2\mathfrak{L}_m(e^{it})\}_1^\infty$, where $\mathfrak{L}_m(e^{it})$ is defined by (4.31). It easy to check (see (4.32)) that this sequence is bounded in $\mathfrak{B}$: $\int_0^{2\pi}|Q_m(e^{it})|^2 G(t)dt = \int_0^{2\pi}|(e^{it}-1)^2\mathfrak{L}_m^2(e^{it})||(e^{it}-1)^2 - e^{it}|\upsilon(t)|^2|dt$. Next,

$$\begin{aligned}\int_0^{2\pi} Q_m(e^{it})\iota_{11}(t)d\sigma(t) &= -\int_0^{2\pi} e^{it}\mathscr{L}_m(e^{it})|\upsilon(t)|^2 d\sigma(t)\\ &= -\int_0^{2\pi} e^{it}\sum_{p=0}^{m^2-1}\left(\frac{m}{m+1}\right)^p e^{ipt}|\upsilon(t)|^2 d\sigma(t)\\ &= -\sum_{p=0}^{m^2-1}\left(\frac{m}{m+1}\right)^p \alpha_{p+1} \to -\infty.\end{aligned}$$

Thus, the corresponding linear functional (4.40) is unbounded.

Generalizing a few the above cases let us consider a collection of functions $\{\upsilon_q(t)\}_{q=1}^m$ such that for every q: $M_0^2(e^{it})\upsilon_q(t) \in L^1_\sigma$ and let us consider the corresponding collection of linear functionals $\{\Upsilon_q\}_{q=1}^m$ defined (compare with (4.40)) on $\mathfrak{L}$:

$$\Upsilon_q(M) = \int_0^{2\pi} M(e^{it})\upsilon_q(t)d\sigma(t), \quad q = 1, 2, \ldots, m. \tag{4.42}$$

DEFINITION 4.1. The collection (4.42) is said to be linear independent modulo $\mathfrak{B}^*$ if its unique linear combination extendable on whole $\mathfrak{B}$ as a continuous linear functional is trivial (i.e. all its coefficients are equal to zero).

LEMMA 4.1. *Let functionals $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m$ be as in (4.42). Then they are linearly independent modulo $\mathfrak{B}^*$ if and only if for all $q = 1, 2, \ldots, m$*

$$\sup_{M\in\bigcap_{j=1, j\neq q}^m \operatorname{Ker}\Upsilon_j,\ \|M\|_{L^2_\nu\cap L^\infty_\sigma}=1} \{|\Upsilon_q(M)|\} = \infty.$$

The proof of Lemma 4.1 is omitted because it is directly follows from Hahn-Banach Theorem (see also the proof of Lemma 2.4 in [17]).

5. A concept of equivalence for some functionals

In [19] the following theorem was proved:

THEOREM 5.1. *Let $\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ be a basic model space for U. Then $U^{-1} \in \operatorname{Alg} U$ if and only if Condition (4.38) is fulfilled.*

Thus, for our aim we need to impose on basic model space of U Condition (4.39). Let us define an additional functional space. First, we put

$$\widehat{G}(\xi) = exp\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+\xi}{e^{it}-\xi}(\frac{1}{2}\ln((1+\sum_{j=1}^{k}\|\widetilde{g}_j(t)\|^2_{\mathfrak{E}})\omega(t)))dt\right]. \tag{5.43}$$

Then $H^2_{\widehat{G}}(\mathfrak{E})$ is by definition the set of holomorphic in $\mathbb{D}$ vector-valued functions $f(\xi)$ with the range in $\mathfrak{E}$, such that $f(\xi)\widehat{G}(\xi) \in H^2(\mathfrak{E})$, by the same way the spaces $L^2_{\widehat{G}}(\mathfrak{E})$ and $L^2_{1/\widehat{G}}(\mathfrak{E})$ are defined, i.e., for instance, $\phi(t) \in L^2_{1/\widehat{G}}(\mathfrak{E})$ if and only if $\phi(t)/\widehat{G}(e^{it}) \in L^2(\mathfrak{E})$.

Since the structure of $\mathfrak{B}^*$ is not yet discussed we try in a first step to compare unbounded functional using $L^1_\sigma + L^2_\eta$. Let us recall Representations (4.21) and (4.33).

DEFINITION 5.1. Two given on $[0,2\pi]$ μ_σ-measurable functions $\varphi(t)$ and $\psi(t)$ are called $\mathfrak{M}$-*equivalent* ($\overset{\mathfrak{M}}{\sim}$) if Condition

$$M_0^2(e^{it})\varphi(t),\, M_0^2(e^{it})\psi(t) \in L^1_\sigma \tag{5.44}$$

is fulfilled and for all polynomials $Q(\xi)$ Equality

$$\int_0^{2\pi} Q(e^{it})M_0^2(e^{it})\{\varphi(t)-\psi(t)\}d\sigma(t) = 0 \tag{5.45}$$

holds, and $\mathfrak{M}'$-*equivalent* ($\overset{\mathfrak{M}'}{\sim}$) if Condition (5.44) is fulfilled and Equality

$$\int_0^{2\pi} Q(e^{it})M_0^2(e^{it})M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\{\varphi(t)-\psi(t)\}d\sigma(t) = 0 \tag{5.46}$$

holds for all polynomials $Q(\xi)$.

REMARK 5.1. The notion of $\mathfrak{M}'$-equivalence lost sense if $\mathfrak{H}_{\mathbb{T}'} = \{0\}$, so in this case we put $M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it}) \equiv 1$. Below it will be proved that $\mathfrak{M}'$-equivalence can be reduced to $\mathfrak{M}$-equivalence in the case $\mathfrak{H}_{\mathbb{T}'} \neq \{0\}$ too. Note also that in Definition 3.15 we take into account Proposition 4.3.

The following lemma describes a special case of $\mathfrak{M}$-equivalence.

LEMMA 5.1. *Let $\omega(t) \equiv 1$ and $\varphi(t)$ be such that*

$$M_0^2(e^{it})\varphi(t) \in L^1. \tag{5.47}$$

Then one can find a function $\psi(t) \in L^1 + L^2_{1/G}$ such that $\varphi(t) \overset{\mathfrak{M}}{\sim} \psi(t)$ if and only if

$$\sup_{N(\xi)\in\mathfrak{X}} \{|\int_0^{2\pi}\varphi(t)N(e^{it})dt|\} < \infty, \tag{5.48}$$

where $\mathfrak{X}$ *is the set of all polynomials from* $\mathfrak{M}$ *such that*

$$\int_0^{2\pi} |\widehat{G}(e^{it})N(e^{it})|^2 dt \leqslant 1, \quad \max_{t\in[0;2\pi]}\{|N(e^{it})|\} \leqslant 1,$$

i.e. $\|N(e^{it})\|_{\mathbf{C}[0,2\pi]\cap L^2_{\widehat{G}}} \leqslant 1.$

Proof. First, let us prove the necessity of Condition (5.48). Thus, let as assume the existence of $\psi(t) \in L^1 + L^2_{1/G}$ such that $\varphi(t) \overset{\mathfrak{M}}{\sim} \psi(t)$. Then due to Definition 5.1 $\int_0^{2\pi} \varphi(t)N(e^{it})dt = \int_0^{2\pi} \psi(t)N(e^{it})dt$ for all $N(\xi) \in \mathfrak{X}$. The integral $\int_0^{2\pi} \psi(t)N(e^{it})dt$ represents a continuous linear functional on $\mathbf{C}[0,2\pi]\cap L^2_{\widehat{G}}$, so the necessity of Condition (5.48) is now evident and we go to its sufficiency. Thus, let (5.48) be fulfilled. Then one can consider the expression

$$\int_0^{2\pi} \varphi(t)N(e^{it})dt$$

as (see (4.35)) a continuous linear functional on $\mathfrak{K} \subset \mathbf{C}[0,2\pi]\cap L^2_{\widehat{G}}$. Due to Hahn-Banach Theorem and the structure of the space dual to $\mathbf{C}[0,2\pi]\cap L^2_{\widehat{G}}$ (see [3] or [9]) there are functions $\vartheta(t)$ and $\varsigma(t)$ such that $\vartheta(0) = 0$, $\vartheta(t)$ has on $[0,2\pi]$ a bounded variation, is continuous from the left and satisfies to the condition

$$\text{if } M_0(e^{it_0}) = 0 \text{ then } \vartheta(t_0+0) = \vartheta(t_0), \tag{5.49}$$

$\varsigma(t) \in L^2_{1/G}$ and for all entire non negative numbers m

$$\int_0^{2\pi} \varphi(t)M_0^2(e^{it})e^{imt}dt = \int_0^{2\pi} M_0^2(e^{it})e^{imt}d\vartheta(t) + \int_0^{2\pi} M_0^2(e^{it})e^{imt}\varsigma(t)dt.$$

Due to F. and M. Riesz Theorem on analytic measures (Garnett [7], Chapter II, Sect.3, Theorem II.3.8, Hoffman [8], Chapter 4, Page 47) we have that the following function of bounded variation

$$\int_0^t \varphi(t)M_0^2(e^{it})dt - \int_0^t M_0^2(e^{it})d\vartheta(t) - \int_0^t M_0^2(e^{it})\varsigma(t)dt$$

is absolutely continuous and analytic, i.e. there is a function $\zeta(\xi) \in H^1(\mathbb{C})$ with boundary values $\zeta(e^{it})$, $\zeta(0) = 0$ such that

$$\int_0^t \zeta(e^{it})dt = \int_0^t \varphi(t)M_0^2(e^{it})dt - \int_0^t M_0^2(e^{it})d\vartheta(t) - \int_0^t M^2(e^{it})\varsigma(t)dt.$$

The latter means that the function of bounded variation $\vartheta(t)$ is absolutely continuous for all segments $[\alpha,\beta] \subset [0,2\pi]$ such that $M(e^{it}) \neq 0$ for $t \in [\alpha,\beta]$. Due to Condition (5.49) it gives that $\vartheta(t)$ is absolutely continuous on the whole interval $[0,2\pi]$. Thus, there is a function $\tau(t) \in L^1$ such that $\vartheta(t) = \int_0^t \tau(t)dt$ and $\varphi(t) = \zeta(e^{it})M_0^{-2}(e^{it}) + \tau(t) + \varsigma(t)$. Moreover, for all $N(\xi) \in \mathfrak{K}$ we have

$$\int_0^{2\pi} N(e^{it})\zeta(e^{it})M_0^{-2}(e^{it})dt = \int_0^{2\pi} Q(e^{it})\zeta(e^{it})dt = 2\pi Q(0)\zeta(0) = 0,$$

so we can put $\psi(t) = \tau(t) + \varsigma(t)$ because in this case $\psi(t) \in L^1 + L^2_{1/G}$. □

DEFINITION 5.2. Let all functions of the collection $\{\varphi_j(t)\}_1^m$ satisfy Condition (5.44). This collection is called $\mathfrak{M}$*-linearly independent* ($\mathfrak{M}'$*-linearly independent*) *with respect to* $L^1_\sigma + L^2_\eta$, if for every non-trivial linear combination $\varphi(t)$ of these functions there is no one function $\psi(t) \in L^1_\sigma + L^2_\eta$ such that

$$\varphi(t) \overset{\mathfrak{M}}{\sim} \psi(t) \ (\varphi(t) \overset{\mathfrak{M}'}{\sim} \psi(t)).$$

The reference to the space $L^1_\sigma + L^2_\eta$ can be omitted.

PROPOSITION 5.1. *If* $\varphi(t) \overset{\mathfrak{M}}{\sim} \psi(t)$ *or* $\varphi(t) \overset{\mathfrak{M}'}{\sim} \psi(t)$ *then* $\mu_{\sigma_s}\{t\colon \varphi(t) \neq \psi(t)\} = 0$.

Proof. Due to F. and M. Riesz Theorem on analytic measures and the conditions of Proposition 5.1 the complex valued Borel measures $M_0^2(e^{it})\{\varphi(t) - \psi(t)\}d\sigma(t)$ and $M_0^2(e^{it})M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\{\varphi(t) - \psi(t)\}d\sigma(t)$ are absolutely continuous, so their singular parts

$$M_0^2(e^{it})\{\varphi(t) - \psi(t)\}d\sigma_s(t) \text{ and } M_0^2(e^{it})M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\{\varphi(t) - \psi(t)\}d\sigma_s(t)$$

must be equal to zero. Thanks to (3.16) the rest is straightforward. □

PROPOSITION 5.2. *Let Condition (4.38) be fulfilled for the measure* μ_σ. *If functions* $\varphi(t)$ *and* $\psi(t)$ *are* $\mathfrak{M}$*-equivalent or* $\mathfrak{M}'$*-equivalent then* $\mu_\sigma\{t\colon \varphi(t) \neq \psi(t)\} = 0$.

Proof. Due to Proposition 5.1 we can assume that $\sigma(t) = \sigma_c(t)$. Next, let $\vartheta(t)$ be an arbitrary function such that $\vartheta(t) \in L^1_\sigma$. Then thanks to Szegö Theorem (see [7], Chapter IV, Theorem 3.1 or [8], Chapter 4) and Condition (4.38)

$$\inf_P \int_0^{2\pi} |1 - e^{it}Q(e^{it})|^2 |\vartheta(t)| d\sigma(t) = 0,$$

where $Q(t)$ runs throw the set of all polynomials. Thus, there is a sequence of polynomials $\{Q_m\}_1^\infty$ such that $\lim_{m\to\infty} \int_0^{2\pi} |1 - e^{it}Q_m(e^{it})|^2 |\vartheta(t)| d\sigma(t) = 0$. Now let us assume that $\int_0^{2\pi} e^{ikt}\vartheta(t)d\sigma(t) = 0,\ k = 0, 1, 2, \dots$.

Then

$$\left|\int_0^{2\pi}(e^{-it}-Q_m(e^{it}))\vartheta(t)d\sigma(t)\right|\leqslant\int_0^{2\pi}|e^{-it}-Q_m(e^{it})||\vartheta(t)|d\sigma(t)$$
$$=\int_0^{2\pi}|1-e^{it}Q_m(e^{it})||\vartheta(t)|d\sigma(t)$$
$$\leqslant\left\{\int_0^{2\pi}|1-e^{it}Q_m(e^{it})|^2|\vartheta(t)|d\sigma(t)\right\}^{1/2}\left\{\int_0^{2\pi}|\vartheta(t)|d\sigma(t)\right\}^{1/2}\longrightarrow 0,$$

so $\int_0^{2\pi}e^{-it}\vartheta(t)d\sigma(t)=\int_0^{2\pi}(e^{-it}-Q_m(e^{it}))\vartheta(t)d\sigma(t)=0$. The same reasoning applying to the expression $\int_0^{2\pi}(e^{-2it}-e^{-it}Q_m(e^{it}))\vartheta(t)d\sigma(t)$ gives $\int_0^{2\pi}e^{-2it}\vartheta(t)d\sigma(t)=0$, etc. Thus, all Fourier coefficients of the measure $\vartheta(t)d\sigma(t)$ are equal to zero. As it is well known (see, for instance, [14], Chapter VII, Part 1, Item 5) a function in L^1 can be uniquely restored via Fejér sums, the latter brings (let us recall (4.37)) the equality $\vartheta(t)\omega(t)\equiv 0$. Now one can conclude the proof taking as $\vartheta(t)$ in the first step $M_0^2(e^{it})\{\varphi(t)-\psi(t)\}$ and $M_0^2(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})\{\varphi(t)-\psi(t)\}$ in the second one. □

The following proposition is a direct corollary of Proposition 5.2.

PROPOSITION 5.3. *Let μ_σ satisfy Condition (4.38) and functions from a system $\{\varphi_j(t)\}_1^m$ satisfy Condition (5.44). Then the latter system be $\mathfrak{M}$-linearly independent ($\mathfrak{M}'$-linearly independent) with respect to $L_\sigma^1+L_\eta^2$ if and only if it is linearly independent modulo $L_\sigma^1+L_\eta^2$, i.e.*

$$\textit{if}\ \sum_1^m\alpha_j\varphi_j(t)\in L_\sigma^1+L_\eta^2\ \textit{then}\ \alpha_1=\alpha_2=\ldots=\alpha_m=0.$$

LEMMA 5.2. *Let $\phi(t)$ satisfy Condition (5.44) and $\omega(t)$ satisfy Condition (4.39). Then $\phi(t)$ has an $\mathfrak{M}$-equivalent function $\psi(t)\in L_\sigma^1+L_\eta^2$ if and only if*

$$\sup_{N(\xi)\in\mathfrak{X}}\left\{\left|\int_0^{2\pi}N(e^{it})\phi(t)d\sigma(t)\right|\right\}<\infty, \tag{5.50}$$

where $\mathfrak{X}$ is the set of all polynomials $N(\xi)=M_0^2(\xi)Q(\xi)$ such that $\|N(e^{it})\|_{L_\sigma^\infty\cap L_\nu^2}\leqslant 1$.

Proof. The necessity of (5.51) has the same justification as the similar one in Lemma 5.1, so we go to its sufficiency. Denote

$$c=\sup_{N(\xi)\in\mathfrak{X}}\left\{\left|\int_0^{2\pi}N(e^{it})\phi(t)d\sigma(t)\right|\right\}. \tag{5.51}$$

Next, let X be a number set such that

$$X\subset[0,2\pi],\quad X\text{ is closed},\ \mu X=0,\quad \mu_{\sigma_s}X>0,\quad X\cap\sigma(U_{00})=\emptyset. \tag{5.52}$$

Let us choose an arbitrary continuous on $\{e^{it}\}_{t\in X}$ function $\theta_X(\xi)$ such that

$$\max_{t\in X}\{|\theta_X(e^{it})|\} \leqslant 1, \quad \int_X |\theta_X(e^{it})|^2 d\nu(t) \leqslant 1\,. \tag{5.53}$$

Due to Rudin-Carleson Theorem on a peak interpolation set (see [7], Part III, Section "Exercises and further results" or the original papers) there is a function $\theta(\xi)$ holomorphic on $\mathbb{D}$ and continuous on $\mathbb{D}\cup\mathbb{T}$ such that $\|\theta\|_{H^\infty} = \mathrm{Max}_{t\in X}\{|\theta_X(e^{it})|\}$. Note that, generally speaking, $\theta(e^{it}) \notin L^2_\nu(\mathbb{C})$ but $\theta(e^{it})\varphi^p(e^{it}) \in L^2_\nu(\mathbb{C})$ for a suitable natural p, $\varphi(\xi)$ was introduced in Lemma 2.1, $\{\nu_j\}_1^m = \sigma(U_{00})$, $\mathscr{E} = X$. Moreover, for every $\varepsilon > 0$ there is a natural number p_ε such that $\int_0^{2\pi} |\theta(e^{it})\varphi^p(e^{it})|^2 d\nu(t) \leqslant (1+\varepsilon)\int_X |\theta_X(e^{it})|^2 d\nu(t)$ for $p \geqslant p_\varepsilon$. Next, for sufficiently big p the function $\frac{\varphi^p(\xi)}{M_0(\xi)}$ is holomorphic on $\mathbb{D}$ and continuous (see (2.13), item c)) on $\mathbb{D}\cup\mathbb{T}$. Thus, the latter can be uniformly approximated by polynomials $Q(e^{it})$, so the functions $\varphi^p(e^{it})$ can be approximated by polynomials from the set $(1+\varepsilon)\mathfrak{X}$ mentioned in (5.51). Finally, the sequence $\{\theta(e^{it})\varphi^p(e^{it})\}$ has the pointwise limit equal to $\theta_X(e^{it})$ if $t\in X$ and 0 if $t \in [0,2\pi]\backslash X$. Taking into account all these steps one can attain the following estimation

$$\sup_{\theta_X\in\mathfrak{V}_X} \{|\int_X \theta_X(e^{it})\phi(e^{it})d\sigma(t)|\} < c, \tag{5.54}$$

where X is under Conditions (5.52), c is the constant defined by (5.51), $\mathfrak{V}_X$ is the set of all continuous on X functions subordinate to Conditions (5.53). The latter means that

$$\phi(e^{it})\chi_X(t) \in L^1_\sigma + L^2_\eta\,, \tag{5.55}$$

where, as usual, $\chi_X(t)$ denotes the indicator of the set X. Now let us split the segment $[0,2\pi]$ into two subsets $X^{(1)}$ and $X^{(2)}$ in such a way that separates absolutely continuous and singular parts of the measure: $X^{(1)}\cap X^{(2)} = \emptyset$, $\chi_{X^{(1)}}(t)d\sigma(t) = \omega(t)dt$, $\chi_{X^{(2)}}(t)d\sigma(t) = \omega_s(t)dt$. Then (5.55) yields $\chi_{X^{(2)}}(t)\phi(t) \in L^1_{\sigma_s} + L^2_{\eta_s}$, so after this observation we can assume $\sigma(t) = \sigma_c(t)$. Under this hypothesis and Condition (4.39) Condition (5.50) takes the form

$$\sup_{N(\xi)\in\mathfrak{X}_c} \{|\int_0^{2\pi} N(e^{it})\phi(t)\omega(t)dt|\} < \infty. \tag{5.56}$$

where $\mathfrak{X}_c$ denotes the set of all polynomials admitting the representation $N(\xi) = M_0^2(\xi)Q(\xi)$ such that $\|N(e^{it})\|_{\mathbf{C}[0,2\pi]\cap L^2_\nu} \leqslant 1$. The next steps that use Inequality (5.56) are similar to the corresponding ones applied during the proof of Lemma 5.1 so we give their short description only. Due to (5.56) one can consider the expression

$$\int_0^{2\pi} \phi(t)N(e^{it})d\sigma(t)$$

as a continuous linear functional on $\mathfrak{K} \subset \mathbf{C}[0,,2\pi]\cap L^2_{\nu_c}$. Thanks to Hahn-Banach Theorem and the structure of the space dual to $\mathbf{C}[0,2\pi]\cap L^2_{\nu_c}$ (see [3] or [9]) there are functions $\vartheta(t)$ and $\varsigma(t)$ such that $\vartheta(0)=0$, $\vartheta(t)$ has on $[0,2\pi]$ a bounded variation, is continuous from the left and satisfies to the condition

$$\text{if } M_0(e^{it_0})=0 \text{ then } \vartheta(t_0+0)=\vartheta(t_0), \tag{5.57}$$

$\varsigma(t)\in L^2_{\eta_c}$ and for every entire non negative number m

$$\int_0^{2\pi}\phi(t)M_0^2(e^{it})e^{imt}d\sigma(t)=\int_0^{2\pi}M_0^2(e^{it})e^{imt}d\vartheta(t)+\int_0^{2\pi}M_0^2(e^{it})e^{imt}\varsigma(t)d\sigma(t).$$

Due to F. and M. Riesz Theorem on analytic measures we have that the following function of bounded variation

$$\int_0^t\phi(t)M_0^2(e^{it})d\sigma(t)-\int_0^t M_0^2(e^{it})d\vartheta(t)-\int_0^t M_0^2(e^{it})\varsigma(t)d\sigma(t)$$

is absolutely continuous and analytic, i.e. there are function $\zeta(\xi)\in H^1(\mathbb{C})$ with boundary values $\zeta(e^{it})$, $\zeta(0)=0$ and $\tau(t)\in L^1$ such that

$$\zeta(e^{it})=\phi(t)M_0^2(e^{it})\omega(t)-M_0^2(e^{it})\tau(t)-M_0^2(e^{it})\varsigma(t)\omega(t),$$

i.e.

$$\phi(t)=\frac{\zeta(e^{it})}{\omega(t)M_0^2(e^{it})}+\frac{\tau(t)}{\omega(t)}+\varsigma(t).$$

Next, $\int_0^{2\pi}N(e^{it})\zeta(e^{it})M_0^{-2}(e^{it})\omega^{-1}(t)d\sigma(t)=\int_0^{2\pi}Q(e^{it})\zeta(e^{it})dt=2\pi Q(0)\zeta(0)=0$ for all $N(\xi)\in\mathfrak{K}$, so we can put $\psi(t)=\tau(t)\omega^{-1}(t)+\varsigma(t)$ because in this case $\psi(t)\in L^1_\sigma+L^2_\eta$ and the difference $\phi(t)-\psi(t)$ is "orthogonal" to $\mathfrak{K}$. □

LEMMA 5.3. *Let a function $\phi(t)$ satisfy Condition (5.44) and let the weight function $\omega(t)$ satisfy Condition (4.39). Then $\phi(t)\overset{\mathfrak{M}'}{\sim}\psi(t)$ for some $\psi(t)\in L^1_\sigma+L^2_\eta$ if and only if*

$$\sup_{N(\xi)\in\mathfrak{Y}}\left\{\left|\int_0^{2\pi}N(e^{it})\phi(t)d\sigma(t)\right|\right\}<\infty, \tag{5.58}$$

where $\mathfrak{Y}$ is the set of polynomials $N(\xi)$ with Representation (4.34) and such that $\|N(e^{it})\|_{L^\infty_\sigma\cap L^2_\nu}\leqslant 1$. Moreover, if Condition (5.58) holds then there is (maybe) another function $\psi_0(t))\in L^1_\sigma+L^2_\eta$ such that that $\phi(t)\overset{\mathfrak{M}}{\sim}\psi(t)$.

Proof. There are two positive constant a and b, $a\leqslant b$ such that $a\leqslant|M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})|\leqslant b$ for all $t\in[0,2\pi]$. Taking into account this fact it is easy to transform Condition (5.58) to a condition like (4.25)

$$\sup_{N(\xi)\in\mathfrak{X}}\left\{\left|\int_0^{2\pi}N(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})\phi(t)d\sigma(t)\right|\right\}<\infty, \tag{5.59}$$

where $\phi(t)$ is replaced by $M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\phi(t)$. Due to Lemma 5.2 the latter means that the necessity of Condition (5.58) was already proved, moreover, for the proof of the sufficiency of the same condition we can assume that $\sigma(t) = \sigma_c(t)$. Under this hypothesis Condition (5.59) takes the form

$$\sup_{N(\xi)\in\mathfrak{X}} \{|\int_0^{2\pi} N(e^{it}) M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\phi(t)\omega(t)dt|\} < \infty.$$

The latter and Lemma 5.2 bring $M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\phi(t) \overset{\mathfrak{M}}{\sim} \psi(t) \in L^1_\sigma + L^2_\eta$, so due to mentioned above F. and M. Riesz Theorem on analytic measures there is a function $\zeta(\xi) \in H^1(\mathbb{C})$ such that

$$M_0^2(e^{it})[M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\phi(t) - \psi(t)]\omega(t) = e^{it}\zeta(e^{it})$$

or

$$\phi(t) = \frac{\psi(t)}{M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})} + \frac{e^{it}\zeta(e^{it})}{M_0^2(e^{it}) M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\omega(t)} .$$

Note that $\frac{\psi(t)}{M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})} \in L^1_\sigma + L^2_\eta$, so we need to analyze the second summand only. The function $\zeta_0(\xi) = \frac{\zeta(\xi)}{M_0^2(\xi)}$ is holomorphic on $\mathbb{D}$ and the function $\zeta_1(\xi) = \frac{\zeta_0(\xi)}{M_{\overset{\triangledown}{\mathbb{T}'}}(\xi)}$ is meromorphic on $\mathbb{D}$ and has here finitely many pols. Let $K(\xi)$ be the Hermite-Birkhoff interpolant for $\zeta_0(\xi)$ with the set of interpolation points that coincides with the zero set of $M_{\overset{\triangledown}{\mathbb{T}'}}(\xi)$ taking into account the multiplicity of zeros. Due to this choice the function $\gamma(\xi) = \frac{\zeta_0(\xi) - K(\xi)}{M_{\overset{\triangledown}{\mathbb{T}'}}(\xi)}$ has within $\mathbb{D}$ removable singularities only. Then

$$\frac{e^{it}\zeta(e^{it})}{M_0^2(e^{it}) M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\omega(t)} = \frac{e^{it}}{\omega(t)}\gamma(e^{it}) + \frac{e^{it}K(e^{it})}{\omega(t) M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})}, \quad \frac{e^{it}K(e^{it})}{M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\omega(t)} \in L^1_\sigma ,$$

$$\frac{\gamma(e^{it})M_0^2(e^{it})}{\omega(t)} = \frac{\zeta(e^{it}) - K(e^{it})M_0^2(e^{it})}{M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\omega(t)} \in L^1_\sigma ,$$

and $\int_0^{2\pi} e^{it}\gamma(e^{it})\omega^{-1}(t)N(e^{it})d\sigma(t) = \int_0^{2\pi} e^{it}\gamma(e^{it})M_0^2(e^{it})Q(e^{it})dt = 0$ for all $N(\xi) \in \mathfrak{K}$. The rest is straightforward. □

Taken together, Lemma 4.1 and Lemma 5.3 yield the following result.

COROLLARY 5.1. *Let a function set $\{\upsilon_q(t)\}_{q=1}^m$ be such that $M_0^2(e^{it})\upsilon_q(t) \in L^1_\sigma$ for all q and let the corresponding set of linear functionals $\{\Upsilon_q\}_{q=1}^m$ be defined by* (4.42)*. This function set is $\mathfrak{M}'$-linear independent with respect to $L^1_\sigma + L^2_\eta$ if and only if for all $q = 1, 2, \ldots, m$:*

$$\sup_{M\in\bigcap_{j=1, j\neq q}^m \operatorname{Ker}\Upsilon_j,\ \|M\|_{L^2_\nu \cap L^\infty_\sigma}=1} \{|\Upsilon_q(M)|\} = \infty.$$

COROLLARY 5.2. *Let a function set* $\{\upsilon_q(t)\}_{q=1}^m$ *be such that* $M_0^2(e^{it})\upsilon_q(t)\in L^1_\sigma$ *for all* q. *Then this set is* $\mathfrak{M}'$*-linear independent with respect to* $L^1_\sigma+L^2_\eta$ *if and only if it is* $\mathfrak{M}$*-linear independent with respect to the same space.*

6. The case in which $\operatorname{Alg}U$ is not a $WJ^\star$-algebra

Let $\mathfrak{S}(U)$ be the collection of operators $\{S\}$ that can be presented (see Decomposition (4.22)) as

$$S=\begin{pmatrix} 0 & 0\,0\,0\,0\\ S_{10} & 0\,0\,0\,0\\ 0 & 0\,0\,0\,0\\ 0 & 0\,0\,0\,0\\ 0 & 0\,0\,0\,0\end{pmatrix}, \tag{6.60}$$

where $S_{01}\colon \mathfrak{G}_0\to\mathfrak{G}_1$ runs through all operators such that (compare with (4.26))

$$\text{if } \sum_{j,q=1}^{k}\alpha_{jq}\iota_{jq}(t)\overset{\mathfrak{M}'}{\sim}\psi(t)\in L^1_\sigma+L^2_\eta, \text{ then } \sum_{j,q=1}^{k}\alpha_{jq}(Sh_j,e_q)=0. \tag{6.61}$$

Note that if Condition (5.44) is fulfilled then due to Proposition 5.3 Condition (6.61) can be replaced by the following condition

$$\text{if } \sum_{j,q=1}^{k}\alpha_{jq}\iota_{jq}(t)\in L^1_\sigma+L^2_\eta, \text{ then } \sum_{j,q=1}^{k}\alpha_{jq}(S_{10}h_j,e_q)=0.$$

The latter was used for the definition of $\mathfrak{S}(U)$ in the paper [19] that was mainly concerned with the case (5.44).

Let $\varphi(t)$ be μ_σ-measurable bounded function defined on $[0,2\pi]$. Let us denote by $\mathscr{G}_\varphi(U)$ the set of operators $B\in\operatorname{Alg}U$ any of which has $\varphi(t)$ as a portrait (see Definition 1.2 and Formula (1.10)). It can occur that $\mathscr{G}_\varphi(U)=\emptyset$ for some $\varphi(t)$, but, for instance, $\mathscr{G}_\varphi(U)\neq\emptyset$ for $\varphi(t)\equiv 0$.

PROPOSITION 6.1. $\mathfrak{S}(U)=\mathscr{G}_0(U)\cap\operatorname{Alg}_M U$.

Proof. If Condition (4.38) is fulfilled then due to Proposition 5.3 the statement of Proposition 6.1 was proved in some previous papers (see, for instance, [17] and [19]). On the other hand a way that we apply here does not depend, roughly speaking, of Condition (4.38).

Let $S\in\operatorname{Alg}_M U$. Then for any collection of vectors $x_1,x_2,\dots,x_k,y,z\in\mathfrak{G}_2$ and the collection $h_1,h_2,\dots,h_k,e_1,e_2,\dots,e_k$ defined by (4.23) there is a sequence of polynomials $\{Q_l(\xi)\}_1^\infty$ such that

$$\left.\begin{aligned}
&\lim_{l\to\infty}(\overset{\diamond}{M_0^2}(U)M_{\underset{\mathbb{T}'}{\triangledown}}(U)Q_l(U)y,z)=(Sy,z),\\
&\lim_{l\to\infty}(\overset{\diamond}{M_0^2}(U)M_{\underset{\mathbb{T}'}{\triangledown}}(U)Q_l(U)h_j,x_j)=(Sh_j,x_j),\\
&\lim_{l\to\infty}(\overset{\diamond}{M_0^2}(U)M_{\underset{\mathbb{T}'}{\triangledown}}(U)Q_l(U)h_j,e_q)=(Sh_j,e_q),\ j,q=1,2,\dots,k.
\end{aligned}\right\} \tag{6.62}$$

Since $S \in \mathscr{G}_0(U)$ then

$$(Sy,z)=0,\ (Sh_j,x_j)=0,\ j=1,2,\dots,k. \tag{6.63}$$

As it was shown in Proposition 3.1) every function $\psi(t) \in L^1_\sigma + L^2_\eta$ accepts the representation

$$\int_0^{2\pi} Q_l(e^{it})M(e^{it})\psi(t)d\sigma(t) = (M(U)Q_l(U)y,z)+\sum_{j=1}^{k}(M(U)Q_l(U)h_j,x_j),$$

so (6.61), (6.62) and (6.63) yield $\mathscr{G}_0(U)\cap \mathrm{Alg}_M U \subset \mathfrak{S}(U)$. Now let $S\in\mathfrak{S}(U)$ and let Ξ be a subset of the set $\{(j,q)\}_{j,q=1}^k$ such that $\{\iota_{jq}(t)\}_{(j,q)\in\Xi}$ is a maximal linear $\mathfrak{M}'$-independent subset of the set $\{\iota_{jq}(t)\}_{j,q=1}^k$. Due to Corollary 5.1 for any $(j_0,q_0)\in\Xi$ there is a polynomial sequence $\{M_0^2(\xi)M_{\overset{\triangledown}{\mathbb{T}'}}(\xi)Q_l^{(j_0,q_0)}(\xi)\}_{l=1}^\infty$ such that

$$\left.\begin{aligned}
&\lim_{l\to\infty}\|M_0^2(e^{it})M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})Q_l^{(j_0,q_0)}(e^{it})\|_{L^2_\nu\cap L^\infty_\sigma}=0,\\
&\lim_{l\to\infty}(\overset{\diamond}{M_0^2}(U)M_{\overset{\triangledown}{\mathbb{T}'}}(U)Q_l^{(j_0,q_0)}(U)h_{j_0},e_{q_0})=(Sh_{j_0},e_{q_0}),\\
&\lim_{l\to\infty}(\overset{\diamond}{M_0^2}(U)M_{\overset{\triangledown}{\mathbb{T}'}}(U)Q_l^{(j_0,q_0)}(U)h_j,e_q)=0,\ (j,q)\in\Xi,\ (j,q)\neq(j_0,q_0).
\end{aligned}\right\}$$

The latter yields $S\in\mathscr{G}_0(U)\cap\mathrm{Alg}_M U$. □

In [19] the following result was obtained.

THEOREM 6.1. *If U is such that (4.38) is fulfilled, then $\mathscr{G}_\varphi(U)\cap\mathrm{Alg}_M U\neq\emptyset$ if and only if $\varphi(t)\in L^\infty_\sigma\cap L^2_\nu$; if B_0 is a fixed operator from $\mathscr{G}_\varphi(U)\cap\mathrm{Alg}_M U$, then $\mathscr{G}_\varphi(U)\cap\mathrm{Alg}_M U = \{B_0+S\}_{S\in\mathfrak{S}(U)}$.*

Before going further, let us make a simple observation concerning the space $H^\infty(\mathbb{C})$. It can be treated as a space of uniformly bounded holomorphic functions on $\mathbb{D}$. These functions have bonded values: $f(e^{it})=\lim_{r\to1-0}f(re^{it})$, the limit exists a.e. with respect to the standard Lebesgue measure, one can say also that $f(re^{it})$ converges to $f(e^{it})$ in the weak* topology on L^∞, see, for instance, [8], P.33. The latter gives a possibility to treat the space $H^\infty(\mathbb{C})$ as a subspace of L^∞. The analogous remark on the couple $H^2_{\widehat{G}}(\mathbb{C})$ and $L^2_{\nu_c}$ is valid too. The latter gives a possibility to consider the space $(H^2_{\widehat{G}}\cap H^\infty(\mathbb{C}))\dot{+}(L^\infty_{\sigma_s}\cap L^2_{\nu_s})$ as a subspace of $L^\infty_\sigma\cap L^2_\nu$.

THEOREM 6.2. *If U is such that (4.39) is fulfilled, then $\mathscr{G}_\varphi(U)\cap\mathrm{Alg}_M U\neq\emptyset$ if and only if $\varphi(t) = M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})\psi(t)$, where $\psi(t)\in(H^2_{\widehat{G}}\cap H^\infty(\mathbb{C}))\dot{+}(L^\infty_{\sigma_s}\cap L^2_{\nu_s})$; if B_0 is a fixed operator from $\mathscr{G}_\varphi(U)\cap\mathrm{Alg}_M U$, then $\mathscr{G}_\varphi(U)\cap\mathrm{Alg}_M U = \{B_0+S\}_{S\in\mathfrak{S}(U)}$.*

Proof. Due to Theorems 4.1, 5.1, 6.1 and Relation (5.55) we can assume $\sigma(t)=\sigma_c(t)$. If (4.39) is fulfilled and d is as in (1.7), then $\frac{e^{int}}{\sqrt{\omega(t)}}d\in L^2_{\widehat{\sigma}}(\mathscr{E})$, where $n=$

$0,\pm1,\pm2,\dots$. If $\varphi(e^{it})$ is a portrait of $B\in \mathrm{Alg}_M(U)$, then for every natural N there is a polynomial sequence $\{M_l(\xi)=M_0^2(\xi)M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)Q_l^{(N)}(\xi)\}_{l=1}^{\infty}$ such that

$$\left[BW\frac{e^{int}}{\sqrt{\omega(t)}}d,W\frac{1}{\sqrt{\omega(t)}}d\right]=\int_0^{2\pi}\varphi(e^{it})e^{int}dt=\lim_{l\to\infty}\int_0^{2\pi}M_l(e^{it})e^{int}dt\,,$$

where $n=0,\pm1,\pm2,\dots,\pm N$. This yields $\int_0^{2\pi}\varphi(e^{it})e^{int}dt=0$ for all $n=1,2,\dots$. So, there is the holomorphic in $\mathbb{D}$ function $\varphi(\xi)$ such that a. e. $\lim_{\rho\to1-0}\varphi(\rho e^{it})=\varphi(e^{it})$. Due to Proposition 3.1 $\varphi(e^{it})\in L^\infty_\sigma\cap L^2_\nu$, in particular, $\int_0^{2\pi}|\varphi(e^{it})|^2G(t)\omega(t)dt<\infty$. The latter thanks to $\varphi(e^{it})\in L^\infty_\sigma$ can be transformed into the condition $\int_0^{2\pi}|\varphi(e^{it})|^2(1+\sum_{j=1}^k\|\widetilde{g}_j(t)\|^2_{\mathfrak{E}})\omega(t))dt<\infty$. So (see (5.43)), $\varphi(\xi)\in(H^2_{\widehat{G}}\cap H^\infty(\mathbb{C}))$. Suppose now that ξ_0 is a zero of $M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)$ with a multiplicity m_0. Then $\frac{e^{it}}{(e^{it}-\xi_0)^j\sqrt{\omega(t)}}d\in L^2_{\vec{\sigma}}(\mathscr{E})$, $j=1,2,\dots,m_0$ and there is a polynomial sequence $\{M_l(\xi)=M_0^2(\xi)M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)Q_l(\xi)\}_{l=1}^{\infty}$ such that

$$\left[BW\frac{e^{it}}{(e^{it}-\xi_0)^j\sqrt{\omega(t)}}d,W\frac{1}{\sqrt{\omega(t)}}d\right]=\int_0^{2\pi}\frac{\varphi(e^{it})e^{it}}{(e^{it}-\xi_0)^j}dt=\lim_{l\to\infty}\int_0^{2\pi}\frac{M_l(e^{it})e^{it}}{(e^{it}-\xi_0)^j}dt\,,$$

where $j=1,2,\dots,m_0$. But for the same j

$$\varphi^{(j-1)}(\xi_0)=\frac{(j-1)!}{2\pi}\int_0^{2\pi}\frac{\varphi(e^{it})e^{it}}{(e^{it}-\xi_0)^j}dt \text{ and } \int_0^{2\pi}\frac{M_l(e^{it})e^{it}}{(e^{it}-\xi_0)^j}dt=0\,,$$

The latter yields $\frac{\varphi(\xi)}{M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)}\in(H^2_{\widehat{G}}\cap H^\infty(\mathbb{C}))$. This concludes the first part of the proof. Now we need to prove that if $\varphi(\xi)=\psi(\xi)M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)$, where

$$\psi(\xi)\in(H^2_{\widehat{G}}\cap H^\infty(\mathbb{C})), \tag{6.64}$$

then $\mathscr{G}_\varphi(U)\cap\mathrm{Alg}_M U\neq\emptyset$. As a previous step let us consider the function $\phi(\xi)=\psi(\xi)M_{\underset{\mathbb{T}'}{\triangledown}}(\xi)M_0^4(\xi)M_3^4(\xi)$. Let $C_n(\rho e^{it})$ be the Cesaro means of the Fourier series for $\psi(\rho e^{it})$. Condition $\psi(\xi)\in(H^2_{\widehat{G}}\cap H^\infty(\mathbb{C}))$ means that $C_n(\rho e^{it})$ is a polynomial of $\xi=\rho e^{it}$. Take an arbitrary collection of vectors $x_1,x_2,\dots,x_k,y,z\in\mathfrak{G}_2$ and the collection $h_1,h_2,\dots,h_k,e_1,e_2,\dots,e_k$ defined by (4.23). Put $f_1(t)=W^{-1}x_1, f_2(t)=W^{-1}x_2,\dots,f_k(t)=W^{-1}x_k$, $u(t)=W^{-1}y$, $v(t)=W^{-1}z$. Then

$$[C_n(U)M_{\underset{\mathbb{T}'}{\triangledown}}(U)M_0^4(U)M_3^4(U)y,z]$$

$$=\int_0^{2\pi}C_n(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})M_0^4(e^{it})M_3^4(e^{it})(u(t),v(t))_{\mathscr{E}}d\sigma(t),$$

$$[C_n(U)M_{\underset{\mathbb{T}'}{\triangledown}}(U)M_0^4(U)M_3^4(U)h_j,x_j]$$

$$=\int_0^{2\pi}C_n(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})M_0^4(e^{it})M_3^4(e^{it})(\widetilde{g}_j(t),f_j(t))_{\mathscr{E}}d\sigma(t),\ j=1,2,\dots,k,$$

$$[C_n(U)M_{\underset{\mathbb{T}'}{\triangledown}}(U)M_0^4(U)M_3^4(U)h_j,e_q]$$
$$= \int_0^{2\pi} C_n(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})M_0^4(e^{it})M_3^4(e^{it})\iota_{jq}(t)d\sigma(t),\ j,q=1,2,\ldots,k.$$

The functions $(u(t),v(t))_{\mathscr{E}}\omega(t), M_0(e^{it})(\widetilde{g}_j(t),f_j(t))_{\mathscr{E}}\omega(t), M_0^2(e^{it})\iota_{jq}(t)\omega(t)$ are absolutely integrable and, as is well known (see, for instance, [8], page 19), the Cesaro means $C_n(e^{it})$ converges to $\psi(e^{it})$ in weak-* topology. These yield

$$\overset{\diamond}{\phi}(U) = \mathrm{w}-\lim_{n\to\infty} C_n(U)M_{\underset{\mathbb{T}'}{\triangledown}}(U)M_0^4(U)M_3^4(U).$$

Now let us apply the scheme like in Proposition 4.2: using the polynomials (4.31) we construct a sequence of polynomials $Q_m(\xi)$ such that

- $\lim_{m\to\infty} Q_m(e^{it})M_0^4(e^{it})M_3^4(e^{it}) = 1$ if $t \notin \sigma(U_{00})\cup\sigma(U_{33})$;
- $\lim_{m\to\infty} Q_m(e^{it})M_0^4(e^{it})M_3^4(e^{it}) = 0$ if $t \in \sigma(U_{00})\cup\sigma(U_{33})$;
- There is a constant $c>0$ such that $|Q_m(e^{it})M_0^4(e^{it})M_3^4(e^{it})| \leqslant c$ for all $t\in[0,2\pi]$ and $m=1,2,\ldots$.

Now let us define the sequence of operators $\{\overset{\diamond}{\phi}_m(U)\}_{m=1}^{\infty}$:

$$\overset{\diamond}{\phi}_m(U) = \mathrm{w}-\lim_{n\to\infty} C_n(U)Q_m(U)M_{\underset{\mathbb{T}'}{\triangledown}}(U)M_0^4(U)M_3^4(U). \tag{6.65}$$

Then

$$[\overset{\diamond}{\phi}_m(U)y,z] = \int_0^{2\pi} \psi(e^{it})Q_m(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})M_0^4(e^{it})M_3^4(e^{it})(u(t),v(t))_{\mathscr{E}}d\sigma(t),$$

$$[\overset{\diamond}{\phi}_m(U)h_j,x_j]$$
$$= \int_0^{2\pi} \psi(e^{it})Q_m(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})M_0^4(e^{it})M_3^4(e^{it})(\widetilde{g}_j(t),f_j(t))_{\mathscr{E}}d\sigma(t),\ j=1,2,\ldots,k,$$

Due to Condition (6.64) the functions

$$(\psi(e^{it})u(t),v(t))_{\mathscr{E}}\omega(t) \text{ and } (\psi(e^{it})\widetilde{g}_j(t),f_j(t))_{\mathscr{E}}\omega(t)$$

are absolutely integrable, so,

$$\lim_{m\to\infty}[\overset{\diamond}{\phi}_m(U)y,z] = \int_0^{2\pi} \psi(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})(u(t),v(t))_{\mathscr{E}}d\sigma(t) \text{ and}$$

$$\lim_{m\to\infty}[\overset{\diamond}{\phi}_m(U)h_j,x_j] = \int_0^{2\pi} \psi(e^{it})M_{\underset{\mathbb{T}'}{\triangledown}}(e^{it})(\widetilde{g}_j(t),f_j(t))_{\mathscr{E}}d\sigma(t),\ j=1,2,\ldots,k.$$

On the other hand, the convergence of the sequences

$$\left\{[\overset{\diamond}{\phi}_m(U)h_j,e_q]\right\}_{m=1}^{\infty},\ j,q=1,2,\ldots,k.$$

cannot, generally speaking, be garanteed, therefore we need to modify (6.65). Acting as in Proposition 6.1 fix a subset Ξ of the set $\{(j,q)\}_{j,q=1}^{k}$ such that $\{\iota_{jq}(t)\}_{(j,q)\in\Xi}$ is a maximal linear $\mathfrak{M}'$-independent subset of the set $\{\iota_{jq}(t)\}_{j,q=1}^{k}$. Select an arbitrary set of numbers $\{b_{jq}\}_{(j,q)\in\Xi}$. Due to Corollary 5.1 for every $(j_0,q_0)\in\Xi$ there is a polynomial sequence $\{M_0^2(\xi)M_{\overset{\triangledown}{\mathbb{T}'}}(\xi)Q_m^{(j_0,q_0)}(\xi)\}_{m=1}^{\infty}$ such that

$$\left.\begin{aligned}
&\lim_{m\to\infty}\|M_0^2(e^{it})M_{\overset{\triangledown}{\mathbb{T}'}}(e^{it})Q_m^{(j_0,q_0)}(e^{it})\|_{L_\nu^2\cap L_\sigma^\infty}=0,\\
&\lim_{m\to\infty}\left(\{\overset{\diamond}{\phi}_m(U)+\overset{\diamond}{M_0^2}(U)M_{\overset{\triangledown}{\mathbb{T}'}}(U)Q_m^{(j_0,q_0)}(U)\}h_{j_0},e_{q_0}\right)=b_{j_0q_0},\\
&\lim_{m\to\infty}(\overset{\diamond}{M_0^2}(U)M_{\overset{\triangledown}{\mathbb{T}'}}(U)Q_m^{(j_0,q_0)}(U)h_j,e_q)=0,\ (j,q)\in\Xi,\ (j,q)\neq(j_0,q_0).
\end{aligned}\right\}$$

The rest is straightforward. □

THEOREM 6.3. *Let $B\in \operatorname{Alg} U$. Then $B=Q(U)+F$, where $Q(\xi)$ is a polynomial and $F\in \operatorname{Alg}_M U$.*

We omite the proof of this theorem because it can be realised on the base of Theorem 6.2 by the same way that was applied in [17] proving Theorem 4.23 on the base of Theorem 4.7.

Closing remarks

The results concerning a similar circle of problem for normal and, in particular, for unitary operators in Hilbert spaces are well known, see [12] for detail analysis. The proof of Lemma 2.1 (including Proposition 2.1) is presented here for the first time. The treatment of unbounded elements in Section 5 differs from one in [19] due to Condition (4.39) that yields $U^{-1}\notin \operatorname{Alg} U$. A similar remark is valid for the notions and results of Section 6.

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