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# TOPOLOGICAL PROPERTIES OF THE BLOCK NUMERICAL RANGE OF OPERATOR MATRICES

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*Abstract.* We show that the block numerical range of an $n\times n$-operator matrix $\mathscr{A}$ corresponding to an operator $A$ on the Banach space $X$ with respect to a decomposition $X=\prod X_j$ has at most $n$ connected components. We then characterize operator matrices with finite block numerical range. As an important tool we prove an inclusion theorem for the block numerical ranges of the principal submatrices of $\mathscr{A}$.

## 1. Introduction and notation

The *block numerical range* of a bounded linear operator $A$ on a Hilbert space with respect to an *orthogonal* decomposition was introduced by C. Tretter and M. Wagenhofer in [9] (see also [10, Chapter 1.11]) in order to obtain a better estimate for the spectrum of $A$ than the one given by the numerical range. Generalizing this work as well as [5] we developed an approach to block numerical ranges of operator matrices with respect to *arbitrary* decompositions of *arbitrary* Banach spaces in [7].

Our starting point in the present paper is the following result of M. Wagenhofer concerning orthogonal decompositions of Hilbert spaces:

Let $\mathscr{D}: H=H_1\times\cdots\times H_n$ be an orthogonal decomposition of the Hilbert space $H$ with unit sphere $S_H$, and let $A$ be a bounded linear operator on $H$. For each $\vec{u}=(u_1,\ldots,u_n)\in\prod_j S_{H_j}=:S_{\times H}$ we refer to the matrix $(Au_i|u_j)_{i,j}$ as $A_{\vec{u}}$. Moreover, $\sigma(A_{\vec{u}})$ denotes the spectrum of $A_{\vec{u}}$, and $n(\lambda)$ the algebraic multiplicity of $\lambda\in\sigma(A_{\vec{u}})$. In this particular case the block numerical range of $A$ is the set $V_{\mathscr{D}}(A)=\bigcup_{\vec{u}\in S_{\times H}}\sigma(A_{\vec{u}})$.

Wagenhofer's result reads as follows:

PROPOSITION 1.1. [11, Proposition 1.15] *Let $H$ and $A$ be as above.*

*The block numerical range $V_{\mathscr{D}}(A)$ consists of at most $n$ path-connected components $W_1,\ldots,W_s$. Moreover, the following equation holds for each $k\leqslant s$, $\vec{u},\vec{v}\in S_{\times H}$:*

$$\sum_{\lambda\in\sigma(A_{\vec{u}})\cap W_k} n(\lambda)=\sum_{\lambda'\in\sigma(A_{\vec{v}})\cap W_k} n(\lambda'). \tag{1}$$

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In the present paper we show in Section 2 that the block numerical range of an $n\times n$-operator matrix $\mathscr{A}$ corresponding to an operator $A$ on an arbitrary Banach space $X$ has at most $n$ *connected* components $C_1,\ldots,C_s$, and an equation analogous to (1) holds. So in particular the block numerical range is either uncountable or finite. Moreover, concerning path-connectedness we generalize Wagenhofer's proposition to arbitrary smoothly normed and arbitrary uniformly convex Banach spaces. The $L^p$-spaces $(1<p<\infty)$ belong to both of these classes (for details see e. g. [3]).

In Sections 4 and 5 we characterize operator matrices with finite block numerical range, thus generalizing a result for $2\times 2$ matrices corresponding to an *orthogonal* decomposition of a *finite-dimensional Hilbert space* ([8, Theorem 2.2]).

As a tool we prove an inclusion theorem concerning the block numerical ranges of principal submatrices of a given operator matrix in section 3 thereby generalizing [9, Theorem 3.1].

Finally, in the appendix we establish a result on the set of zeros of a connected set of polynomials which will be needed in Section 2. Moreover, we show that for some common Banach spaces $X$ the hypotheses of a restricted version of the generalisation of Wagenhofer's result hold even though their norm is neither smooth nor uniformly convex (see Proposition 2.6).

Let us point out that our results are new even in the case of finite dimensional spaces.

### 1.1. Decomposition of Banach spaces

Our starting point is a complex Banach space $(X,\|\cdot\|)$ which is decomposed into the direct product of $n\geqslant 1$ closed subspaces $X_1,\ldots,X_n$ of $X$: $X=\prod_1^n X_k$ and referred to as $\mathscr{D}$. The number $n=n(\mathscr{D})$ is called the *order* of $\mathscr{D}$. The canonical projection onto $X_j$ with kernel $\prod_{k\neq j}X_k$ is denoted $P_j$ $(j=1\ldots n)$.

The dual of the Banach space $X$ is $X'$. We set $S_{att}(X)=\{(x,\varphi):x\in X,\ \varphi\in X',\ \|x\|=\|\varphi\|=\varphi(x)=1\}$. To explain the notation $S_{att}$ we observe that a normalized pair $(x,\varphi)\in X\times X'$ is in $S_{att}(X)$ if $\varphi$ *attains* its norm on $x$, while $x$ considered as an element in the bidual $X''$ of $X$ *attains* its norm on $\varphi$. Moreover, we denote $S_{\mathscr{D}}=\prod_{i=1}^n S_{att}(X_i)$. To define a topology $\mathscr{T}_{att}$ on $S_{att}(X)$ we first equip $X$ with the norm topology and $X'$ with the weak$^*$-topology and then consider the product topology on $X\times X'$. Restricting this topology to $S_{att}(X)$ gives us $\mathscr{T}_{att}$. We take the product topology $\mathscr{T}_{\mathscr{D}}$ on $S_{\mathscr{D}}$, i. e. $(S_{\mathscr{D}},\mathscr{T}_{\mathscr{D}})=\prod_{i=1}^n(S_{att}(X_i),\mathscr{T}_{att})$.

For every Banach space $X$, $(S_{att}(X),\mathscr{T}_{att})$ is connected (see e. g. [1, p. 101, Theorem 4]), and so is the product $(S_{\mathscr{D}},\mathscr{T}_{\mathscr{D}})$ for every decomposition $\mathscr{D}$ of $X$.

OPEN PROBLEM. *Is $S_{att}(X)$ even* path-*connected for every Banach space $X$?* If this was true, then Proposition 2.4 below (a generalisation of Proposition 1.1 above) would hold for every Banach space.

### 1.2. The block numerical range of an operator

In the following, $\mathscr{L}(X)$ denotes the set of bounded linear operators on the complex Banach space $X$. Moreover we refer to the unit sphere of the Banach space $X$ as

$S_X = \{x \in X : \|x\| = 1\}$.

Let $X = \prod_1^n X_k$ be an arbitrary decomposition of $X$ into closed subspaces $X_k$, denoted by $\mathscr{D}$. For $A \in \mathscr{L}(X)$ and $i, j \in \{1, \ldots, n\}$ we define bounded linear operators $A_{ij} := P_i A P_j|_{X_j}$. Then $A$ can be written as an $n \times n$ operator matrix $\mathscr{A} = (A_{ij})_{i,j=1}^n$ such that for all $x = (x_1, \ldots, x_n) \in X$ the following equation holds:

$$Ax_k = AP_k x = \sum_{j=1}^n A_{jk} P_k x = \sum_{j=1}^n A_{jk} x_k.$$

For $d = ((u_1, \varphi_1), \ldots, (u_n, \varphi_n)) = \prod_{k=1}^n (u_k, \varphi_k) \in S_{\mathscr{D}}$ we define

$$B_d = (\varphi_i(A_{ij} u_j))_{i,j=1}^n = (\varphi_i(P_i A u_j))_{i,j=1}^n.$$

For details we refer to [7, Section 1.2].

The next lemma is obvious but useful in the remainder of the paper.

LEMMA 1.2. *The mapping $d \mapsto B_d$ from $S_{\mathscr{D}}(X)$ into the space $M_n(\mathbb{C})$ of all complex $n \times n$ matrices is continuous. In particular, the image $\mathscr{M}_{\mathscr{D}}$, say, is connected and even path-connected, whenever $S_{\mathscr{D}}(X)$ is path-connected.*

*Proof.* Recall that the mapping $(X, \mathscr{T}_{\|.\|}) \times (X', \mathscr{T}_{w^*}) \ni (x, \varphi) \mapsto \varphi(x)$ is jointly continuous on bounded sets. This fact implies the Lemma, since $S_{\mathscr{D}}(X)$ is connected (see Section 1.1). $\square$

For a bounded operator $T$ the set $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ not bijective}\}$ denotes its spectrum. The block numerical range is defined as follows:

DEFINITION 1.3. [5, p. 8], [7, Definition 1.1] Let $\mathscr{D}$ be a decomposition of the complex Banach space $X$ and let $A \in \mathscr{L}(X)$. The *block numerical range of $A$ with respect to $\mathscr{D}$* is the set

$$V_{\mathscr{D}}(A) := \bigcup_{d \in S_{\mathscr{D}}} \sigma(B_d) = \bigcup_{d \in S_{\mathscr{D}}} \{\lambda : \det(\lambda I - B_d) = 0\}. \tag{2}$$

Observe that for the trivial decomposition $X = X$, denoted by $\mathscr{D}_0$, the block numerical range $V_{\mathscr{D}_0}(A)$ is nothing else than the spatial numerical range (see e. g. [1]) which is connected by Lemma 1.2:

$$V(A) = \{\varphi(Au) : (u, \varphi) \in S_{att}(X)\}. \tag{3}$$

Moreover, if $H$ is a Hilbert space and $\mathscr{D}$ is an orthogonal decomposition of $H$, then $V_{\mathscr{D}}(A)$ coincides with the block numerical range introduced in [9] and already mentioned in the introduction.

Let us recall the following facts on the block numerical range: we have always $V_{\mathscr{D}}(A) \subset V_{\mathscr{D}'}(A') \subset \overline{V_{\mathscr{D}}(A)}$ ([7, Proposition 2.6]). It follows that

$$V_{\mathscr{D}'}(A') \subset V_{\mathscr{D}''}(A'') \subset \overline{V_{\mathscr{D}'}(A')} = \overline{V_{\mathscr{D}}(A)}, \tag{4}$$

where $\mathscr{D}'$, $\mathscr{D}''$ are the corresponding decompositions of $X'$, $X''$, respectively. The importance of the block numerical range lies among others in the relation $\sigma(A) \subset \overline{V_{\mathscr{D}}(A)}$ ([7, Theorem 2.8]).

## 2. Topological properties of the block numerical range

In this section we show that on arbitrary Banach spaces the block numerical range consists of at most $n$ connected components where $n$ is the order of the decomposition. Moreover, an equation analogue to (1) holds. Assuming then that the norm of the underlying Banach space is smooth, i. e. for each $x \in S_X$ there exists a unique $\varphi \in X'$, $\|\varphi\| = 1$, with $\varphi(x) = 1$, we even obtain a corresponding decomposition into path-connected components.

In the following, for each $B \in M_n(\mathbb{C})$ the number $n(z)$ refers to the algebraic multiplicity of $z \in \sigma(B)$.

LEMMA 2.1. *Let $\mathscr{M} \subset M_n(\mathbb{C})$ be connected and $W(\mathscr{M}) := \bigcup_{B\in\mathscr{M}} \sigma(B)$. Then the following holds.*

*(a) $W(\mathscr{M})$ consists of at most $n$ connected components $C_1,\dots,C_s$, and on each component $C_k$ the following equation holds:*

$$\sum_{z\in\sigma(B)\cap C_k} n(z) = \sum_{z'\in\sigma(C)\cap C_k} n(z') \text{ for all } B,C \in \mathscr{M}. \tag{5}$$

*(b)* ([11, Prop. 1.10 (3)]) *If $\mathscr{M}$ is even path-connected then $W(\mathscr{M})$ consists of at most $n$ path-connected components $C_k$ and equation (5) holds also for these components.*

*Proof.* (a) Let $\mathbb{P}_n$ be the set of all monic polynomials of degree $n$ equipped with the topology of uniform convergence on compacta. The mapping $\mathscr{M} \ni A \mapsto \det(z - A) \in \mathbb{P}_n$ is obviously continuous, and hence the image $\mathscr{P}$ of $\mathscr{M}$ is connected. So the assertion follows from Proposition 6.1 in the appendix.

(b) See the given reference. □

REMARKS.

(i) We point out that the proof of part (b) is quite different from that one of part (a), and none of the two assertions can be deduced from the other.

(ii) In general the connected components and the path-connected components are different as the following simple example shows:

$$\mathscr{M} = \left\{ \begin{pmatrix} t + i(\sin(\frac{2\pi}{t})) & 0 \\ 0 & i(2t-1) \end{pmatrix} : 0 < t \leqslant 1 \right\}.$$

The lemma above leads immediately to our first result.

PROPOSITION 2.2. *Let $X = \prod_{j=1}^n X_j$ be an arbitrary decomposition $\mathscr{D}$ of the Banach space $X$, and let $A \in \mathscr{L}(X)$ be arbitrary. Then the block numerical range $V_{\mathscr{D}}(A)$ consists of at most $n$ connected components $C_1,\dots,C_s$ ($s \leqslant n$). Moreover, on each component $C_k$ the following equation holds:*

$$\sum_{z\in\sigma(B_d)\cap C_k} n(z) = \sum_{z'\in\sigma(B_{d'})\cap C_k} n(z'),\ d,d' \in S_{\mathscr{D}}.$$

*Proof.* Apply Lemmata 1.2 and 2.1 (a) bearing equation (2) in mind. □

Now we direct our attention to the announced generalization of Proposition 1.1 resulting in path-connected components of $V_{\mathscr{D}}(A)$. To this end we need the following lemma:

LEMMA 2.3. *If the norm on $X$ is smooth then $(S_{att}(X), \mathscr{T}_{att})$ is path-connected.*

*Proof.* Since the norm is smooth, to every $x \in S_X$ there exists a unique $\varphi_x \in S_{X'}$ satisfying $\varphi_x(x) = 1$. The mapping $x \mapsto \varphi_x$ is continuous with respect to the norm topology on $S_X$ and the weak*-topology on $S_{X'}$ (for details see [3, Theorem 2.2]).

Let $(x, \varphi_x),\ (y, \varphi_y) \in S_{att}(X)$ be arbitrary.

(i) $y \neq -x$: Then $x + t(y-x) \neq 0$ for all $0 \leqslant t \leqslant 1$. Set $z(t) = \frac{x+t(y-x)}{\|x+t(y-x)\|}$. Then $[0,1] \ni t \mapsto v(t) = (z(t), \varphi_{z(t)})$ is a path from $(x, \varphi_x)$ to $(y, \varphi_y)$.

(ii) $y = -x$: Choose a third point $z \neq x, -x$, and apply (i) to $x, z$ and $z, -x$. □

PROPOSITION 2.4. *Let $X$ be a Banach space equipped with a smooth norm, and let $X = \prod_{j=1}^{n} X_j$ be an arbitrary decomposition $\mathscr{D}$ of $X$. Let $A \in \mathscr{L}(X)$ be arbitrary. Then the block numerical range $V_{\mathscr{D}}(A)$ consists of at most $n$ path-connected components $W_1, \dots, W_s$ ($s \leqslant n$). Moreover on each path-connected component $W_k$ the following equation holds:*

$$\sum_{z \in \sigma(B_d) \cap W_k} n(z) = \sum_{z' \in \sigma(B_{d'}) \cap W_k} n(z'),\ d, d' \in S_{\mathscr{D}}.$$

*Proof.* By Lemma 2.3 all $S_{att}(X_j)$ are path-connected, since the norm is smooth on the subspaces $X_j$. Hence $S_{\mathscr{D}}$ is path-connected, too. Now apply Lemmata 1.2 and 2.1 (b). □

COROLLARY 2.5. *Let $X$ be uniformly convex. Then the assertions of the proposition hold for every bounded operator $A$ and every decomposition $\mathscr{D}$ of $X$.*

*Proof.* The hypothesis on $X$ implies that $X$ is reflexive and that the norm on the dual space $X'$ is (uniformly) smooth (see e. g. [6, Propositions 1.e.2, 1.e.3]). Since $X$ is reflexive, we have $V_{\mathscr{D}}(A) = V_{\mathscr{D}'}(A')$ where $\mathscr{D}'$ is the decomposition of $X'$ induced by $\mathscr{D}$, (see equation (4)). □

The proof of Proposition 2.4 shows that its conclusion holds whenever $S_{\mathscr{D}}$ is path-connected. In the appendix we show, that $S_{att}(X)$ is path-connected for $X = C(K)$, $C_0(L)$, and $L^1(\Omega, \Sigma, \mu)$ even though the corresponding norms are not smooth. So we obtain the following proposition:

PROPOSITION 2.6. *Assume $X = \prod_j X_j$ where each $X_j$ is isometrically isomorphic to one of the spaces mentioned above. Then the assertions of Proposition 2.4 hold for every bounded linear operator $A$ on $X$.*

OPEN PROBLEM. *Let $X$ be a Banach space with smooth norm. Is it possible that $V_{\mathscr{D}}(A)$ is connected but not path-connected?* Note that $\mathscr{M}_{\mathscr{D}}$ is path-connected and $V_{\mathscr{D}}(A) = W(\mathscr{M}_{\mathscr{D}})$, cf. Remark (ii) after Lemma 2.1.

## 3. Inclusion

In this section we show that under a rather weak restriction on the decomposition the block numerical range of a principal operator submatrix is contained in the closure of the block numerical range. If the underlying Banach space is reflexive, the closure can be omitted.

In the remainder of this section, $X = X_1 \times \ldots \times X_n$ is a fixed decomposition $\mathscr{D}$ while $A \in \mathscr{L}(X)$ is arbitrary.

LEMMA 3.1. [5, Lemma 1.3] *Let $\pi$ be a permutation of $\{1,\ldots,n\}$. Abbreviate the decomposition $X_\pi = \prod X_{\pi(i)}$ by $\mathscr{D}_\pi$, and let $S_\pi : X \to X_\pi$ be given by $S_\pi(x_1,\ldots,x_n) = (x_{\pi(1)},\ldots,x_{\pi(n)}) =: x_\pi$. Moreover, set $\|x_\pi\| = \|S_\pi(x)\| = \|x\|$. Then $A_\pi := S_\pi A S_\pi^{-1} = (A_{\pi(i)\pi(k)})_{i,k=1,\ldots,n}$ and $V_{\mathscr{D}_\pi}(S_\pi A S_\pi^{-1}) = V_{\mathscr{D}}(A)$.*

*Proof.* Obvious. □

DEFINITION 3.2. The decomposition $\mathscr{D} : X = \prod_1^n X_j$ of order $n$ is called *regular* if $\dim(X_j) \geqslant n$ for all $j = 1,\ldots,n$.

THEOREM 3.3. (cf. [9, Theorem 3.1]) *Let $\mathscr{D}$ be regular. Let $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and let $P = \sum_1^k P_{i_j}$ be the projection of $X$ onto $X_{i_1} \times \ldots \times X_{i_k} =: Y$. Abbreviate the induced decomposition on $Y$ as $\mathscr{D}_Y$. Then $V_{\mathscr{D}_Y}(PAP|_Y) \subset \overline{V_{\mathscr{D}}(A)}$. If moreover $X$ is reflexive then $V_{\mathscr{D}_Y}(PAP|_Y) \subset V_{\mathscr{D}}(A)$.*

REMARK. Let $A = (A_{ik})_{i,k=1,\ldots,n}$ be an $n \times n$ (operator) matrix. A *principal submatrix of size $r$* is a submatrix of the form $\left(A_{i_k i_l}\right)_{k,l=1,\ldots,r}$ where $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$ holds. Using this notion the theorem asserts that *the block numerical range of every principal submatrix $C$ of $A$ is contained in $\overline{V_{\mathscr{D}}(A)}$ (in $V_{\mathscr{D}}(A)$ if $X$ is reflexive).*

Our proof is inspired by that one of [9, Theorem 3.1].

In the following we identify $X$ with its canonical image in $X''$, and $X'$ with its canonical image in $X'''$.

*Proof of Theorem 3.3.* On account of Lemma 3.1 we may assume $Y = X_{n-k+1} \times \cdots \times X_n$. Assume $k = n-1$ and let $\mu \in V_{\mathscr{D}_Y}(PAP|_Y)$ be arbitrary. Then there exists $d_Y = ((v_j,\psi_j))_{j=2}^n$ with $\mu \in \sigma(B_{d_Y})$. Since $\dim(X_1) \geqslant n$, there exists $\varphi \in X_1'$ of norm 1 satisfying $\varphi(A_{1k}v_k) = 0$ for $k = 2,\ldots,n$. To this $\varphi$ there exists $\xi \in X''$ ($\in X$ if $X$ is reflexive) such that $(\xi,\varphi) \in S_{att}(X'')$ We set $d'' = ((\xi,\varphi),(v_2,\psi_2),\ldots,(v_n,\psi_n))$ and obtain

$$B_{d''} = \begin{pmatrix} \varphi(A''_{11}\xi) & 0 & \cdots & 0 \\ \psi_2(A_{21}x) & \psi_2(A_{22}v_2) & \cdots & \psi_2(A_{2n}v_n) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_n(A_{n1}x) & \psi_n(A_{n2}v_2) & \cdots & \psi_n(A_{nn}v_n) \end{pmatrix},$$

from which $\mu \in \sigma(B_{d''}) \subset V_{\mathscr{D}''}(A'') \subset \overline{V_{\mathscr{D}}(A)}$ follows by equation (4).

The remainder of the proof follows by induction. □

COROLLARY 3.4. *Let $\mathscr{D}$ be regular. Then $V(A_{ii}) \subset \overline{V_{\mathscr{D}}(A)}$ ( $V_{\mathscr{D}}(A)$ in case $X$ is reflexive) for all $i$.*

## 4. Operators with finite block numerical range

By Proposition 2.2 the block numerical range is either uncountable or finite. In this section we characterize operators with finite block numerical range whenever the decomposition is *regular*.

DEFINITION 4.1. Let $\mathscr{D} : X = \prod_1^n X_n$ be a given decomposition of $X$, and let $A = (A_{ik})$ be the corresponding operator matrix of an operator $A \in \mathscr{L}(X)$. It is called *upper triangular* if $A_{ik} = 0$ for all $1 \leqslant k < i \leqslant n$. The matrix is called *p-similar to an upper triangular operator matrix* if there exists a permutation $\pi$ of the index set $N = \{1,\dots,n\}$ such that the operator matrix $\left(A_{\pi(i)\pi(k)}\right)$ corresponding to the decomposition $\mathscr{D}_\pi$ is upper triangular.

The set of eigenvalues of $A$ is denoted by $\sigma_p(A)$. Our main theorem reads as follows:

THEOREM 4.2. *Let $\mathscr{D} : X = \prod_1^n X_k$ be a regular decomposition with the canonical projections $P_j : X \mapsto X_j$, and for $A \in \mathscr{L}(X)$ let $\mathscr{A} = (A_{ik})$ be the corresponding representation of $A$ as an operator matrix. The following assertions are equivalent:*

*(a) The block numerical range $V_{\mathscr{D}}(A)$ is finite.*

*(b) $\mathscr{A}$ is p-similar to an upper triangular operator matrix and for all $i$ there exists $\lambda_i \in \mathbb{C}$ such that $A_{ii} = \lambda_i P_i$ holds.*

*If one of these assertions holds (and therefore the other), then $V_{\mathscr{D}}(A) = \sigma(A) = \sigma_p(A) = \{\lambda_1,\dots,\lambda_n\}$.*

The proof will be given in the next section.

REMARKS.

(i) The theorem shows that $\sigma(A) \neq V_{\mathscr{D}}(A)$ implies that $V_{\mathscr{D}}(A)$ is uncountable, even if $X$ is finite-dimensional.

(ii) The theorem does not hold for decompositions which are not regular. A counterexample may be found in [8].

## 5. The proof of Theorem 4.2

### 5.1. Auxiliary results

The following characterisation of operator matrices which are p-similar to upper triangular operator matrices is a direct consequence *of the proof* of the result of R. Brualdi [2, Theorem 1.4.2] restricted to the class of $(0,1)$-matrices the diagonal of which consists completely of ones. It says the following[1]:

*Let $J$ be an $n \times n$-matrix of $0$ and $1$ and assume that $J_{ii} = 1$ for $i = 1, \dots, n$. Then its permanent* $\mathrm{perm}(J) := \sum_{\pi \text{permutation}} \prod_i J_{i\pi(i)}$ *equals $1$ if and only if there exists a permutation $\pi$ of the set $N = \{1, \dots, n\}$ such that the matrix $(J_{\pi(i)\pi(j)})_{i,j \in N}$ is lower triangular.*

Consider the permutation $\sigma = \begin{pmatrix} 1, & 2, & \dots, n \\ n, & n-1, & \dots, 1 \end{pmatrix}$. Then $\sigma \circ \pi =: \rho$ transforms $J$ into an upper triangular matrix $J_\rho$.

PROPOSITION 5.1. *The operator matrix $\mathscr{A} = (A_{ik})$ is p-similar to an upper triangular operator matrix if (and only if) to every permutation $\pi \neq id$ there exists $k$ such that $k \neq \pi(k)$ and $A_{k\pi(k)} = 0$.*

*Proof.* We consider the incidence matrix $J$ corresponding to $\mathscr{A}$, which is defined by $J_{ik} = \begin{cases} 1, & A_{ik} \neq 0, \text{ or } i = k \\ 0, & \text{else.} \end{cases}$ By hypothesis its permanent $\mathrm{perm}(J)$ satisfies $\mathrm{perm}(J) = 1$. So it is p-similar to an upper triangular matrix $J_\rho$ by the previous paragraph. But then the same is true for $(A_{\rho(i)\rho(k)})$. The remainder is obvious. □

LEMMA 5.2. *Let $X, Y, Z$ be Banach spaces and let $S : X \mapsto Y$, $T : Y \mapsto Z$ be two bounded non-zero linear operators. Then there exists a pair $(u, \varphi) \in S_{att}(Y)$ such that $S'\varphi \neq 0$ as well as $Tu \neq 0$.*

*Proof.* By hypothesis there exists $v \in X$ such that $Sv \neq 0$. Set $x = \frac{Sv}{\|Sv\|}$. Then there exists $\psi \in Y'$ such that $(x, \psi) \in S_{att}(Y)$.

(I) If $Tx \neq 0$ then $(x, \psi)$ is the desired pair.

(II) Let $Tx = 0$. By hypothesis there exists $z \in S_Y$ such that $Tz \neq 0$. $x$ and $z$ are linearly independent. We set $w_\lambda = \frac{\lambda x + z}{\|\lambda x + z\|}$ for $\lambda \in \mathbb{C}$. To every $\lambda$ there exists $w'_\lambda$ such that $(w_\lambda, w'_\lambda) \in S_{att}(Y)$. Then $Tw_\lambda = \frac{Tz}{\|\lambda x + z\|} \neq 0$, and we have only to show that there exists $\lambda$ such that $S'w'_\lambda \neq 0$ holds. Assuming the contrary we obtain from $x = \|x + z/\lambda\| \cdot w_\lambda - z/\lambda$ for $\lambda > 0$

$$\begin{aligned} 0 &= S'w'_\lambda(v) = w'_\lambda(Sv) = \|Sv\| w'_\lambda(x) \\ &= \|Sv\|(\|x + z/\lambda\| \cdot 1 - w'_\lambda(z)/\lambda) \to \|Sv\| \text{ for } \lambda \to \infty, \end{aligned}$$

a contradiction. □

[1] In the proof mentioned above on [2, p. 18] the matrix in question is $A'$.

COROLLARY 5.3. *Let $Y$, $Z$ be Banach spaces, $0 \neq v \in Y$, and $0 \neq T \in \mathscr{L}(Y,Z)$. Then there exists $(u,\varphi) \in S_{att}(Y)$ such that $\varphi(v) \neq 0$, and $0 \neq Tu$.*

*Proof.* Apply the lemma for $X = \mathbb{C}$ and $S(\lambda) = \lambda \cdot v$. $\square$

LEMMA 5.4. *Let $X_1,\dots,X_n$ be arbitrary Banach spaces. For $k = 1,\dots,n-1$ let $T_k$ be a bounded linear mapping from $X_k$ to $X_{k+1}$, and let $T_n$ be a bounded linear mapping from $X_n$ into $X_1$. If $T_k \neq 0$ holds for all $k$ then there exist pairs $(u_k,\varphi_k) \in S_{att}(X_k)$ such that*

$$\varphi_1(T_n u_n) \cdot \prod_{k=1}^{n-1} \varphi_{k+1}(T_k u_k) \neq 0.$$

*Proof.* (I) We apply Lemma 5.2 to $X = X_n$, $Y = X_1$, and $Z = X_2$. There exists $(u_1,\varphi_1) \in S_{att}(X_1)$ such that $T_1 u_1 \neq 0$, and $T_n' \varphi_1 \neq 0$.

(II) By Corollary 5.3 there exists $(u_2,\varphi_2) \in S_{att}(X_2)$ such that $\varphi_2(T_1 u_1) \neq 0$, as well as $T_2 u_2 \neq 0$. Repeating this construction up to $k = n-1$ we obtain $(u_2,\varphi_2),\dots(u_{n-1},\varphi_{n-1})$ satisfying $\varphi_k(T_{k-1}u_{k-1}) \neq 0$, and $T_k u_k \neq 0$, for $k = 2,\dots,n-1$.

(III) We apply Corollary 5.3 again to $Y = X_n$, $v = T_{n-1}u_{n-1}$, and $Z = \mathbb{C}$, $T = T_n'\varphi_1$. We get $(u_n,\varphi_n) \in S_{att}(X_n)$ satisfying

$$\varphi_n(v) \neq 0 \text{ and } 0 \neq Tu_n = (T_n'\varphi_1)(u_n) = \varphi_1(T_n u_n).$$

The proof is complete. $\square$

The next lemma should be known, but since we could not find any reference for it, we include a proof:

LEMMA 5.5. *Let $A \in \mathscr{L}(X)$ be arbitrary. If its numerical range $V(A)$ is finite then it consists of only one element $\lambda$ and $A = \lambda I$ holds.*

*Proof.* Since $V(A)$ is connected (see equation (3)), it consists of only one element $\lambda$, say. Assume first of all that $\lambda = 0$. Then the numerical radius $v(A) = 0$, hence $A = 0$ on account of the Bohnenblust-Karlin inequality $\|A\| \leqslant \exp(1)v(A)$ , see [1, p. 7]. If $\lambda \neq 0$ consider the operator $A - \lambda I = B$. Then $V(B) = \{0\}$, and the assertion follows. $\square$

LEMMA 5.6. *Let $\mathscr{D} : X = \prod_{j=1}^{n} X_j$ be a regular decomposition with the canonical projections $P_i : X \mapsto X_i$, and let $\mathscr{A} = (A_{ik})$ be the corresponding block operator matrix. Suppose that $V_{\mathscr{D}}(A)$ is finite. Then the following assertions hold:*

*(a) To every $i \in \{1,\dots,n\}$ there exists $\lambda_i$ such that $A_{ii} = \lambda_i P_i$.*

*(b) The function $\mathbb{C} \times S_{\mathscr{D}} \ni (\lambda,d) \mapsto \det(\lambda - B_d) =: Q_d(\lambda)$ is constant with respect to $d$; in other words all characteristic polynomials are equal.*

*Proof.* By Theorem 3.3 $V(A_{ii}) \subset \overline{V_{\mathscr{D}}(A)}$. Since $V(A_{ii})$ is connected (see equation (3)), it consists of one element. Lemma 5.5 yields the first assertion. Let $Q_d(\lambda) = \sum_{k=0}^{n} a_k(d)\lambda^k$. Let $k$ be arbitrary. $d \mapsto a_k(d)$ is a polynomial in the elements of $B_d$ hence continuous. On the other hand $a_k(d)$ is the $k$-th elementary symmetric function of the roots of $Q_d(\lambda)$ lying in $V_{\mathscr{D}}(A)$. Hence the range of $d \mapsto a_k(d)$ is finite. Since $S_{\mathscr{D}}$ is connected, $d \mapsto a_k(d)$ is constant. The assertion follows. □

COROLLARY 5.7. *Let $V_{\mathscr{D}}(A)$ be finite. Then we have:*

*(a)* $\sigma(B_d) = \sigma(B_{d'}) = V_{\mathscr{D}}(A)$ *for all* $d, d' \in S_{\mathscr{D}}$.

*(b) Each $\lambda_i$ from Lemma 5.6 (a) is an eigenvalue of all $B_d$.*

*Proof.* Use equation (2), Lemma 5.6 (a), and Theorem 3.3. □

LEMMA 5.8. *Let $\mathscr{D} : X = X_1 \times X_2$ be a regular decomposition, let $A \in \mathscr{L}(X)$ be arbitrary, and let $\mathscr{A} = (A_{ik})$ be the corresponding operator matrix. If $V_{\mathscr{D}}(A)$ is finite then $\mathscr{A}$ is p-similar to an upper triangular operator matrix and $V_{\mathscr{D}}(A) = \sigma(A) = \sigma_p(A)$ consists of at most 2 elements.*

*Proof.* Let $P_j$ be the projection from $X$ onto $X_j$ with kernel $X_k$ $(k \neq j)$. By Lemma 5.6 we have $A_{ii} = \lambda_i P_i$. Let now $d = ((u_1, \varphi_1), (u_2, \varphi_2)) \in S_{\mathscr{D}}$ be arbitrary. Then the associated matrix

$$B_d = (\varphi_i(A_{ij}u_j)) = \begin{pmatrix} \lambda_1 & \varphi_1(A_{12}u_2) \\ \varphi_2(A_{21}u_1) & \lambda_2 \end{pmatrix}$$

possesses the characteristic polynomial $Q_d(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) - \varphi_1(A_{12}u_2) \cdot \varphi_2(A_{21}u_1)$. By Corollary 5.7 $\lambda_j$ is a root of $Q_d(\lambda)$ for $j = 1, 2$, hence $\varphi_1(A_{12}u_2) \cdot \varphi_2(A_{21}u_1) = 0$. Since $d$ was arbitrary, Lemma 5.4 implies that $A_{12} = 0$ or $A_{21} = 0$ and the assertion follows from Proposition 5.1. □

## 5.2. Proof of Theorem 4.2

The implication (b) ⇒ (a) is obvious.

(a) ⇒ (b): *Proof by induction on $n \geqslant 2$:*

(I) The implication holds for $n = 2$ by Lemma 5.8.

(II) Suppose it holds for all regular decompositions of order $n-1$, $(n \geqslant 3)$.

*Claim:* Theorem 4.2 holds for all regular decompositions of order $n$.

*Proof.* We set $N = \{1, \dots, n\}$.

(i) By Lemma 5.6 $A_{ii} = \lambda_i P_i$ for all $1 \leqslant i \leqslant n$ where $\lambda_i \in \sigma_p(A)$.

(ii) Let $Y = \prod_{i=1}^{n-1} X_i$ and $A_{n-1} = (A_{ik})_{i,k=1,\dots,n-1}$. Theorem 3.3 implies that $V_{\mathscr{D}_Y}(A_{n-1}) \subset \overline{V_{\mathscr{D}}(A)} = V_{\mathscr{D}}(A)$. By the induction hypothesis $A_{n-1}$ is p-similar to an upper triangular operator matrix, that means: there exists a permutation $\tilde{\sigma}$ of the set $N \setminus \{n\}$ such that the matrix $(A_{\tilde{\sigma}(i)\tilde{\sigma}(k)})_{i,k \leqslant n-1}$ satisfies

$$A_{\tilde{\sigma}(i)\tilde{\sigma}(k)} = 0 \text{ for } 1 \leqslant \tilde{\sigma}(k) < \tilde{\sigma}(i) \leqslant n-1. \tag{6}$$

Setting $\sigma(k) = \tilde{\sigma}(k)$ for $k < n$ and $\sigma(n) = n$ the matrix $S_\sigma \mathscr{A} S_\sigma^{-1}$ satisfies equation (6), too (where $\tilde{\sigma}$ is replaced by $\sigma$). Hence, we may assume without loss of generality that

$$A_{ik} = 0 \text{ holds for all } i,\ k \text{ satisfying } 1 \leqslant k < i \leqslant n-1. \tag{7}$$

(iii) Let $id \neq \pi$ be an arbitrary permutation of the set $N$. By Proposition 5.1 it is enough to show that there exists $j$ such that $j \neq \pi(j)$ as well as $A_{j\pi(j)} = 0$. Assume first that

$$\pi \neq \rho := \begin{pmatrix} 1\ 2\ 3\ \dots\ n \\ 2\ 3\ 4\ \dots\ 1 \end{pmatrix}.$$

Then there exists $k \leqslant n-1$ satisfying $\pi(k) \leqslant k$ If $\pi(k) = k$ then the induced permutation $\tilde{\pi}$ on $N \setminus \{k\} =: N_k$ is not the identity. Consider the principal submatrix $\mathscr{A}_k = (A_{jl})_{j,l \neq k}$. By the induction hypothesis there exists $j \in N_k$ with $\tilde{\pi}(j) \neq j$ as well as $A_{j\tilde{\pi}(j)} = 0 = A_{j\pi(j)}$.

Now assume that $\pi(k) < k \leqslant n-1$ holds. But then $A_{k\pi(k)} = 0$ by equation (7).

(iv) So it remains to prove that to $\rho$ there exists $j$ satisfying $A_{j\rho(j)} = 0$. Let $d = ((u_k, \varphi_k))_k$ be arbitrary. Then the characteristic polynomial $Q_d(\lambda)$ satisfies

$$Q_d(\lambda) = \det(\lambda - B_d) = \prod_{j=1}^{n} (\lambda - \lambda_j) + \operatorname{sign}(\rho) \prod_{j=1}^{n} \varphi_j(A_{j\rho(j)} u_{\rho(j)})$$

since all other products $\prod_{j=1}^{n} \varphi_j(A_{j\pi(j)} u_{\pi(j)})$ vanish on account of what has been proved so far. By Corollary 5.7(a) $Q_d(\lambda_j) = 0$. Since $d$ was arbitrary Lemma 5.4 implies the existence of $j$ such that $A_{j\rho(j)} = 0$.

The proof by induction is complete. The remainder follows from representing the Matrix $(A_{ik})$ as an upper triangular matrix. □

## 6. Appendix

### 6.1. On the set of zeros of a connected set of monic polynomials

Let $\mathbb{P}_n$ be the set of all monic polynomials $P(z) = z^n + \sum_{k=1}^{n} a_k z^{n-k}$ of degree $n$ equipped with the topology of uniform convergence on compacta. For each polynomial $P$ let $N(P)$ be the set of its zeros and for $z \in N(P)$ the number $n(z)$ denotes its multiplicity. The main result of this subsection reads as follows:

PROPOSITION 6.1. *Let $\mathscr{P} \subset \mathbb{P}_n$ be a connected subset and let $W(\mathscr{P}) = \bigcup_{P \in \mathscr{P}} N(P)$ be the set of its zeros. Then $W(\mathscr{P})$ has at most $n$ connected components $C_1, \dots, C_s$ ($s \leqslant n$). Moreover for every $k \leqslant s$ and $P, Q \in \mathscr{P}$ the following equation holds:*

$$\sum_{z \in N(P) \cap C_k} n(z) = \sum_{z' \in N(Q) \cap C_k} n(z') =: n(C_k). \tag{8}$$

In order to prove this proposition we need the Theorem of Rouché (see e. g. [4, Theorem III.7.7, p. 177]). To this end let $P \in \mathbb{P}_n$ be arbitrary. Let $z_1, \dots, z_s$ be the distinct zeros of $P$. If $s = 1$ then let $\varepsilon > 0$ be arbitrary. Else let $\delta(P) = \min\{|z_i - z_j| : i \neq j\}$ and let $\varepsilon$ satisfy $0 < \varepsilon < \delta(P)/3$.

Set $B(z_j,\varepsilon)=\{z\in\mathbb{C}:|z-z_j|<\varepsilon\}$, $m(P,z_j,\varepsilon)=\min\{|P(z)|:|z-z_j|=\varepsilon\}$, and

$$U_j(P,\varepsilon)=\{Q\in\mathbb{P}_n:\sup\{|Q(z)-P(z)|:|z-z_j|=\varepsilon\}<m(P,z_j,\varepsilon)\}.$$

The following lemma is a special case of Rouché's Theorem.

LEMMA 6.2. *(a) For all $Q\in U(P,z_j,\varepsilon)$ we have $\sum_{z\in N(Q)\cap B(z_j,\varepsilon)}n(z)=n(z_j)$, in particular $N(Q)\cap B(z_j,\varepsilon)\neq\emptyset$.*

*(b) For all $Q\in\bigcap_{j=1,\ldots,s}U(P,z_j,\varepsilon)=:U(P,\varepsilon)$ and all $z\in N(Q)$ there exists exactly one $z_j\in N(P)$ such that $|z-z_j|<\varepsilon$.*

*Proof.* (a) This is a direct application of Rouché's Theorem to $f=P$ and $g=Q-P$ (we use the notation of the cited reference).

(b) follows from

$$n=\sum_{j\leqslant s}n(z_j)=\sum_{j\leqslant s}\underbrace{\left(\sum_{z\in N(Q)\cap B(z_j,\varepsilon)}n(z)\right)}_{=n(z_j)}. \qquad \square$$

Furthermore we need the following lemma:

LEMMA 6.3. *Let $\mathscr{P}\neq\emptyset$ be a connected subset of $\mathbb{P}_n$. Let $\emptyset\neq C$ be a (relatively) open and closed subset of $W(\mathscr{P})$. Then the following assertions hold:*

*(a) For every $R\in\mathscr{P}$ we have $N(R)\cap C\neq\emptyset$.*

*(b) The function $f_C:R\mapsto\sum_{z\in N(R)\cap C}n(z)=f_C(R)$ is constant on $\mathscr{P}$.*

*Proof.* Without loss of generality assume $C\neq W(\mathscr{P})$. Set $D=W(\mathscr{P})\setminus C$ and $\mathscr{P}_C=\{R\in\mathscr{P}:N(R)\cap C\neq\emptyset\}$, $\mathscr{P}^D=\{R\in\mathscr{P}:N(R)\subset D\}$. Since $C\neq\emptyset$ there exists $R\in\mathscr{P}$ such that $N(R)\cap C\neq\emptyset$ which implies that $\mathscr{P}_C\neq\emptyset$.

*Claim: $\mathscr{P}_C$ is open and closed.* Let $R\in\mathscr{P}_C$ be arbitrary, let $N(R)=\{z_1,\ldots,z_s\}$, $N(R)\cap C=\{z_1,\ldots,z_r\}$, $(r\leqslant s)$. Since $C$ is open there exists $\varepsilon>0$ such that $\bigcup_{j\leqslant r}B(z_j,\varepsilon)\cap W(\mathscr{P})\subset C$. Without loss of generality assume that $\varepsilon<\delta(R)/3$. For $Q\in V(R):=U(R,\varepsilon)\cap\mathscr{P}$ and $j\leqslant r$ Lemma 6.2 yields $\emptyset\neq N(Q)\cap B(z_j,\varepsilon)\subset W(\mathscr{P})\cap B(z_j,\varepsilon)\subset C$, hence $Q\in\mathscr{P}_C$, which proves that $V(R)\subset\mathscr{P}_C$, thus $\mathscr{P}_C$ is open. Likewise it follows that $\mathscr{P}^D$ is open. Since $\mathscr{P}\setminus\mathscr{P}_C=\mathscr{P}^D$ the claim is proved. Since $\mathscr{P}_C$ is nonvoid and $\mathscr{P}$ is connected it follows $\mathscr{P}_C=\mathscr{P}$.

Let $R\in\mathscr{P}$ and $Q\in V(R)$ be arbitrary. For $z\in N(Q)\cap C$ there exists $z_j$ such that $j\leqslant r$ and $z\in B(z_j,\varepsilon)$ by Lemma 6.2 (b). Thus we obtain

$$f_C(Q)=\sum_{z\in N(Q)\cap C}n(z)=\sum_{j=1}^{r}\left(\sum_{z\in N(Q)\cap B(z_j,\varepsilon)}n(z)\right) \tag{9}$$

$$=\sum_{j=1}^{r}n(z_j)=f_C(R), \tag{10}$$

which implies that $f_C$ is locally constant, thus constant since $\mathscr{P}$ is connected. □

*Proof of Proposition 6.1. Claim: $W(\mathscr{M})$ has at most $n$ connected components:*
Let $P_0 \in \mathscr{M}$ be fixed. Then there exist at most $n$ connected components $C_j$ of $W(\mathscr{M})$ ($j = 1,\dots,s \leqslant n$) such that $\emptyset \neq N(P_0) \cap C_j$ since $N(P_0)$ has at most $n$ distinct elements.

We apply the Lemma above to $C = W(\mathscr{M}) \setminus \bigcup_{j \leqslant s} C_j$. If $C \neq \emptyset$ then it satisfies the conditions of the Lemma, hence $N(P_0) \cap C \neq \emptyset$, a contradiction, and the claim is proved.

Finally we apply part b) of the foregoing Lemma to $C = C_j$ and obtain the second assertion of our Proposition. □

## 6.2. Path-connectedness of $S_{att}(X)$ for three spaces with non-smooth norm

(I) Let $X = C(K)$ denote the space of continuous functions on the compact space $K$. Its dual is the space of all Radon-measures on $K$. Let $x_0 \in K$ be arbitrary and $(f,\mu),(g,\nu) \in S_{att}(X)$.

(i) Assume $x_0 \in U = \{x \in K : |f(x)| < 1\}$.

Then there exists a continuous function $h : K \to [0,1]$ such that $h(x_0) = 1$ and $h(K \setminus U) = 0$. Since $\mu(f) = 1 = |\mu|(1_K)$, where $1_K$ is the constant one function, we obtain $\mu(U) = 0$. Now $[0,1] \ni t \mapsto (h((1-t)f + th) + (1-h)f, \mu)$ connects $(f,\mu)$ with a pair $(f_1,\mu)$ where $f_1(x_0) = 1$.

Assume now that $f(x_0) = \exp(ia)$ for some $a \in \mathbb{R}$.

Then $[0,1] \ni t \mapsto (\exp(-ita)f), \exp(ita)\cdot\mu))$ is a path in $S_{att}(X)$ connecting $(f,\mu)$ with a pair $(f_1,\mu_1)$, where again $f_1(x_0) = 1$.

(ii) In both cases $[0,1] \ni t \mapsto (f_1,(1-t)\mu_1 + t\delta_{x_0})$ is a path in $S_{att}(X)$ from $(f_1,\mu_1)$ to $(f_1,\delta_{x_0})$ where $\delta_{x_0}$ denotes the Dirac measure in $x_0$.

(iii) Analogously, we find a function $g_1$ and a path connecting $(g_1,\delta_{x_0})$ and $(g,\nu)$.

(iv) $[0,1] \ni t \mapsto ((1-t)f_1 + tg_1, \delta_{x_0})$ is a path in $S_{att}(X)$ connecting $(f_1,\delta_{x_0})$ and $(g_1,\delta_{x_0})$ where $f_1(x_0) = g_1(x_0) = 1$.

(II) Almost the same arguments work for $X = C_0(L)$ where $L$ is locally compact.

(III) Let $X = L^1(\Omega,\Sigma,\mu)$ be the space of integrable functions $f$ (modulo null functions) on a complete measure space $(\Omega,\Sigma,\mu)$. The functions of the dual space $L^\infty(\Omega,\Sigma,\mu)$ are referred to as $\varphi$, $\psi,\dots$. For a set $M \subset \Omega$, $1_M$ denotes the indicator function and $M^c = \Omega \setminus M$ refers to the complement of $M$.

$(f,\varphi) \in S_{att}(X)$ holds if and only if $\int |f| d\mu = 1$, $|\varphi| \leqslant 1_\Omega$, and $\varphi(x) = \overline{f(x)}/|f(x)|$ whenever $f(x) \neq 0$. We set $Z_f = f^{-1}(\{0\})$, $U_f = Z_f^c$.

Let $(f,\varphi)$, $(g,\psi) \in S_{att}(X)$ be arbitrary.

(i) Define $\varphi_1 = 1_{U_g \cap Z_f} \cdot \psi + 1_{Z_g \cup U_f} \cdot \varphi$ and analogously $\psi_1$. Then $[0,1] \ni t \mapsto (f,(1-t)\varphi + t\varphi_1)$ connects $(f,\varphi)$ and $(f,\varphi_1)$. Likewise $(g,\psi)$ and $(g,\psi_1)$ are connected by a path.

(ii) Assume that $A := \{x \in U_f : g(x)/f(x) < 0\}$ is empty. Then $(1-t)f(x) + tg(x) \neq 0$ for $0 < t < 1$ and $x \in U_f \cup U_g$. For $0 \leqslant t \leqslant 1$ set $f_t = \frac{(1-t)f+tg}{\|(1-t)f+tg\|}$. For $0 <$

$t < 1$ define $\chi_t = 1_{U_f \cup U_g} \frac{(1-t)\overline{f}+t\overline{g}}{|(1-t)f+tg|} + 1_{Z_f \cap Z_g}((1-t)\varphi_1 + t\psi_1)$. Set $\chi_0 = \varphi_1$, $\chi_1 = \psi_1$. Then $t \mapsto (f_t, \chi_t)$ connects $(f, \varphi_1)$ and $(g, \psi_1)$.

(iii) If $A \neq \emptyset$ take $h = 1_A \cdot \mathrm{i}g + 1_{A^c} \cdot g$ and $\xi$ such that $(h, \xi) \in S_{att}(X)$. Now use $(h, \xi)$ as an intermediate point and apply (i) and (ii) twice.

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