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GROUP INVERSE OF FINITE POTENT ENDOMORPHISMS ON ARBITRARY VECTOR SPACES

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Abstract. The aim of this work is to introduce the group inverse of a finite potent endomorphism on an infinite-dimensional vector space that generalizes the notion of group inverse of a square finite matrix. The existence and uniqueness of this inverse is proved, several properties are offered and the relations with Drazin inverse, CMP inverse and DMP inverses are studied.

1. Introduction

For an arbitrary $(n\times n)$-matrix A with entries in the complex numbers, the index of A, $i(A)\geqslant 0$, is the smallest integer such that $\mathrm{rk}(A^{i(A)})=\mathrm{rk}(A^{i(A)+1})$.

It is known that, given a matrix $A\in \mathrm{Mat}_{n\times n}(k)$, the system of equations

$$\begin{aligned} AXA &= A\\ XAX &= X\\ AX &= XA \end{aligned}$$

has a solution if and only if $i(A)\leqslant 1$ and the solution is unique. This solution is the "group inverse" of A and is denoted $A^{\#}$ and satisfies several properties (Subsection 2.5).

This notion can be immediately extended to endomorphisms of finite-dimensional vector spaces over $\mathbb{C}$. Thus, given a finite-dimensional $\mathbb{C}$-vector space E, an endomorphism $f\in \mathrm{End}_{\mathbb{C}}E$ has index $i(f)\leqslant 1$ $(\mathrm{Im}\, f^2=\mathrm{Im}\, f)$ if and only if there exists an endomorphism $f^{\#}\in \mathrm{End}_{\mathbb{C}}E$ such that:

$$\begin{aligned} f\circ f^{\#}\circ f &= f;\\ f^{\#}\circ f\circ f^{\#} &= f^{\#};\\ f^{\#}\circ f &= f\circ f^{\#}. \end{aligned}$$

The endomorphism $f^{\#}$ is the "group inverse" of f.

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If V is an infinite-dimensional vector space over an arbitrary ground field k, an endomorphism $\varphi \in \operatorname{End}_k(V)$ is "finite potent" when there exists $n \in \mathbb{N}$ such that $\dim_k \varphi^n < \infty$. The index of a finite potent endomorphism φ is the smaller $n \in \mathbb{N}$ such that $\dim_k \operatorname{Im} \varphi^n = \dim_k \operatorname{Im} \varphi^{n+1} < \infty$.

The aim of this work is to give an affirmative answer to the following question: is it possible to extend the notion of group inverse to finite potent endomorphisms on infinite-dimensional vector spaces such that $\varphi^{\#}$ exists if and only if $i(\varphi) \leqslant 1$?.

Given an arbitrary k-vector space V, we prove the existence and uniqueness of the group inverse $\varphi^{\#}$ of a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ with $i(\varphi) \leqslant 1$. Moreover, several properties of $\varphi^{\#}$ are offered and its relations with Drazin inverse, CMP inverse and DMP inverses are studied.

In particular, for every matrix $A \in \operatorname{Mat}_{n\times n}(\mathbb{C})$, from the statement of this paper, we deduce that:

- A is $EP \Longleftrightarrow A^{c\dagger} = A^{\#}$;
- $A^{\#} = A^{d,\dagger}$ if and only if $N(A^{\dagger}) \subseteq N(A^D)$;
- $A^{\#} = A^{\dagger,d}$ if and only if $R(A) \subseteq R(A^*)$;

where $A^{c\dagger}$ is the CMP inverse of A, $A^{d,\dagger}$ and $A^{\dagger,d}$ are the DMP inverses of A and A^D is the Drazin inverse of A.

The paper is organized as follows. In section 2 we recall the basic definitions of this work and a summary of the statements of the articles [2], [8] and [10].

Section 3 contains the main results of this work: in Subsection 3.1 we study the Drazin inverse of two commuting linear maps; the goal of Subsection 3.2 is to prove the existence and uniqueness of the group inverse of a finite potent endomorphism (Theorem 3.5) and to show several properties of this inverse; Subsection 3.3 is devoted to offer new results on EP finite potent endomorphisms and, finally, in Subsection 3.4 we obtain the relations between the group inverse and the DMP inverses of finite potent endomorphisms.

Finally, bearing in mind the relationship that exists between linear maps and matrices, we wish to remark that the results of this work are valid for some infinite matrices (associated with finite potent endomorphisms).

2. Preliminaries

This section is added for the sake of completeness.

2.1. Finite potent endomorphisms

Let k be an arbitrary field, and let V be a k-vector space.

Let us now consider an endomorphism φ of V. We say that φ is "finite potent" if $\varphi^n V$ is finite dimensional for some n. This definition was introduced by J. Tate in [11] as a basic tool for his elegant definition of Abstract Residues.

In 2007 M. Argerami, F. Szechtman and R. Tifenbach showed in [1] that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|U_\varphi}$ is nilpotent, W_φ is finite dimensional, and $\varphi_{|W_\varphi} : W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k, we may view V as an $k[x]$-module via φ, and the explicit definition of the above φ-invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } x^m v = 0 \text{ for some m }\}$.
- $W_\varphi = \{v \in V \text{ such that } p(x)v = 0 \text{ for some } p(x) \in k[x] \text{ relative prime to } x\}$.

Note that if the annihilator polynomial of φ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \operatorname{Ker} \varphi^m$ and $W_\varphi = \operatorname{Ker} p(\varphi)$.

Hence, this decomposition is unique. We shall call this decomposition the φ-invariant AST-decomposition of V.

Moreover, we shall call "index of φ", $i(\varphi)$, to the nilpotent order of $\varphi_{|U_\varphi}$, which coincides with the smaller $n \in \mathbb{N}$ such that $\operatorname{Im} \varphi^n = W_\varphi$. One has that $i(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism.

For a finite potent endomorphism φ, a trace $\operatorname{Tr}_V(\varphi) \in k$ may be defined as $\operatorname{Tr}_V(\varphi) = \operatorname{Tr}_{W_\varphi}(\varphi_{|W_\varphi})$. Furthermore, a determinant for every finite potent endomorphism is defined as follows:

$$\det{}^k_V(1+\varphi) := \det{}^k_{W_\varphi}(1+\varphi_{|W_\varphi}).$$

2.2. Drazin inverse of finite potent endomorphisms

Let V be an arbitrary k-vector space, and let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism of V. Let us consider the AST-decomposition $V = U_\varphi \oplus W_\varphi$ induced by φ.

For each finite potent endomorphism φ there exists a unique finite potent endomorphism φ^D that satisfies:

1. $\varphi^{k+1} \circ \varphi^D = \varphi^k$;
2. $\varphi^D \circ \varphi \circ \varphi^D = \varphi^D$;
3. $\varphi^D \circ \varphi = \varphi \circ \varphi^D$;

where k is the index of φ.

The map φ^D is the Drazin inverse of φ and is the unique linear map such that:

$$\varphi^D(v) = \begin{cases} (\varphi_{|W_\varphi})^{-1}(v) & \text{if} \quad v \in W_\varphi \\ 0 & \text{if} \quad v \in U_\varphi \end{cases}.$$

Moreover, φ^D satisfies the following properties:

- $(\varphi^D)^D = \varphi$ if and only if the $i(\varphi) \leqslant 1$;

- $\varphi = \varphi^D$ if and only if $\varphi_{|U_\varphi} = 0$ and $(\varphi_{|W_\varphi})^2 = \mathrm{Id}_{|W_\varphi}$;
- $\mathrm{Tr}_V(\varphi + \varphi^D) = \mathrm{Tr}_V(\varphi) + \mathrm{Tr}_V(\varphi^D)$;
- If ψ is a projection finite potent endomorphism, then $\psi^D = \psi$.

2.3. CN decomposition of a finite potent endomorphism

Given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \mathrm{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \leqslant 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

According to [8, Theorem 3.2], if φ^D is the Drazin inverse of φ, one has that $\varphi_1 = \varphi \circ \varphi^D \circ \varphi$ is the core part of φ. Also, φ_2 is named the nilpotent part of φ.

Moreover, one has that

$$\varphi = \varphi_1 \Longleftrightarrow U_\varphi = \mathrm{Ker}\,\varphi \Longleftrightarrow W_\varphi = \mathrm{Im}\,\varphi \Longleftrightarrow (\varphi^D)^D = \varphi \Longleftrightarrow i(\varphi) \leqslant 1. \quad (2.1)$$

2.4. Moore-Penrose inverse of a linear map over arbitrary vector spaces

2.4.1. Moore-Penrose inverse of an $(n \times m)$-matrix

Let $\mathbb{C}$ be the field. Given a matrix $A \in \mathrm{Mat}_{n\times m}(\mathbb{C})$, the Moore-Penrose inverse of A is a matrix $A^\dagger \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ such that:

- $AA^\dagger A = A$;
- $A^\dagger AA^\dagger = A^\dagger$;
- $(AA^\dagger)^* = AA^\dagger$;
- $(A^\dagger A)^* = A^\dagger A$;

B^* being the conjugate transpose of the matrix B.

The Moore-Penrose inverse of A always exists, it is unique, $[A^\dagger]^\dagger = A$, and, if $A \in \mathbb{C}^{n\times n}$ is non-singular, then the Moore-Penrose inverse of A coincides with the inverse matrix A^{-1}.

For details, readers are referred to [3].

2.4.2. Moore-Penrose inverse of a linear map over arbitrary vector spaces

Let (V,g) and $(W,\bar{g})$ be inner product vector spaces over k, with $k=\mathbb{C}$ or $k=\mathbb{R}$.

Given a linear map $f\colon V\to W$, a linear map $f^+\colon W\to V$ is a reflexive generalized inverse of f when

- $f\circ f^+\circ f=f$;
- $f^+\circ f\circ f^+=f^+$.

DEFINITION 2.1. Given a linear map $f\colon V\longrightarrow W$, we say that f is admissible for the Moore-Penrose inverse when $V=\operatorname{Ker} f\oplus[\operatorname{Ker} f]^\perp$ and $W=\operatorname{Im} f\oplus[\operatorname{Im} f]^\perp$.

If (V,g) and $(W,\bar{g})$ are inner product spaces over k and $f\colon V\to W$ is a linear map admissible for the Moore-Penrose inverse, according to [2, Theorem 3.11] there exists a unique linear map $f^\dagger\colon W\to V$ such that:

1. $f^\dagger$ is a reflexive generalized inverse of f;
2. $f^\dagger\circ f$ and $f\circ f^\dagger$ are self-adjoint, that is:
   - $g([f^\dagger\circ f](v),v')=g(v,[f^\dagger\circ f](v'))$;
   - $\bar{g}([f\circ f^\dagger](w),w')=\bar{g}(w,[f\circ f^\dagger](w'))$;

for all $v,v'\in V$ and $w,w'\in W$. The operator $f^\dagger$ is named the Moore-Penrose inverse of f and it is the unique linear map satisfying that

$$f^\dagger(w)=\begin{cases}(f_{|_{[\operatorname{Ker} f]^\perp}})^{-1}(w) & \text{if} \quad w\in\operatorname{Im} f\\ 0 & \text{if} \quad w\in[\operatorname{Im} f]^\perp\end{cases}.$$

The Moore-Penrose inverse $f^\dagger\colon W\to V$ also satisfies the following properties:

- $f^\dagger$ is admissible for the Moore-Penrose inverse and $(f^\dagger)^\dagger=f$;
- If $f\in\operatorname{End}_k(V)$ and f is an isomorphism, then $f^\dagger=f^{-1}$;
- $f^\dagger\circ f=P_{[\ker f]^\perp}$;
- $f\circ f^\dagger=P_{\operatorname{Im} f}$;

where $P_{[\ker f]^\perp}$ and $P_{\operatorname{Im} f}$ are the projections induced by the decompositions $V=\operatorname{Ker} f\oplus[\operatorname{Ker} f]^\perp$ and $W=\operatorname{Im} f\oplus[\operatorname{Im} f]^\perp$ respectively.

For details on this Moore-Penrose inverse readers are referred to [2].

2.5. Group inverse of a finite matrix

Given a matrix $A \in \mathrm{Mat}_{n\times n}(k)$, the system of equations

$$\begin{aligned} AXA &= A \\ XAX &= X \\ AX &= XA \end{aligned}$$

has a solution if and only if $i(A) \leqslant 1$ and the solution is unique. This solution is the "group inverse" of A and is denoted $A^{\#}$.

If $A \in \mathrm{Mat}_{n\times n}(k)$ with $i(A) \leqslant 1$, then the group inverse $A^{\#}$ satisfies the following properties:

- If A is nonsingular, then $A^{\#} = A^{-1}$.
- $(A^{\#})^{\#}$.
- $(A^t)^{\#} = (A^{\#})^t$, where A^t is the transpose of A.
- If $n \in \mathbb{Z}^+$, then $(A^n)^{\#} = (A^{\#})^n$.
- A is EP if and only if $A^{\#} = A^{\dagger}$.

3. Group inverse of a finite potent endomorphism

In this final section we shall characterize the conditions under which the group inverse of a finite potent endomorphism exits. Moreover, we shall offer the explicit expression of this linear map and we shall study its properties.

3.1. Commuting finite potent endomorphisms

Similar to above, if $\varphi \in \mathrm{End}_k(V)$ is finite potent, we denote $\varphi = \varphi_1 + \varphi_2$ to the core-nilpotent decomposition of φ, where φ_1 is the core part and φ_2 is the nilpotent part.

LEMMA 3.1. *Let $\psi, \varphi \in \mathrm{End}_k(V)$, where φ is finite potent and $\psi \circ \varphi = \varphi \circ \psi$. If $V = U_\varphi \oplus W_\varphi$ is the AST φ-invariant decomposition of V, then:*

1. *U_φ and W_φ are ψ-invariant;*
2. *$\varphi^D \circ \psi = \psi \circ \varphi^D$;*

φ^D being the Drazin inverse of φ.
Moreover, if ψ is finite potent, then

$$\varphi^D \circ \psi^D = \psi^D \circ \varphi^D.$$

Proof. Let us assume that $\operatorname{Im} \varphi^n = W_\varphi$. If $u \in U_\varphi$, one has that

$$\varphi^n(\psi(u)) = \psi(\varphi^n(u)) = \psi(0) = 0$$

and $\psi(u) \in U_\varphi$.

Moreover, if $w \in W_\varphi$, writing $\varphi^n(w') = w$ with $w' \in W_\varphi$, then

$$\psi(w) = \psi(\varphi^n(w')) = \varphi^n(\psi(w')) \in W_\varphi .$$

Hence, the first assertion is proved and, bearing in mind that

$$\varphi^D(v) = \begin{cases} 0 & \text{if} \quad v \in U_\varphi \\ (\varphi_{|W_\varphi})^{-1}(v) & \text{if} \quad v \in W_\varphi \end{cases},$$

the second statement is also deduced. □

A direct consequence of Lemma 3.1 is

COROLLARY 3.2. *If $\psi, \varphi \in \operatorname{End}_k(V)$, φ is finite potent, $\psi \circ \varphi = \varphi \circ \psi$ and $\varphi = \varphi_1 + \varphi_2$ is the core-nilpotent decomposition of φ, then*

- $\varphi_1 \circ \psi = \psi \circ \varphi_1$*;*
- $\varphi_2 \circ \psi = \psi \circ \varphi_2$*.*

3.2. Group inverse of a finite potent endomorphism

Let $\varphi \in \operatorname{End}_k(V)$ be again a finite potent endomorphism on an arbitrary k-vector space V.

DEFINITION 3.3. We say that a linear map $\varphi^\# \in \operatorname{End}_k(V)$ is a group inverse of φ when it satisfies the following properties:

- $\varphi \circ \varphi^\# \circ \varphi = \varphi$;
- $\varphi^\# \circ \varphi \circ \varphi^\# = \varphi^\#$;
- $\varphi^\# \circ \varphi = \varphi \circ \varphi^\#$.

Thus, a group inverse $\varphi^\#$ is a reflexive generalized inverse of φ that commutes with it.

LEMMA 3.4. *Given a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, if there exists a group inverse $\varphi^\# \in \operatorname{End}_k(V)$, then $i(\varphi) \leqslant 1$.*

Proof. Let $V = U_\varphi \oplus W_\varphi$ be the AST decomposition of V induced by φ and let us assume that $i(\varphi) = s$. If $u \in U_\varphi$ and a group inverse $\varphi^\# \in \mathrm{End}_k(V)$ exists, bearing in mind that

$$\varphi \circ \varphi^\# \circ \varphi = \varphi \quad \text{and} \quad \varphi^\# \circ \varphi = \varphi \circ \varphi^\#,$$

one has that $\varphi = (\varphi^\#)^{n-1} \circ \varphi^n$ for every $n \in \mathbb{N}$, and then

$$\varphi(u) = [(\varphi^\#)^{s-1} \circ \varphi^s](u) = 0,$$

because $(\varphi^s)_{|U_\varphi} = 0$.

Hence, $\mathrm{Ker}\,\varphi = U_\varphi$ and we conclude that $i(\varphi) \leqslant 1$. □

THEOREM 3.5. *If $\varphi \in \mathrm{End}_k(V)$ is a finite potent endomorphism with $i(\varphi) \leqslant 1$, then $\varphi^D = \varphi^\#$ is the unique group inverse of φ, where φ^D is its Drazin inverse.*

Proof. Since $i(\varphi) \leqslant 1$, then $\varphi = \varphi_1$. Let φ^D be the Drazin inverse of φ.

Bearing in mind that $\varphi^D \circ \varphi \circ \varphi^D = \varphi^D$ and $\varphi^D \circ \varphi = \varphi \circ \varphi^D$, it follows from

$$\varphi_1 = \varphi \circ \varphi^D \circ \varphi$$

that φ^D is a group inverse of φ.

Conversely, let us assume that $\varphi^\#$ is a group inverse of φ and let $V = U_\varphi \oplus W_\varphi$ be again the AST decomposition of V induced by φ.

Since $\varphi^\# \circ \varphi = \varphi \circ \varphi^\#$, one deduces from Lemma 3.1 that U_φ and W_φ are invariant under the action of $\varphi^\#$.

Accordingly, if $i(\varphi) = s$ and $u \in U_\varphi$, it follows from $\varphi^\# \circ \varphi \circ \varphi^\# = \varphi^\#$ that

$$\varphi^\#(u) = [(\varphi^\#)^{s+1} \circ \varphi^s](u) = 0.$$

Moreover, since $\varphi_{|W_\varphi} \in \mathrm{Aut}_k(W_\varphi)$, one has that

$$\begin{aligned} &(\varphi \circ \varphi^\# \circ \varphi)(w) = \varphi(w) \text{ for all } w \in W_\varphi \Longleftrightarrow \\ &(\varphi \circ \varphi^\#)_{|W_\varphi} = \mathrm{Id}_{|W_\varphi} \Longleftrightarrow \\ &(\varphi^\#)_{|W_\varphi} = (\varphi_{|W_\varphi})^{-1}. \end{aligned}$$

Hence, $\varphi^\# = \varphi^D$ and the statement is proved. □

EXAMPLE 1. Let V be a vector space of countable dimension over k. Let $\{v_1, v_2, v_3, \dots\}$ be a basis of V indexed by the natural numbers. If $(x_i)_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathbb{N}} k$, since $x_i = 0$ for all but finitely many $i \in \mathbb{N}$, we shall write $x = (x_i)$ to denote the well-defined vector

$$x = \sum_{i\in\mathbb{N}} x_i \cdot v_i \in V.$$

If we consider the finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ defined as:

$$\varphi(v_i) = \begin{cases} 0 & \text{if i is odd} \\ v_{10j-4} & \text{if } i = 10j-8 \text{ and } j \in \{1,2,3,4\} \\ v_{10j} - v_{10j-8} & \text{if } i = 10j-6 \text{ and } j \in \{1,2,3,4\} \\ v_{10j-2} & \text{if } i = 10j-4 \text{ and } j \in \{1,2,3,4\} \\ v_{10j-6} + v_{10j} & \text{if } i = 10j-2 \text{ and } j \in \{1,2,3,4\} \\ v_{10j-8} & \text{if } i = 10j \text{ and } j \in \{1,2,3,4\} \\ 0 & \text{if } i = 2s \text{ and } s \geqslant 21 \end{cases},$$

it is clear that $i(\varphi) = 1$ and its group inverse $\varphi^{\#}$ is the unique linear map such that

$$\varphi^{\#}(v_i) = \begin{cases} 0 & \text{if i is odd} \\ v_{10j} & \text{if } i = 10j-8 \text{ and } j \in \{1,2,3,4\} \\ v_{10j-2} - v_{10j-6} - v_{10j} & \text{if } i = 10j-6 \text{ and } j \in \{1,2,3,4\} \\ v_{10j-8} & \text{if } i = 10j-4 \text{ and } j \in \{1,2,3,4\} \\ v_{10j-4} & \text{if } i = 10j-2 \text{ and } j \in \{1,2,3,4\} \\ v_{10j-6} + v_{10j} & \text{if } i = 10j \text{ and } j \in \{1,2,3,4\} \\ 0 & \text{if } i = 2s \text{ and } s \geqslant 21 \end{cases}.$$

From the basic properties of the Drazin inverse [10], for every finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ with $i(\varphi) \leqslant 1$, the group inverse $\varphi^{\#}$ immediately satisfies the following properties:

- $(\varphi^{\#})^{\#} = \varphi$;
- $\varphi = \varphi^{\#}$ if and only if $(\varphi_{|W_\varphi})^2 = \mathrm{Id}_{|W_\varphi}$;
- $\mathrm{Tr}_V(\varphi + \varphi^{\#}) = \mathrm{Tr}_V(\varphi) + \mathrm{Tr}_V(\varphi^{\#})$.

LEMMA 3.6. *If $\varphi \in \mathrm{End}_k(V)$ is a finite potent endomorphism with $i(\varphi) \leqslant 1$ and $n \in \mathbb{Z}^+$, then $(\varphi^n)^{\#} = (\varphi^{\#})^n$.*

Proof. Let $V = U_\varphi \oplus W_\varphi$ be the AST decomposition of V induced by φ. For all $n \in \mathbb{Z}^+$, it is clear that $U_{\varphi^n} = U_\varphi$ and $W_{\varphi^n} = W_\varphi$. Thus, the claim is deduced bearing in mind that

$$([\varphi^n]_{|W_\varphi})^{-1} = [(\varphi_{|W_\varphi})^{-1}]^n. \quad \square$$

REMARK 3.7. Given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, if we denote $\varphi^0 = \varphi \circ \varphi^{\#}$ and $\varphi^{-i} = (\varphi^{\#})^i$ for every $i \in \mathbb{N}$, it follows from Theorem 3.5 and Lemma 3.6 that $X = \{\varphi^j\}_{j \in \mathbb{Z}}$ is an Abelian Group with identity element φ^0.

LEMMA 3.8. *If* $\varphi \in \mathrm{End}_k(V)$ *is a finite potent endomorphism with* $i(\varphi) \leqslant 1$ *and* $\tau \in \mathrm{Aut}_k(V)$, *then* $\tau \circ \varphi \circ \tau^{-1}$ *is finite potent with* $i(\tau \circ \varphi \circ \tau^{-1}) \leqslant 1$ *and*

$$(\tau \circ \varphi \circ \tau^{-1})^{\#} = \tau \circ \varphi^{\#} \circ \tau^{-1}.$$

Proof. Since $(\tau \circ \varphi \circ \tau^{-1})^n = \tau \circ \varphi^n \circ \tau^{-1}$ for every $n \in \mathbb{N}$, if $i(\varphi) \leqslant 1$, it is clear that $\tau \circ \varphi \circ \tau^{-1}$ is also finite potent with $i(\tau \circ \varphi \circ \tau^{-1}) \leqslant 1$.

Moreover, one has that:

- $(\tau \circ \varphi \circ \tau^{-1}) \circ (\tau \circ \varphi^{\#} \circ \tau^{-1}) \circ (\tau \circ \varphi \circ \tau^{-1}) = \tau \circ (\varphi \circ \varphi^{\#} \circ \varphi) \circ \tau^{-1} = \tau \circ \varphi \circ \tau^{-1}$;
- $(\tau \circ \varphi^{\#} \circ \tau^{-1}) \circ (\tau \circ \varphi \circ \tau^{-1}) \circ (\tau \circ \varphi^{\#} \circ \tau^{-1}) = \tau \circ (\varphi^{\#} \circ \varphi \circ \varphi^{\#}) \circ \tau^{-1} = \tau \circ \varphi^{\#} \circ \tau^{-1}$;
- $(\tau \circ \varphi \circ \tau^{-1}) \circ (\tau \circ \varphi^{\#} \circ \tau^{-1}) = \tau \circ (\varphi \circ \varphi^{\#}) \circ \tau^{-1} = \tau \circ (\varphi^{\#} \circ \varphi) \circ \tau^{-1} = (\tau \circ \varphi^{\#} \circ \tau^{-1}) \circ (\tau \circ \varphi \circ \tau^{-1})$;

and we conclude that $(\tau \circ \varphi \circ \tau^{-1})^{\#} = \tau \circ \varphi^{\#} \circ \tau^{-1}$. □

REMARK 3.9. According to the statements of [7], the set of finite potent endomorphisms admits a classification when the group $\mathrm{Aut}_k(V)$ acts by conjugation. In this context, if $\varphi, \psi \in \mathrm{End}_k(V)$ are finite potent and $\varphi \sim \psi$, then it follows from Lemma 3.8 that $\varphi^{\#} \sim \psi^{\#}$.

To finish this part related to the definition and the basic properties of the group inverse of a finite potent endomorphism, we shall study the group inverse of the composition of two commuting linear maps.

LEMMA 3.10. *If* $\varphi, \psi \in \mathrm{End}_k(V)$ *such that* φ *is finite potent and* $\varphi \circ \psi = \psi \circ \varphi$, *then one has that:*

1. $\varphi \circ \psi$ *is finite potent;*
2. $(\varphi \circ \psi)^{\#} = \psi^{-1} \circ \varphi^{\#}$ *when* $\psi \in \mathrm{Aut}_k(V)$;
3. $(\varphi \circ \psi)^{\#} = \psi^{\#} \circ \varphi^{\#}$ *when* ψ *is finite potent.*

Proof.

1. For every linear map $\psi \in \mathrm{End}_k(V)$, since φ is finite potent, one has that

$$(\varphi \circ \psi)^n = \psi^n \circ \varphi^n,$$

from where we deduce that $\varphi \circ \psi$ is finite potent.

2. If $\psi \in \mathrm{Aut}_k(V)$, it is clear that:

   - $(\varphi \circ \psi) \circ (\psi^{-1} \circ \varphi^{\#}) \circ (\varphi \circ \psi) = (\varphi \circ \varphi^{\#} \circ \varphi) \circ \psi = \varphi \circ \psi$;

- $(\psi^{-1}\circ\varphi^{\#})\circ(\varphi\circ\psi)\circ(\psi^{-1}\circ\varphi^{\#}) = \psi^{-1}\circ(\varphi^{\#}\circ\varphi\circ\varphi^{\#}) = \psi^{-1}\circ\varphi^{\#}$;
- Bearing in mind Lemma 3.1 one deduces that

$$\begin{aligned}(\varphi\circ\psi)\circ(\psi^{-1}\circ\varphi^{\#}) &= \varphi\circ\varphi^{\#} = \varphi^{\#}\circ\varphi \\ &= (\varphi^{\#}\circ\psi^{-1})\circ(\psi\circ\varphi) = (\psi^{-1}\circ\varphi^{\#})\circ(\varphi\circ\psi)\end{aligned};$$

and from Theorem 3.5 we conclude that $(\varphi\circ\psi)^{\#} = \psi^{-1}\circ\varphi^{\#}$.

3. With similar arguments to the previous proof, from Corollary 3.2 and Theorem 3.5, we deduce that $(\varphi\circ\psi)^{\#} = \psi^{\#}\circ\varphi^{\#}$ when ψ is finite potent. □

3.3. EP finite potent endomorphisms

Henceforth, (V,g) will be an inner product vector space over k, with $k=\mathbb{C}$ or $k=\mathbb{R}$. According to [4, Lemma 2.1], the following definition makes sense:

DEFINITION 3.11. We say that a linear map $\varphi\in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse is EP when

$$\varphi\circ\varphi^{\dagger} = \varphi^{\dagger}\circ\varphi .$$

We shall now characterize EP finite potent endomorphisms.

LEMMA 3.12. *Let $\varphi\in \operatorname{End}_k(V)$ be a finite potent endomorphism admissible for the Moore-Penrose inverse. If φ is EP, then $i(\varphi)\leqslant 1$.*

Proof. Similar to the proof of Lemma 3.4, let $V = U_\varphi\oplus W_\varphi$ be the AST decomposition of V induced by φ and let us assume that $i(\varphi)=s$. If $u\in U_\varphi$, it follows from the definition of the Moore-Penrose inverse that

$$\varphi(u) = [(\varphi^{\dagger})^{s-1}\circ\varphi^{s}](u) = 0 .$$

Hence, $\operatorname{Ker}\varphi = U_\varphi$ and we conclude that $i(\varphi)\leqslant 1$. □

PROPOSITION 3.13. *A finite potent endomorphism admissible for the Moore-Penrose inverse $\varphi\in \operatorname{End}_k(V)$ is EP if and only if $\varphi^{\#}=\varphi^{\dagger}$.*

Proof. If φ is EP, then $\varphi^{\dagger}$ is a group inverse of φ and it follows from Theorem 3.5 that

$$\varphi^{\dagger} = \varphi^{D} = \varphi^{\#} .$$

On the other hand, if $\varphi^{\#}=\varphi^{\dagger}$, one deduces from Definition 3.3 that φ is EP. □

COROLLARY 3.14. *If $\varphi\in \operatorname{End}_k(V)$ is an EP finite potent endomorphism admissible for the Moore-Penrose inverse, then $\varphi^{\dagger}$ is also EP.*

Proof. Since $\varphi \in \text{End}_k(V)$ is an EP finite potent endomorphism, then $\varphi^\dagger$ is also finite potent. Moreover, it follows from [2, Corollary 3.12] that $\varphi^\dagger$ is admissible for the Moore-Penrose inverse.

Hence, the claim is immediately deduced from Proposition 3.13 bearing in mind that $(\varphi^\#)^\# = \varphi$. □

Moreover, for every finite potent endomorphism admissible for the Moore-Penrose inverse $\varphi \in \text{End}_k(V)$ the author has shown in [8] that there exists a unique finite potent endomorphism $\varphi^{c\dagger} \in \text{End}_k(V)$ satisfying that:

- $\varphi^{c\dagger} \circ \varphi \circ \varphi^{c\dagger} = \varphi^{c\dagger}$;
- $\varphi \circ \varphi^{c\dagger} \circ \varphi = \varphi_1$;
- $\varphi \circ \varphi^{c\dagger} = \varphi_1 \circ \varphi^\dagger$;
- $\varphi^{c\dagger} \circ \varphi = \varphi^\dagger \circ \varphi_1$.

The finite potent endomorphism $\varphi^{c\dagger}$ is the core-Moore-Penrose (CMP) inverse of φ.

LEMMA 3.15. *If $\varphi \in \text{End}_k(V)$ is a finite potent endomorphism admissible for the Moore-Penrose inverse and $\varphi^\# = \varphi^{c\dagger}$, then $i(\varphi) \leqslant 1$.*

Proof. Since $\varphi^\# = \varphi^{c\dagger}$, then

$$\varphi = \varphi \circ \varphi^\# \circ \varphi = \varphi \circ \varphi^{c\dagger} \circ \varphi = \varphi_1,$$

and the statement is deduced. □

Hence, another characterization of EP finite potent endomorphisms is the following:

PROPOSITION 3.16. *A finite potent endomorphism admissible for the Moore-Penrose inverse $\varphi \in \text{End}_k(V)$ is EP if and only if $\varphi^\# = \varphi^{c\dagger}$.*

Proof. Let us firstly assume that φ is EP. Then $\varphi = \varphi_1$ and, since $\varphi^\# = \varphi^\dagger$ (Proposition 3.13), one deduces that $\varphi^\# = \varphi^{c\dagger}$.

Conversely, if $\varphi^\# = \varphi^{c\dagger}$, it follows from Lemma 3.15 that $i(\varphi) \leqslant 1$ and then

$$\varphi \circ \varphi^\dagger = \varphi_1 \circ \varphi^\dagger = \varphi \circ \varphi^\# = \varphi^\# \circ \varphi = \varphi^\dagger \circ \varphi_1 = \varphi^\dagger \circ \varphi,$$

and we conclude that φ is EP. □

REMARK 3.17. Given a matrix $A \in \text{Mat}_{n\times n}(\mathbb{C})$, if $A^{c\dagger}$ is the CMP inverse of A introduced by M. Mehdipour and A.Salemi in [6], it follows from Proposition 3.16 that a new characterization of EP matrices is

$$\text{“}A \text{ is } EP \Longleftrightarrow A^{c\dagger} = A^\#.\text{''}$$

3.4. Drazin-Moore-Penrose inverses

Let (V,g) be again an inner product vector space over k, with $k=\mathbb{C}$ or $k=\mathbb{R}$, and let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism admissible for the Moore-Penrose inverse. According to [9], the left Drazin-Moore-Penrose (lDMP) inverse of φ is the unique finite potent endomorphism $\varphi^{d,\dagger} \in \mathrm{End}_k(V)$ satisfying the following conditions:

- $\varphi^{d,\dagger} \circ \varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger}$;
- $\varphi^m \circ \varphi^{d,\dagger} = \varphi^m \circ \varphi^{\dagger}$ with $m = i(\varphi)$;
- $\varphi^{d,\dagger} \circ \varphi = \varphi^D \circ \varphi$.

LEMMA 3.18. *Given a finite potent endomorphism admissible for the Moore-Penrose inverse $\varphi \in \mathrm{End}_k(V)$ with $i(\varphi) \leqslant 1$, one has that $\varphi^{\#} = \varphi^{d,\dagger}$ if and only if* $\mathrm{Ker}\,\varphi^{\dagger} \subseteq \mathrm{Ker}\,\varphi^D$.

Proof. It follows from [9, Proposition 3.3] that $\varphi \circ \varphi^{d,\dagger} \circ \varphi = \varphi_1$, where φ_1 is the core part of φ. Moreover, from [9, Proposition 3.9], we knew that $\varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger} \circ \varphi$ if and only if $\mathrm{Ker}\,\varphi^{\dagger} \subseteq \mathrm{Ker}\,\varphi^D$.

Accordingly, the statement is deduced from Theorem 3.5. □

Furthermore, the right Drazin-Moore-Penrose (rDMP) inverse of φ is the unique finite potent endomorphism $\varphi^{\dagger,d} \in \mathrm{End}_k(V)$ satisfying the following conditions:

- $\varphi^{\dagger,d} \circ \varphi \circ \varphi^{\dagger,d} = \varphi^{\dagger,d}$;
- $\varphi^{\dagger,d} \circ \varphi^m = \varphi^{\dagger} \circ \varphi^m$ with $m = i(\varphi)$;
- $\varphi \circ \varphi^{\dagger,d} = \varphi \circ \varphi^D$.

LEMMA 3.19. *If $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism admissible for the Moore-Penrose inverse with $i(\varphi) \leqslant 1$ and $V = U_\varphi \oplus W_\varphi$ be the AST decomposition of V induced by φ, then $\varphi^{\#} = \varphi^{\dagger,d}$ if and only if $W_\varphi \subseteq [\mathrm{Ker}\,\varphi]^{\perp}$.*

Proof. Similar to above, the claim is obtained from Theorem 3.5 from the following properties of the rDMP inverse $\varphi^{\dagger,d}$ studied in [9]:

- $\varphi \circ \varphi^{\dagger,d} \circ \varphi = \varphi_1$, where φ_1 is the core part of φ;
- $\varphi \circ \varphi^{\dagger,d} = \varphi^{\dagger,d} \circ \varphi$ if and only if $W_\varphi \subseteq [\mathrm{Ker}\,\varphi]^{\perp}$. □

REMARK 3.20. Given a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, in [5] S. Malik and N. Thome introduced the DMP inverses $A^{d,\dagger}$ and $A^{\dagger,d}$.

Thus, if $i(A) \leqslant 1$, from Lemma 3.18 and Lemma 3.19 we deduce that:

- $A^{\#} = A^{d,\dagger}$ if and only if $N(A^{\dagger}) \subseteq N(A^D)$;
- $A^{\#} = A^{\dagger,d}$ if and only if $R(A) \subseteq R(A^*)$;

where $N(B)$ is the Kernel of B and $R(B)$ is the range of B, for every matrix $B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

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