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RANDOM STURM–LIOUVILLE OPERATORS WITH GENERALIZED POINT INTERACTIONS

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Abstract. In this work we study the point spectra of selfadjoint Sturm-Liouville operators with generalized point interactions, where the two one-sided limits of the solution data are related via a general $\mathrm{SL}(2,\mathbb{R})$ matrix. We are particularly interested in the stability of eigenvalues with respect to the variation of the parameters of the interaction matrix. As a particular application to the case of random generalized point interactions we establish a version of Pastur's theorem, stating that except for degenerate cases, any given energy is an eigenvalue only with probability zero. For this result, independence is important but identical distribution is not required, and hence our result extends Pastur's theorem from the ergodic setting to the non-ergodic setting.

1. Introduction

In this paper we study the point spectra of selfadjoint Sturm-Liouville operators with generalized point interactions. More specifically, we investigate whether varying the parameters of the spectral problem preserves or destroys the fact that a given energy is an eigenvalue. This is of particular interest in the setting of random parameters. In the case of i.i.d. random variables, one can use methods from ergodic theory and it is a classical result due to Pastur [17] that a given energy can be an eigenvalue only with probability zero. However, if the random variables are not identically distributed, Pastur's argument does not apply and it was realized only recently, in the special case of δ and δ' point interactions, that a result in the same spirit still holds [4].

The purpose of the present paper is two-fold. On the one hand, we introduce a new approach to this problem, which is based on geometric ideas and mapping properties of $\mathrm{SL}(2,\mathbb{R})$ matrices. This makes the resulting spectral statement particularly natural and easy to understand. On the other hand, our approach allows us to generalize the setting and pass from δ and δ' point interactions to the whole class of real connecting selfadjoint point interactions and hence develops the theory in the appropriate level of generality.

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The key idea will be the following. Fixing the boundary conditions of the spectral problem and considering an energy that is an eigenvalue for a given collection of parameters, we vary one of them while keeping the others fixed. How to vary the parameter is clear if δ or δ' point interactions are considered, but it is somewhat less clear in the case of general $\mathrm{SL}(2,\mathbb{R})$ matrices connecting the left- and right-limit of the solution data at the point in question. To this end, we will consider the Iwasawa decomposition of an $\mathrm{SL}(2,\mathbb{R})$ matrix, which expresses it as a canonical product of a parabolic, a hyperbolic, and an elliptic factor. This provides the parameters we seek and will vary. The next step is to investigate the stability question for the eigenvalue problem at hand when the parameter is varied. It turns out in most cases that there is a dichotomy. Either the eigenvalue is present for all values of the parameter, or it is present only for the one we started with and not for any other value. To establish this dichotomy we look at the projective action of the $\mathrm{SL}(2,\mathbb{R})$ matrix in question and are able to exhibit this dichotomy via direct and very simple calculations. Once the dichotomy corresponding to a single point interaction has been established, it will then be straightforward to process the entire family and to deduce a global result. The application to the case of random parameters is then also immediate.

Since they are crucial to our discussion, we will include discussions of the essential tools we use in Section 2, even though this material is well known. We hope that this will be useful for those readers who are less familiar with these tools in the context of spectral theory applications. This includes in particular the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$ matrices and their mapping properties on the real projective line. As a warm-up we consider the case of a single δ interaction in Section 3. Although this case has been studied before, we present our new perspective in this simple setting, partly to introduce the ideas, and partly to show how the known result can be proved with our method. In Section 4 we then consider the case of a general connecting point interaction, which is given by an $\mathrm{SL}(2,\mathbb{R})$ matrix. The three parameters describing such a matrix are given, in our representation, by the parameters corresponding to the three factors in the Iwasawa decomposition of the given matrix. We discuss the stability question for a given eigenvalue when two of the three parameters are fixed and the third is varied. Next, Section 5 considers the case of countably many general point interactions located on a discrete set inside the interval. Again, only one parameter for one interaction will be varied, while all other parameters are fixed, and the eigenvalue stability problem is investigated. Finally, we consider the case of countably many general point interactions with *random* parameters in Section 6 and prove a result in the spirit of Pastur and in the appropriate level of generality, that is, without assuming identical distribution. We do, however, make crucial use of independence. An important case where our result holds is when we have a measurable family of operators.

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2. Preliminaries

In this section we collect a few tools, all of which are well known. As usual, $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{GL}(2,\mathbb{R})$ denote the special and general linear groups respectively. We include this material for the sake of the reader. Anyone familiar with these concepts may skip ahead to the next section.

2.1. Transfer matrices

Let us discuss an elementary way to introduce the transfer matrices, which we emphasize is not the standard way of introducing them.

Consider an open interval $I=(a,b)\subset\mathbb{R}$, a L^1_{loc} potential $V:I\to\mathbb{R}$, and an energy $E\in\mathbb{R}$. The associated differential equation is

$$-u''(x)+V(x)u(x)=Eu(x),\quad x\in I. \tag{1}$$

Standard ODE theory shows that for each $x\in I$ and each $(v,d)^T\in\mathbb{R}^2$, there is a unique solution u of (1) with $(u(x),u'(x))^T=(v,d)^T$. Moreover, all real solutions of (1) arise in this way. See for example [16, Thm. 2.2.1]. This has the following immediate consequence.

PROPOSITION 2.1. *The set S_E of real solutions of* (1) *is a two-dimensional real vector space and, for each $x\in I$, the map*

$$M_{x,E}:S_E\to\mathbb{R}^2,\quad u\mapsto\begin{pmatrix}u(x)\\u'(x)\end{pmatrix}$$

is a linear isomorphism.

Proof. It follows directly from the definition of the map $M_{x,E}$ (and the linearity of differentiation) that it is linear. By the standard ODE results quoted above, it is both onto and one-to-one. This also implies the well-known fact that S_E is a two-dimensional real vector space. □

PROPOSITION 2.2. *For $x,y\in I$, there is a matrix $M(x,y;E)\in\mathrm{SL}(2,\mathbb{R})$ such that for every $u\in S_E$, we have*

$$\begin{pmatrix}u(x)\\u'(x)\end{pmatrix}=M(x,y;E)\begin{pmatrix}u(y)\\u'(y)\end{pmatrix}. \tag{2}$$

Proof. If we define $M(x,y;E):=M_{x,E}M_{y,E}^{-1}$, then (2) holds by Proposition 2.1. By construction, $M(x,y;E)\in\mathrm{GL}(2,\mathbb{R})$, so it remains to show that $\det M(x,y;E)=1$.

Consider the two solutions $u_D,u_N\in S_E$ with

$$\begin{pmatrix}u_N(y)&u_D(y)\\u_N'(y)&u_D'(y)\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Then,

$$\begin{aligned} M(x,y;E) &= M(x,y;E)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= M(x,y;E)\begin{pmatrix} u_N(y) & u_D(y) \\ u_N'(y) & u_D'(y) \end{pmatrix} \\ &= \begin{pmatrix} u_N(x) & u_D(x) \\ u_N'(x) & u_D'(x) \end{pmatrix}, \end{aligned}$$

and therefore

$$\begin{aligned} \det M(x,y;E) &= \det \begin{pmatrix} u_N(x) & u_D(x) \\ u_N'(x) & u_D'(x) \end{pmatrix} \\ &= u_N(x)u_D'(x) - u_D(x)u_N'(x) \\ &= u_N(y)u_D'(y) - u_D(y)u_N'(y) \\ &= 1. \end{aligned}$$

Here we used the constancy of the Wronskian, which follows from the fact that u_D, u_N solve (1):

$$\begin{aligned} (u_N(t)u_D'(t) - u_D(t)u_N'(t))' = \\ &= u_N'(t)u_D'(t) + u_N(t)u_D''(t) - u_D'(t)u_N'(t) - u_D(t)u_N''(t) \\ &= u_N(t)[(V(t)-E)u_D(t)] - u_D(t)[(V(t)-E)u_N(t)] \\ &= 0. \quad \square \end{aligned}$$

2.2. The real projective line

Recall that the real projective line $\mathbb{RP}^1$ is given by

$$\mathbb{RP}^1 = \{\text{lines in } \mathbb{R}^2 \text{ through the origin}\}.$$

Note that the elements of $\mathbb{RP}^1$ are equivalence classes with respect to the equivalence relation on $\mathbb{R}^2 \setminus \{0\}$ given by

$$v \sim w \quad \Leftrightarrow \quad \exists \lambda \in \mathbb{R} \setminus \{0\} : v = \lambda w.$$

DEFINITON 2.1. We denote the equivalence class of $v \in \mathbb{R}^2 \setminus \{0\}$ by $[v]$.

REMARK 2.1. Let $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$. Then, $[u] = [v]$ if and only if $arg(u_2 + iu_1) = arg(v_2 + iv_1) + k\pi$, $k \in \mathbb{Z}$.

LEMMA 2.1. *Any* $M \in \mathrm{GL}(2, \mathbb{R})$ *induces a well-defined bijective map from* $\mathbb{RP}^1$ *to* $\mathbb{RP}^1$*, which will be denoted by* $\tilde{M}$*, via*

$$\tilde{M}([v]) = [Mv].$$

Proof. Let $u \sim v$. Then $u = \lambda v$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ and

$$[Mu] = \tilde{M}[u] = \tilde{M}[\lambda v] = [M\lambda v] = [\lambda M v] = [Mv].$$

This shows that $\tilde{M}$ is well defined.

Let $[v] \in \mathbb{RP}^1$ with representative v. Since M is surjective by assumption, there exists $u \in \mathbb{R}^2$ such that $Mu = v$. Since

$$\tilde{M}([u]) = [Mu] = [v],$$

it follows that $\tilde{M}$ is surjective.

Finally, suppose $[Mu] = [Mv]$. Then there exists $k \in \mathbb{R} \setminus \{0\}$ such that $Mu = kMv$, and since M is injective by assumption, $u = kv$. Thus $[u] = [v]$ and $\tilde{M}$ is injective. □

2.3. The Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$ matrices

In this subsection we discuss the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$ matrices; compare [9]. We provide some details on how to obtain this decomposition for the reader's convenience.

We define the following subgroups of $\mathrm{SL}(2,\mathbb{R})$:

$$\mathscr{E} = \left\{ E_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\},$$
$$\mathscr{P} = \left\{ P_\alpha := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\},$$
$$\mathscr{H} = \left\{ H_r := \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} : r > 0 \right\}.$$

THEOREM 2.1. (Iwasawa decomposition) *Every $A \in \mathrm{SL}(2,\mathbb{R})$ can be written in a unique way as $A = P_\alpha H_r E_\theta$, where $P_\alpha \in \mathscr{P}$, $H_r \in \mathscr{H}$ and $E_\theta \in \mathscr{E}$.*

Proof. Consider the complex upper half-plane, $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$. Given $A \in \mathrm{SL}(2,\mathbb{R})$, we consider its action on $\mathbb{C}_+$ given by

$$A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

Note that $A \cdot z$ indeed belongs to $\mathbb{C}_+$ for each $z \in \mathbb{C}_+$ since

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)\Im z}{|cz+d|^2} = \frac{\Im z}{|cz+d|^2} > 0.$$

Moreover, note that

$$(A \cdot B) \cdot z = A \cdot (B \cdot z) \tag{3}$$

for all $A, B \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{C}_+$.

Consider the case $A\cdot i = i$, that is,

$$\frac{ai+b}{ci+d} = i \Leftrightarrow ai+b = di-c \Leftrightarrow a = d \text{ and } b = -c.$$

Thus the condition $\det A = ad - bc = 1$ becomes $a^2+c^2 = 1$ and we can choose $\theta \in \mathbb{R}$ with $a = \cos\theta$ and $c = \sin\theta$, so that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This discussion shows that $A\cdot i = i$ if and only if $A \in \mathscr{E}$.

Given any $A \in \mathrm{SL}(2,\mathbb{R})$, we consider $A\cdot i \in \mathbb{C}_+$ and set

$$\alpha := \Re(A\cdot i), \quad r := (\Im(A\cdot i))^{1/2}.$$

Then,

$$\begin{aligned} A\cdot i &= \alpha + ir^2 \\ &= \begin{pmatrix} r & \alpha/r \\ 0 & 1/r \end{pmatrix}\cdot i \\ &= \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\cdot i \end{aligned}$$

Thus, by (3),

$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}^{-1}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^{-1} A\cdot i = i,$$

which implies that

$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}^{-1}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^{-1} A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for a suitable $\theta \in \mathbb{R}$ by our discussion above. Thus,

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

as desired. This establishes existence.

To show uniqueness, consider the identity

$$\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_1 & 0 \\ 0 & 1/r_1 \end{pmatrix}\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_2 & 0 \\ 0 & 1/r_2 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

with $\alpha_1, \alpha_2, \theta_1, \theta_2 \in \mathbb{R}$ and $r_1, r_2 > 0$.

Applying both sides to $i \in \mathbb{C}_+$, we obtain

$$\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_1 & 0 \\ 0 & 1/r_1 \end{pmatrix}\cdot i = \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_2 & 0 \\ 0 & 1/r_2 \end{pmatrix}\cdot i,$$

which (by an observation above) is equivalent to

$$\alpha_1 + ir_1^2 = \alpha_2 + ir_2^2.$$

This implies $\alpha_1 = \alpha_2$ and $r_1 = r_2$ (since $r_1, r_2 > 0$). Once this holds, we must also have

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix},$$

proving uniqueness. □

REMARK 2.2. Since any matrix in $\mathrm{SL}(2,\mathbb{R})$ can be written as the inverse of the transpose of a matrix in $\mathrm{SL}(2,\mathbb{R})$, we also have the decomposition

$$A = \begin{pmatrix} 1 & 0 \\ -\tilde{\alpha} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\tilde{r}} & 0 \\ 0 & \tilde{r} \end{pmatrix} \begin{pmatrix} \cos\tilde{\theta} & -\sin\tilde{\theta} \\ \sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix}$$

for some $\tilde{\alpha} \in \mathbb{R}$, $\tilde{r} > 0$ and $\tilde{\theta} \in \mathbb{R}$.

2.4. The differential operator and its eigenvalues

For a finite closed interval $I = [a,b]$ and a real-valued $V \in L^1(I)$, consider the associated differential expression defined by

$$\tau f := -f'' + Vf.$$

For all $x, y \in I$, let $M(x,y;E)$ be the transfer matrix defined in Proposition 2.2. Then $M(x,y;E) \in \mathrm{SL}(2,\mathbb{R})$ and for every real solution of $\tau u = Eu$, we have

$$\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = M(x,y;E) \begin{pmatrix} u(y) \\ u'(y) \end{pmatrix}.$$

Let $T_{\theta,\gamma}$ be the selfadjoint operator defined by

$$T_{\theta,\gamma} f = \tau f$$

with domain

$$D(T_{\theta,\gamma}) := \{f \in L^2(I) : f, f' \text{abs. con. on} I, \tau f \in L^2(I)$$
$$f(a)\cos\theta - f'(a)\sin\theta = 0$$
$$f(b)\cos\gamma - f'(b)\sin\gamma = 0\}.$$

See Theorem 8.25 a) and Theorem 8.26 in [18].

As an application of Lemma 2.1 we will prove the following well-known result.

THEOREM 2.2. *Let $E \in \mathbb{R}$, then for each $\theta \in [0,\pi)$ $(\gamma \in [0,\pi))$, there exists a unique $\gamma \in [0,\pi)$ $(\theta \in [0,\pi))$ such that $E \in \sigma_p(T_{\theta,\gamma})$.*

Proof. For $E \in \mathbb{R}$ and $\theta \in [0,\pi)$, there exists a non-trivial solution $u \in L^2(I)$ of $\tau u = Eu$, which is unique up to a non-zero multiple, satisfying

$$u(a)\cos\theta - u'(a)\sin\theta = 0.$$

Since $M(b,a;E) \in \mathrm{SL}(2,\mathbb{R})$, there exists a unique vector $(u(b),u'(b))^T$ satisfying

$$\begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = M(b,a;E)\begin{pmatrix} u(a) \\ u'(a) \end{pmatrix}$$

Let $\gamma := \arctan \frac{u(b)}{u'(b)}$ be the angle of the vector $(u(b),u'(b))^T$. Then,

$$u(b)\cos\gamma - u'(b)\sin\gamma = 0.$$

Therefore, $E \in \sigma_p(T_{\theta,\gamma})$.

Assume $\tilde{\gamma} \in [0,\pi)$, $\tilde{\gamma} \neq \gamma$ and $E \in \sigma_p(T_{\theta,\tilde{\gamma}})$. Then there exists a non-zero $v \in D(T_{\theta,\tilde{\gamma}})$ such that $\tau v = Ev$,

$$v(a)\cos\theta - v'(a)\sin\theta = 0,$$

$$v(b)\cos\tilde{\gamma} - v'(b)\sin\tilde{\gamma} = 0.$$

Thus the angle of the vector $(v(a),v'(a))^T$ is θ and the angle of the vector $(v(b),v'(b))^T$ is $\tilde{\gamma}$. Then by Lemma 2.1,

$$\tilde{M}(a,b;E)[(v(b),v'(b))^T] = [(u(a),u'(a))^T] = \tilde{M}(a,b;E)[(u(b),u'(b))^T]$$

the vectors $(v(b),v'(b))^T$ and $(u(b),u'(b))^T$ must belong to the same element of the real projective line, i.e. they must have the same angle, so that $\gamma = \tilde{\gamma}$. Analogously, for each $\gamma \in [0,\pi)$, there exists a unique $\theta \in [0,\pi)$ such that $E \in \sigma_p(T_{\theta,\gamma})$. □

COROLLARY 2.1. *Let γ fixed and θ such that $E \in \sigma_p(T_{\theta,\gamma})$, then for any $\tilde{\theta} \neq \theta$ one has $E \notin \sigma_p(T_{\tilde{\theta},\gamma})$.*

3. The case of a single δ-interaction

As a warm-up we consider the case of a single δ-interaction.

Let $I = [a,b] \subset \mathbb{R}$ be a closed finite interval, $V \in L^1(I)$ real valued, $p \in I$ an interior point, and $\alpha \in \mathbb{R}$.

We consider the formal differential expressions

$$\tau := -\frac{d^2}{dx^2} + V$$

and

$$\tau_{\alpha,p} := -\frac{d^2}{dx^2} + V + \alpha\delta(x-p).$$

The maximal operator $T_{\alpha,p}$ corresponding to $\tau_{\alpha,p}$ is defined by

$$T_{\alpha,p}f = \tau f$$

$$D(T_{\alpha,p}) = \Big\{ f \in L^2(I) : f, f' \text{ abs. cont in } I \setminus \{p\}, -f'' + Vf \in L^2(I), \begin{pmatrix} f(p+) \\ f'(p+) \end{pmatrix} = A_{\alpha,p} \begin{pmatrix} f(p-) \\ f'(p-) \end{pmatrix} \Big\}.$$

REMARK 3.1. Note that the restriction of any absolutely continuous function to a bounded interval is of bounded variation and therefore the limits from the right and from the left at any point exist, see Section 8.15 in [13].

Here, $A_{\alpha,p}$ is the $\mathrm{SL}(2,\mathbb{R})$ matrix defined by

$$A_{\alpha,p} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}. \tag{4}$$

Let us consider the selfadjoint restriction $H_{\alpha,p}$ of $T_{\alpha,p}$ in $L^2(I)$, see Theorem 5.2 in [3], defined by

$$H_{\alpha,p}f = \tau f \tag{5}$$

$$D(H_{\alpha,p}) = \left\{ f \in D(T_{\alpha,p}) : \begin{matrix} f(a)\cos\theta + f'(a)\sin\theta = 0 \\ f(b)\cos\gamma + f'(b)\sin\gamma = 0 \end{matrix} \right\} \qquad \theta, \gamma \in [0,\pi).$$

THEOREM 3.1. *Let $E \in \sigma_p(H_{\alpha,p})$. Then one of the following holds:*

i) $E \in \sigma_p(H_{\tilde{\alpha},p})$ *for every* $\tilde{\alpha} \in \mathbb{R}$,

ii) $E \notin \sigma_p(H_{\tilde{\alpha},p})$ *for every* $\tilde{\alpha} \in \mathbb{R} \setminus \{\alpha\}$.

Proof. Note first that on the level of transfer matrices, the local point interaction inserts the factor (4) between $M(y,p+;E)$ and $M(p-,x;E)$ for $a \leqslant x < p < y \leqslant b$.

Let $E \in \mathbb{R}$ be such that $E \in \sigma_p(H_{\alpha,p})$. Then there exists a non-zero $u \in D(H_{\alpha,p})$ with $H_{\alpha,p}u = Eu$. In particular, we have

$$\begin{pmatrix} u(p+) \\ u'(p+) \end{pmatrix} = A_{\alpha,p} \begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix}.$$

Suppose *ii*) fails; and hence we have to prove *i*). Let $E \in \sigma_p(H_{\tilde{\alpha},p})$ for some $\tilde{\alpha} \in \mathbb{R} \setminus \{\alpha\}$. There exists a non-zero $v \in D(H_{\tilde{\alpha},p})$ such that $H_{\tilde{\alpha},p}v = Ev$. Since $M = M(p-,a;E) \in \mathrm{SL}(2,\mathbb{R})$ and $[(u(a),u'(a))^T] = [(v(a),v'(a))^T]$, we have

$$[(u(p-),u'(p-))^T] = \tilde{M}([(u(a),u'(a))^T]) = \tilde{M}([(v(a),v'(a))^T]) = [(v(p-),v'(p-))^T].$$

Thus there exists $k \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix} = k \begin{pmatrix} v(p-) \\ v'(p-) \end{pmatrix},$$

and since $u \in D(H_{\alpha,p})$ and $v \in D(H_{\tilde{\alpha},p})$,

$$\begin{pmatrix} u(p-) \\ (\alpha-\tilde{\alpha})u(p-)+u'(p-) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha-\tilde{\alpha} & 1 \end{pmatrix} \begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix} = A_{\tilde{\alpha},p}^{-1} A_{\alpha,p} \begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix}$$

$$= A_{\tilde{\alpha},p}^{-1} \begin{pmatrix} u(p+) \\ u'(p+) \end{pmatrix} = kA_{\tilde{\alpha},p}^{-1} \begin{pmatrix} v(p+) \\ v'(p+) \end{pmatrix} = k \begin{pmatrix} v(p-) \\ v'(p-) \end{pmatrix} = \begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix}$$

and then $(\alpha-\tilde{\alpha})u(p-)+u'(p-) = u'(p-)$. Since $\tilde{\alpha} \neq \alpha$, $u(p-)=0$. Thus $\forall \tilde{\alpha} \in \mathbb{R}$,

$$\begin{pmatrix} u(p+) \\ u'(p+) \end{pmatrix} = \begin{pmatrix} 0 \\ u'(p+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 0 \\ u'(p-) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tilde{\alpha} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ u'(p-) \end{pmatrix} = A_{\tilde{\alpha},p} \begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix}$$

Therefore $u \in \sigma_p(H_{\tilde{\alpha},p})$, $\forall \tilde{\alpha} \neq \alpha$ and $i)$ holds. □

4. The case of a single general point interaction

Now we construct the operator with one general point interaction. The case we are going to consider in this and in the following sections corresponds to the case of connecting real selfadjoint boundary conditions, see Definition 2.5 in [3]. In more general cases, an additional phase factor may appear in front of the transfer matrix, see Theorem 1 in [1].

Let $I = [a,b] \subset \mathbb{R}$ be a closed finite interval. Let $V \in L^1(I)$ be a real-valued function, $p \in I$ an interior point and $A_{\alpha,r,\theta} \in \mathrm{SL}(2,\mathbb{R})$ with Iwasawa decomposition $A_{\alpha,r,\theta} = P_\alpha H_r E_\theta$, where $P_\alpha \in \mathscr{P}$, $H_r \in \mathscr{H}$ and $E_\theta \in \mathscr{E}$. We consider the formal differential expression

$$\tau := -\frac{d^2}{dx^2} + V.$$

DEFINITON 4.1. The corresponding maximal operator $T_{\alpha,r,\theta}$ is defined by

$$T_{\alpha,r,\theta} f = \tau f$$

$$D(T_{\alpha,r,\theta}) = \Big\{ f \in L^2(I) : f, f' \text{ abs. cont in } I \setminus \{p\}, -f'' + Vf \in L^2(I),$$

$$\begin{pmatrix} f(p+) \\ f'(p+) \end{pmatrix} = A_{\alpha,r,\theta} \begin{pmatrix} f(p-) \\ f'(p-) \end{pmatrix} \Big\}.$$

Let us consider the selfadjoint restriction $H^{\delta,\gamma}_{\alpha,r,\theta}$ of $T_{\alpha,r,\theta}$ in $L^2(I)$, see equation (4.3) in [15], defined below

DEFINITON 4.2. Let $H^{\delta,\gamma}_{\alpha,r,\theta}$ be the operator defined by

$$H^{\delta,\gamma}_{\alpha,r,\theta} f = \tau f$$

$$D(H^{\delta,\gamma}_{\alpha,r,\theta}) = \left\{ f \in D(T_{\alpha,r,\theta}) : \begin{matrix} f(a)\cos\delta + f'(a)\sin\delta = 0 \\ f(b)\cos\gamma + f'(b)\sin\gamma = 0 \end{matrix} \right\}, \quad \delta, \gamma \in [0,\pi).$$

LEMMA 4.1. *Let* $\theta,\tilde{\theta}\in\mathbb{R}$ *and fix* $v\in\mathbb{R}^2$. *The following holds:* $\tilde{\theta}\neq\theta+k\pi$, $k\in\mathbb{Z}$ *if and only if* $[A_{\alpha,r,\theta}v]\neq[A_{\alpha,r,\tilde{\theta}}v]$.

Proof.

$\Rightarrow$) Let $\tilde{\theta}\neq\theta+k\pi$, $k\in\mathbb{Z}$ and $v\in\mathbb{R}^2$. Since E_γ acts as a rotation matrix through an angle γ, we have $[E_\theta v]\neq[E_{\tilde{\theta}}v]$. Taking into account that $P_\alpha H_r\in \mathrm{SL}(2,\mathbb{R})$, Lemma 2.1 gives that $[A_{\alpha,r,\theta}v]\neq[A_{\alpha,r,\tilde{\theta}}v]$.

$\Leftarrow$) Suppose now $[A_{\alpha,r,\theta}v]\neq[A_{\alpha,r,\tilde{\theta}}v]$. Recalling the definition introduced in Lemma 2.1,

$$\widetilde{P_\alpha H_r}[E_{\tilde{\theta}}v]=[P_\alpha H_r E_{\tilde{\theta}}v]=[A_{\alpha,r,\tilde{\theta}}v]\neq[A_{\alpha,r,\theta}v]=\widetilde{P_\alpha H_r}[E_\theta v],$$

since $\widetilde{P_\alpha H_r}$ is injective. Thus $[E_\theta v]\neq[E_{\tilde{\theta}}v]$ and hence $\tilde{\theta}\neq\theta+k\pi$, $k\in\mathbb{Z}$. □

LEMMA 4.2. *Let* $r,\tilde{r}>0$, $\tilde{r}\neq r$ *and* $v\in\mathbb{R}^2$. *The following are equivalent:*

i) $[v]=[(\sin\theta,\cos\theta)^T]$ *or* $[v]=[(\cos\theta,-\sin\theta)^T]$,

ii) $[A_{\alpha,r,\theta}v]=[A_{\alpha,\tilde{r},\theta}v]$.

Proof.

ii) $\Rightarrow$ *i*) Assume $[A_{\alpha,r,\theta}v]=[A_{\alpha,\tilde{r},\theta}v]$. Then $\tilde{P}_\alpha[H_rE_\theta v]=\tilde{P}_\alpha[H_{\tilde{r}}E_\theta v]$. Since by Lemma 2.1 $\tilde{P}_\alpha$ is injective, $[H_rE_\theta v]=[H_{\tilde{r}}E_\theta v]$, that is, there exists $k\in\mathbb{R}\setminus\{0\}$ such that $H_rE_\theta v=kH_{\tilde{r}}E_\theta v$. Thus, $H_{\tilde{r}}^{-1}H_rE_\theta v=kE_\theta v$, and therefore $E_\theta v$ is eigenvector of the diagonal matrix $H_{\tilde{r}}^{-1}H_r$. Since $r\neq\tilde{r}$, the eigenvectors are multiples of $[0,1]^T$ or $[1,0]^T$. Then $[E_\theta v]=[(1,0)^T]$ or $[E_\theta v]=[(0,1)^T]$, taking into account that

$$E_\theta^{-1}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

and Lemma 2.1, we obtain $[v]=[(\sin\theta,\cos\theta)^T]$ or $[v]=[(\cos\theta,-\sin\theta)^T]$.

i) $\Rightarrow$ *ii*) Assume $[v]=[(\sin\theta,\cos\theta)^T]$. Then,

$$\begin{aligned}[A_{\alpha,r,\theta}v]&=[P_\alpha H_r(0,1)^T]=\left[\frac{1}{r}P_\alpha(0,1)^T\right]=\left[\frac{1}{\tilde{r}}P_\alpha(0,1)^T\right]\\&=[P_\alpha H_{\tilde{r}}(0,1)^T]=[A_{\alpha,\tilde{r},\theta}v].\end{aligned}$$

When $[v]=[(\cos\theta,-\sin\theta)^T]$, the result follows in an analogous way. □

LEMMA 4.3. *Let* $\alpha,\tilde{\alpha}\in\mathbb{R}$, $\tilde{\alpha}\neq\alpha$ *and* $v\in\mathbb{R}^2$. *The following are equivalent:*

i) $[v]=[(\cos\theta,-\sin\theta)^T]$,

ii) $[A_{\alpha,r,\theta}v]=[A_{\tilde{\alpha},r,\theta}v]$.

Proof.

ii) $\Rightarrow$ *i)* Assume $[A_{\alpha,r,\theta}v] = [A_{\tilde{\alpha},r,\theta}v]$, that is, there exists $k \in \mathbb{R}\setminus\{0\}$ such that $P_\alpha H_r E_\theta v = kP_{\tilde{\alpha}} H_r E_\theta v$. Then $P_{\tilde{\alpha}}^{-1} P_\alpha H_r E_\theta v = kH_r E_\theta v$. Thus $H_r E_\theta v$ is an eigenvector of the matrix $P_{\tilde{\alpha}}^{-1} P_\alpha$. Since $\alpha \neq \tilde{\alpha}$, $P_{\tilde{\alpha}}^{-1} P_\alpha \neq I$ and its eigenvectors are multiples of $(1,0)^T$. Then $[H_r E_\theta v] = [(1,0)^T]$, and taking into account that

$$(H_r E_\theta)^{-1} = \begin{pmatrix} \frac{1}{r}\cos\theta & r\sin\theta \\ -\frac{1}{r}\sin\theta & r\cos\theta \end{pmatrix}$$

as well as Lemma 2.1, we obtain $[v] = [\frac{1}{r}(\cos\theta, -\sin\theta)^T] = [(\cos\theta, -\sin\theta)^T]$.

i) $\Rightarrow$ *ii)* Assume $[v] = [(\cos\theta, -\sin\theta)]$. Then

$$[A_{\alpha,r,\theta}v] = [P_\alpha r(1,0)^T] = [r(1,0)^T] = [P_{\tilde{\alpha}} r(1,0)^T] = [A_{\tilde{\alpha},r,\theta}v]. \quad \square$$

THEOREM 4.1. *Let $E \in \mathbb{R}$. If $E \in \sigma_p(H^{\delta,\gamma}_{\alpha,r,\theta})$, then:*

a) $E \in \sigma_p(H^{\delta,\gamma}_{\alpha,r,\tilde{\theta}})$ if and only if $\tilde{\theta} = \theta + k\pi$, $k \in \mathbb{Z}$.

b) One of the following holds:

i) $E \notin \sigma_p(H^{\delta,\gamma}_{\alpha,\tilde{r},\theta})$ for every $\tilde{r} \neq r$.

ii) $E \in \sigma_p(H^{\delta,\gamma}_{\alpha,\tilde{r},\theta})$ for every $\tilde{r} > 0$.

c) One of the following holds:

i) $E \notin \sigma_p(H^{\delta,\gamma}_{\tilde{\alpha},r,\theta})$ for every $\tilde{\alpha} \neq \alpha$.

ii) $E \in \sigma_p(H^{\delta,\gamma}_{\tilde{\alpha},r,\theta})$ for every $\tilde{\alpha} \in \mathbb{R}$.

Proof. Since $E \in \sigma_p(H^{\delta,\gamma}_{\alpha,r,\theta})$, there exists $u \in L^2(a,b)$, $u \neq 0$, such that $u \in D(H^{\delta,\gamma}_{\alpha,r,\theta})$ and $H^{\delta,\gamma}_{\alpha,r,\theta}u = Eu$.

a) Suppose now that $E \in \sigma_p(H^{\delta,\gamma}_{\alpha,r,\tilde{\theta}})$ for some $\tilde{\theta} \neq \theta$. Then there exists $v \in L^2(a,b)$, $v \neq 0$, such that $v \in D(H^{\delta,\gamma}_{\alpha,r,\tilde{\theta}})$ and $H^{\delta,\gamma}_{\alpha,r,\tilde{\theta}}v = Ev$. We will now consider the matrices $M(p-,a;E)$ and $M(b,p+;E)$, which do not depend on α, r and θ. Since $M := M(p-,a;E) \in \mathrm{SL}(2,\mathbb{R})$ and $[(u(a),u'(a))^T] = [(v(a),v'(a))^T]$, we have

$$\begin{aligned}[(u(p-),u'(p-))^T] &= \tilde{M}([(u(a),u'(a))^T]) = \tilde{M}([(v(a),v'(a))^T]) \\ &= [(v(p-),v'(p-))^T].\end{aligned}$$

Thus there exists $\lambda \in \mathbb{R}\setminus\{0\}$ such that

$$\begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix} = \lambda \begin{pmatrix} v(p-) \\ v'(p-) \end{pmatrix}.$$

Analogously, since $M^{-1}(b,p+;E) \in \mathrm{SL}(2,\mathbb{R})$, there exists $\mu \in \mathbb{R}\setminus\{0\}$ such that

$$\begin{pmatrix} u(p+) \\ u'(p+) \end{pmatrix} = \mu \begin{pmatrix} v(p+) \\ v'(p+) \end{pmatrix}.$$

Since

$$\begin{pmatrix} u(p+) \\ u'(p+) \end{pmatrix} = A_{\alpha,r,\theta} \begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} v(p+) \\ v'(p+) \end{pmatrix} = A_{\alpha,r,\tilde{\theta}} \begin{pmatrix} v(p-) \\ v'(p-) \end{pmatrix},$$

we have

$$[A_{\alpha,r,\theta}(u(p-),u'(p-))^T] = [A_{\alpha,r,\tilde{\theta}}(v(p-),v'(p-))^T] = [A_{\alpha,r,\tilde{\theta}}(u(p-),u'(p-))^T].$$

By Lemma 4.1 this happens if and only if $\tilde{\theta} = \theta + k\pi$, $k \in \mathbb{Z}$.

b) Let us assume that $i)$ is false. Then for some $r_0 \neq r$, there is $E \in \sigma_p(H^{\delta,\gamma}_{\alpha,r_0,\theta})$. Therefore there exists a non-zero $v \in L^2(a,b)$ such that $v \in D(H^{\delta,\gamma}_{\alpha,r_0,\theta})$ and $H^{\delta,\gamma}_{\alpha,r_0,\theta} v = Ev$. As in case $a)$ above we conclude

$$[A_{\alpha,r,\theta}(u(p-),u'(p-))^T] = [A_{\alpha,r_0,\theta}(v(p-),v'(p-))^T] = [A_{\alpha,r_0,\theta}(u(p-),u'(p-))^T].$$

By Lemma 4.2 this happens if and only if $[(u(p-),u'(p-))^T] = [(\sin\theta,\cos\theta)^T]$ or $[(u(p-),u'(p-))^T] = [(\cos\theta,-\sin\theta)^T]$.
Let us assume that $[(u(p-),u'(p-))^T] = [(\sin\theta,\cos\theta)^T]$. If $[(u(p-),u'(p-))^T] = [(\cos\theta,-\sin\theta)^T]$, the argument proceeds analogously. There exists $c \in \mathbb{R}\setminus\{0\}$ such that $(u(p-),u'(p-))^T = c(\sin\theta,\cos\theta)^T$. We normalize and take $c=1$. Let us verify that for each $\tilde{r} > 0$, $E \in \sigma(H^{\delta,\gamma}_{\alpha,\tilde{r},\theta})$ with eigenvector

$$w(x) := \begin{cases} \frac{\tilde{r}}{r} u(x) \text{ if } a \leqslant x < p \\ u(x) \ \text{ if } p < x \leqslant b \end{cases}$$

First notice that w satisfies the conditions at a and b of the functions in $D(H^{\delta,\gamma}_{\alpha,r,\theta})$ since u satisfies these conditions too. Now

$$\begin{aligned} A_{\alpha,\tilde{r},\theta} \begin{pmatrix} w(p-) \\ w'(p-) \end{pmatrix} &= A_{\alpha,\tilde{r},\theta} \left(\frac{\tilde{r}}{r} \begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix} \right) = \frac{\tilde{r}}{r} A_{\alpha,\tilde{r},\theta} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \\ &= \frac{\tilde{r}}{r}\frac{1}{\tilde{r}} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \frac{1}{r} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = A_{\alpha,r,\theta} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} u(p+) \\ u'(p+) \end{pmatrix} = \begin{pmatrix} w(p+) \\ w'(p+) \end{pmatrix}. \end{aligned}$$

The first and second equalities hold by definition of w and u, the next three equalities are straightforward calculations. The equality before the last one follows because $u \in D(H^{\delta,\gamma}_{\alpha,r,\theta})$ and the last one follows because $w = u$ to the right of p. Therefore $w \in D(H^{\delta,\gamma}_{\alpha,\tilde{r},\theta})$, $\tau w = Ew$ in $[a,b] \setminus \{p\}$, and E is an eigenvalue for $H^{\delta,\gamma}_{\alpha,\tilde{r},\theta}$, $\tilde{r} > 0$.

c) Let us assume that $i)$ is false. Then for some $\alpha_0 \neq \alpha$, there is $E \in \sigma_p(H^{\delta,\gamma}_{\alpha_0,r,\theta})$. Therefore there exists $v \in L^2(a,b)$, $v \neq 0$, such that $v \in D(H^{\delta,\gamma}_{\alpha_0,r,\theta})$ and $H^{\delta,\gamma}_{\alpha_0,r,\theta} v = Ev$. As in case a) above we conclude

$$[A_{\alpha,r,\theta}(u(p-),u'(p-))^T] = [A_{\alpha_0,r,\theta}(v(p-),v'(p-))^T] = [A_{\alpha_0,r,\theta}(u(p-),u'(p-))^T].$$

By Lemma 4.3 this happens if and only if $[(u(p-),u'(p-))^T] = [(\cos\theta, -\sin\theta)^T]$. There exists $c \in \mathbb{R} \setminus \{0\}$ such that $(u(p-),u'(p-))^T = c(\sin\theta, \cos\theta)^T$. We normalize and take $c = 1$. For all $\tilde{\alpha} \in \mathbb{R}$,

$$A_{\tilde{\alpha},r,\theta}\begin{pmatrix} u(p-) \\ u'(p-) \end{pmatrix} = A_{\tilde{\alpha},r,\theta}\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = r\begin{pmatrix} 1 & \tilde{\alpha} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, for every $\tilde{\alpha} \in \mathbb{R}$, $u \in D(H^{\delta,\gamma}_{\tilde{\alpha},r,\theta})$ and $H^{\delta,\gamma}_{\tilde{\alpha},r,\theta} u = Eu$. $\square$

5. The case of countably many general point interactions

Let $-\infty \leqslant a < b \leqslant \infty$ and let $V \in L^1_{\mathrm{loc}}(a,b)$ be a real-valued function. Fix a set of points $M = \{x_n\}_{n \in \mathfrak{I}} \subset (a,b)$, where $\mathfrak{I} \subseteq \mathbb{Z}$. We assume that the discrete set M accumulates at most at a or b. Let $\Lambda := \{\alpha_n\} \subset \mathbb{R}$, $R := \{r_n\} \subset (0,\infty)$ and $\Theta := \{\theta_n\} \subset \mathbb{R}$.

DEFINITON 5.1. Let $A_{\alpha_n,r_n,\theta_n} \in \mathrm{SL}(2,\mathbb{R})$ with Iwasawa decomposition $A_{\alpha_n,r_n,\theta_n} = P_{\alpha_n} H_{r_n} E_{\theta_n}$, where $P_{\alpha_n} \in \mathscr{P}$, $H_{r_n} \in \mathscr{H}$ and $E_{\theta_n} \in \mathscr{E}$ for every $n \in \mathfrak{I}$.

We consider the formal differential expression

$$\tau := -\frac{d^2}{dx^2} + V.$$

DEFINITON 5.2. The maximal operator $T_{\Lambda,R,\Theta}$ is defined by

$$T_{\Lambda,R,\Theta} f = \tau f$$

$$D(T_{\Lambda,R,\Theta}) = \Big\{ f \in L^2(a,b) : f, f' \text{ abs. cont in } (a,b) \setminus M, -f'' + Vf \in L^2(a,b),$$
$$\begin{pmatrix} f(x_n+) \\ f'(x_n+) \end{pmatrix} = A_{\alpha_n,r_n,\theta_n}\begin{pmatrix} f(x_n-) \\ f'(x_n-) \end{pmatrix} \forall n \in \mathfrak{I} \Big\}$$

(See Remark 3.1)

DEFINITON 5.3. Given $g \in L^1_{\text{loc}}(a,b)$ and $z \in \mathbb{C}$, we call f a solution of $(\tau_{\Lambda,R,\Theta} - z)f = g$ if f and f' are absolutely continuous in $(a,b) \setminus M$ with $-f'' + Vf - zf = g$ and

$$\begin{pmatrix} f(x_n+) \\ f'(x_n+) \end{pmatrix} = A_{\alpha_n,r_n,\theta_n} \begin{pmatrix} f(x_n-) \\ f'(x_n-) \end{pmatrix} \forall n \in \mathfrak{J}.$$

DEFINITON 5.4. We define the Wronskian of two solutions u_1 and u_2 of $(\tau_{\Lambda,R,\Theta} - z)f = 0$ by

$$W_x(u_1,u_2) = u_1(x+)u_2'(x+) - u_1'(x+)u_2(x+).$$

DEFINITON 5.5. For f and g absolutely continuous in $(a,b) \setminus M$, we define the Lagrange bracket by

$$[f,g]_x := \overline{f(x+)}g'(x+) - \overline{f'(x+)}g(x+) \qquad x \in (a,b)$$

If $f,g \in D(T_{\Lambda,R,\Theta})$, then the limits

$$[f,g]_a := \lim_{x \to a+} [f,g]_x \quad \text{and} \quad [f,g]_b := \lim_{x \to b-} [f,g]_x$$

exist; see [3, Theorem 2.2].

A solution of $(\tau_{\Lambda,R,\Theta} - z)f = 0$ is said to lie right (resp., left) in $L^2(a,b)$ if f is square integrable in a neighborhood of b (resp., a).

DEFINITON 5.6.

i) $\tau_{\Lambda,R,\Theta}$ is in the *limit circle case* (lcc) at b if for every $z \in \mathbb{C}$, all solutions of $(\tau_{\Lambda,R,\Theta} - z)f = 0$ lie right in $L^2(a,b)$.

ii) $\tau_{\Lambda,R,\theta}$ is in the *limit point case* (lpc) at b if for every $z \in \mathbb{C}$, there is at least one solution of $(\tau_{\Lambda,R,\Theta} - z)f = 0$ not lying right in $L^2(a,b)$.

The same definition applies to the endpoint a.

According to the *Weyl alternative*, see [3, Theorem 4.4], we have always either *i)* or *ii)*.

In Theorem 5.2 in [3] is stated that the operator $H_{\Lambda,R,\Theta}$, defined below, is a self-adjoint restriction of the maximal operator $T_{\Lambda,R,\Theta}$ introduced in Definition 5.2.

DEFINITON 5.7. Let $H_{\Lambda,R,\Theta}$ be the operator defined as

$$H_{\Lambda,R,\Theta} f = \tau f$$

$$D(H_{\Lambda,R,\Theta}) = \left\{ f \in D(T_{\Lambda,R,\Theta}) : \begin{array}{l} [v,f]_a = 0 \text{ if } \tau_{\Lambda,R,\Theta} \text{ lcc at } a \\ [w,f]_b = 0 \text{ if } \tau_{\Lambda,R,\Theta} \text{ lcc at } b \end{array} \right\},$$

where v and w are non-trivial real solutions of $(\tau_{\Lambda,R,\Theta} - \lambda)v = 0$ near a and near b, respectively, $\lambda \in \mathbb{R}$.

If $\tau_{\Lambda,R,\Theta}$ is in the lcc at b, then w lies right in $L^2(a,b)$ and therefore $[w,f]_b$ is well defined and the same holds for a and v. If $\tau_{\Lambda,R,\Theta}$ is in the lpc at a or b we do not require any conditions at those points.

REMARK 5.1. For the existence of the boundary values $[v,f]_a$ and $[w,f]_b$ see Theorem 2.2 a) in [3]. In Theorem 5.2 in [3], the selfadjoint restrictions are characterized using unitary matrices. The case we are treating corresponds to the particular case of the connecting real selfadjoint boundary conditions. For construction of selfadjoint restrictions see also [10, pp. 216], [5, Section 15], [6, Theorem 2.2], [14, Section 3] and [1, Theorem 1].

REMARK 5.2. Whenever we fix a parameter, we do not write it. For example if we fix R and Θ we shall just write H_Λ and analogously for the other cases.

DEFINITON 5.8. We say that $\tau_{\Lambda,R,\Theta}$ is regular at a if a is finite, $V \in L^1_{\text{loc}}[a,b)$ and a is not an accumulation point of M. The same definition applies to the endpoint b.

If $\tau_{\Lambda,R,\Theta}$ is regular at a, then $\tau_{\Lambda,R,\Theta}$ is lcc at a and the condition $[v,f]_a = 0$ can be replaced by

$$f(a)\cos\psi + f'(a)\sin\psi = 0$$

for $\psi \in [0,\pi)$. The same holds for b.

In the rest of this section for the operator $H_{\Lambda,R,\Theta}$ of Definition 5.7 we are going to fix the values of α_n, r_n and θ_n for $n \neq n_0$. Set $\alpha = \alpha_{n_0}$, $r = r_{n_0}$ and $\theta = \theta_{n_0}$. This operator will be denoted by $H_{\alpha,r,\theta}$, where only the values that Λ, R and Θ take on n_0 are made explicit.

Let $E \in \mathbb{R}$ be fixed and define

$$P(E) := \{(\alpha,r,\theta) \in \mathbb{R} \times (0,\infty) \times \mathbb{R} : E \in \sigma_p(H_{\alpha,r,\theta})\}.$$

Now let us take the operator $H^{\delta,\gamma}_{\alpha,r,\theta}$ introduced in Definition 4.2 with $p = x_{n_0}$ and $I = [c,d] \subset (a,b)$ such that $[c,d] \cap M = \{x_{n_0}\}$ (Note that we have changed the notation for the endpoints of the interval I).

LEMMA 5.1. *There exist fixed* $\delta_0,\ \gamma_0 \in [0,\pi)$ *such that for all* $(\alpha,r,\theta) \in P(E)$, *it happens that* $E \in \sigma_p(H^{\delta_0,\gamma_0}_{\alpha,r,\theta})$.

Proof. If $(\alpha_1,r_1,\theta_1) \in P(E)$, then for some *non-zero* $\varphi \in D(H_{\alpha_1,r_1,\theta_1})$, $H_{\alpha_1,r_1,\theta_1}\varphi = E\varphi$.

Let us fix the points $\delta_0, \gamma_0 \in [0,\pi)$ *where*

$$\begin{aligned} \varphi(c)\cos\delta_0 + \varphi'(c)\sin\delta_0 &= 0 \\ \varphi(d)\cos\gamma_0 + \varphi'(d)\sin\gamma_0 &= 0 \end{aligned} \tag{6}$$

If $(\alpha,r,\theta)\in P(E)$ *is such that* $(\alpha,r,\theta)=(\alpha_1,r_1,\theta_1)$*, the assertion follows.*

If $(\alpha,r,\theta)\in P(E)$ *but* $(\alpha,r,\theta)\neq(\alpha_1,r_1,\theta_1)$*, then* $H_{\alpha,r,\theta}\psi=E\psi$ *for some non-zero* $\psi\in D(H_{\alpha,r,\theta})$*. Therefore, there exist* $\delta,\gamma\in[0,\pi)$ *which satisfy the boundary conditions at c and d for* ψ*, similar to (6). If we prove that* $\delta=\delta_0$ *and* $\gamma=\gamma_0$*, then* $H^{\delta_0,\gamma_0}_{\alpha,r,\theta}\psi=E\psi$ *and therefore* $E\in\sigma(H^{\delta_0,\gamma_0}_{\alpha,r,\theta})$*.*

Let us prove that $\gamma=\gamma_0$*. The proof for* δ *is analogous.*

a) Assume $\tau_{\alpha,r,\theta}$ *is in the limit circle case at* b*.*

The Wronskian satisfies $W_x(w,\varphi)=[w,\varphi]_x$ *and* $W_x(w,\psi)=[w,\psi]_x$ *because w is real. It is constant for* $x\in[d,b)$ *since* w,ψ *and* φ *are solutions of* $\tau_{\alpha,r,\theta}f=Ef$ *in the interval* $[d,b)$ *because* x_{n_0} *does not intersect* $[d,b)$*. By hypothesis, the functions* φ *and* ψ *satisfy the lcc condition at b. This implies*

$$0=[w,\psi]_b=\lim_{x\to b-}W_x(w,\psi)\qquad\text{and}\qquad 0=[w,\varphi]_b=\lim_{x\to b-}W_x(w,\varphi)$$

Therefore $W_x(w,\psi)=W_x(w,\varphi)=0$ *and then* $W_x(\varphi,\psi)=0$*. Thus* φ *and* ψ *are linearly dependent and* $\varphi=K\psi$ *for some non-zero constant* $K\in\mathbb{R}$*. Hence* $\gamma=\gamma_0$*. See Lemma 4.2* [3].

b) Assume $\tau_{\alpha,r,\theta}$ *is in the limit point case at* b*. If* $\gamma_0\neq\gamma$*, then* φ *and* ψ *are linearly independent in* $[d,b)$*, since if there exists a non-zero constant* $K\in\mathbb{R}$ *such that* $\psi=K\varphi$ *then* $\gamma=\gamma_0$*. Therefore every solution* f *of* $\tau_{\alpha,r,\theta}f=Ef$ *in* $[d,b)$ *can be written as* $u=c_1\varphi+c_2\psi$*. But, since* $\varphi,\psi\in L^2(a,b)$*, then* $u\in L^2(a,b)$ *and we get a contradiction to the limit point case.* □

THEOREM 5.1. *We have the following cases:*

a) If $\alpha=\alpha_0$ *and* $r=r_0$ *are fixed, then* $\{(\alpha_0,r_0,\theta)\in P(E)\}$ *is empty or is countable.*

b) If $\alpha=\alpha_0$ *and* $\theta=\theta_0$ *are fixed, then* $\{(\alpha_0,r,\theta_0)\in P(E)\}$ *has at most one element or* $\{(\alpha_0,r,\theta_0)\in P(E)\}=\{\alpha_0\}\times(0,\infty)\times\{\theta_0\}$*.*

c) If $r=r_0$ *and* $\theta=\theta_0$ *are fixed, then* $\{(\alpha,r_0,\theta_0)\in P(E)\}$ *has at most one element or* $\{(\alpha,r_0,\theta_0)\in P(E)\}=\mathbb{R}\times\{r_0\}\times\{\theta_0\}$*.*

Proof.

a) Suppose that for some θ, $(\alpha_0,r_0,\theta)\in P(E)$. Then by Lemma 5.1, $E\in\sigma_p(H^{\delta_0,\gamma_0}_{\alpha_0,r_0,\theta})$. By Theorem 4.1 *a*), this implies $(\alpha_0,r_0,\tilde\theta)\in P(E)$ if and only if $\tilde\theta=\theta+k\pi$, $k\in\mathbb{Z}$. Therefore the set $\{(\alpha_0,r_0,\theta)\in P(E)\}$ is countable.

b) Suppose that for some r, $(\alpha_0,r,\theta_0)\in P(E)$. Then by Lemma 5.1, $E\in\sigma_p(H^{\delta_0,\gamma_0}_{\alpha_0,r,\theta_0})$. By Theorem 4.1 *b*), one has $(\alpha_0,\tilde r,\theta_0)\notin P(E)$, $\forall\tilde r\neq r$ or $(\alpha_0,\tilde r,\theta_0)\in P(E)$, $\forall\tilde r>0$. Therefore the assertion follows.

c) Suppose that for some α, $(\alpha, r_0, \theta_0) \in P(E)$. Then by Lemma 5.1, $E \in \sigma_p(H^{\delta_0,\gamma_0}_{\alpha,r_0,\theta_0})$. By Theorem 4.1 c), one has $(\tilde{\alpha}, r_0, \theta_0) \notin P(E)$, $\forall \tilde{\alpha} \neq \alpha$ or $(\tilde{\alpha}, r_0, \theta_0) \in P(E)$, $\forall \tilde{\alpha} \in \mathbb{R}$. Therefore the assertion follows. $\square$

REMARK 5.3. Observe that in b) of Theorem 5.1, if the eigenvector associated to E is such that $u(x_{n_0}-) = \cos\theta_{n_0}$ and $u'(x_{n_0}-) = -\sin\theta_{n_0}$ or $u(x_{n_0}-) = \sin\theta_{n_0}$ and $u'(x_{n_0}-) = \cos\theta_{n_0}$, then $\{(\alpha_0, r, \theta_0) \in P(E)\} = \{\alpha_0\} \times (0,\infty) \times \{\theta_0\}$, otherwise $\{(\alpha_0, r, \theta_0) \in P(E)\}$ has at most one element. In case c) of the same theorem, if the eigenvector associated to E is such that $u(x_{n_0}-) = \cos\theta_{n_0}$ and $u'(x_{n_0}-) = -\sin\theta_{n_0}$, then $\{(\alpha, r_0, \theta_0) \in P(E)\} = \mathbb{R} \times \{r_0\} \times \{\theta_0\}$, otherwise $\{(\alpha, r_0, \theta_0) \in P(E)\}$ has at most one element.

6. Sturm-Liouville operators with random point interactions

In this section we use the previously obtained results to study the random case. First the probability space Ω where the sequences of coupling constants live is constructed and then our random operators are defined.

The space of real valued sequences $\{\omega_n\}_{n\in\mathfrak{I}}$, where $\mathfrak{I} \subseteq \mathbb{Z}$, will be denoted by $\mathbb{R}^{\mathfrak{I}}$. We introduce a measure in $\mathbb{R}^{\mathfrak{I}}$ in the following way. Let $\{p_n\}_{n\in\mathfrak{I}}$ be a sequence of probability measures in $\mathbb{R}$ and consider the product measure $\mathbb{P} = \times_{n\in\mathfrak{I}} p_n$ defined on the product σ-algebra $\mathscr{F}$ of $\mathbb{R}^{\mathfrak{I}}$ generated by the cylinder sets, that is, by the sets of the form $\{\omega : \omega(i_1) \in A_1, \ldots, \omega(i_n) \in A_n\}$ for $i_1, \ldots, i_n \in \mathfrak{I}$, where $A_1, \ldots, A_n$ are Borel sets in $\mathbb{R}$. In this way a measure space $\Omega = (\mathbb{R}^{\mathfrak{I}}, \mathscr{F}, \mathbb{P})$ is constructed. See chapter 1, section 1 in [12]. In some cases we may require for the measure space Ω to be complete, i.e. subsets of sets of measure zero are measurable. Every measurable space can be completed, see Theorem 1.36 [13].

If we fix R and Θ, and let $\Lambda \in \mathbb{R}^{\mathfrak{I}}$, we denote the operator $H_{\Lambda,R,\Theta}$ as H_Λ and analogously H_R and H_Θ when the parameters Λ and Θ or Λ and R are fixed respectively, see Remark 5.2. Assume moreover the limit point occurs at a or that $\tau_{\Lambda,R,\Theta}$ is regular at a and the same possibilities for b (see Definition 5.8).

Let $\Omega_1 = (\mathbb{R}^{\mathfrak{I}}, \mathscr{F}_1, \mathbb{P}_1)$, $\Omega_2 = ((0,\infty)^{\mathfrak{I}}, \mathscr{F}_2, \mathbb{P}_2)$ and $\Omega_3 = (\mathbb{R}^{\mathfrak{I}}, \mathscr{F}_3, \mathbb{P}_3)$ be probability spaces constructed as described above.

DEFINITON 6.1. For any $E \in \mathbb{R}$, we define

$$P_{R,\Theta}(E) := \{\Lambda \in \mathbb{R}^{\mathfrak{I}} : E \in \sigma_p(H_\Lambda)\}$$

$$P_{\Lambda,\Theta}(E) := \{R \in (0,\infty)^{\mathfrak{I}} : E \in \sigma_p(H_R)\}$$

$$P_{\Lambda,R}(E) := \{\Theta \in \mathbb{R}^{\mathfrak{I}} : E \in \sigma_p(H_\Theta)\}$$

We shall prove the following theorem. Recall that a continuous measure p is a measure such that $p(\{x\}) = 0$, for any point x.

THEOREM 6.1. *Assume Ω_1 is complete and $\mathbb{P}_1 = \times_{n\in I} p_n$ is such that p_n are continuous measures for all $n \in I$. Let $E \in \mathbb{R}$ fixed and $B \subset P_{R,\Theta}(E)$, for any measurable set $B \in \mathscr{F}_1$. Then one of the following options holds:*

i) $\mathbb{P}_1(B) = 0$

ii) $P_{R,\Theta}(E) = \mathbb{R}^{\mathfrak{I}}$

REMARK 6.1. We will show that in some cases there is always a set of point interactions M where option *ii*) happens. See Theorem 6.5 below.

REMARK 6.2. An analogous result holds for $P_{\Lambda,\Theta}(E)$: either $\mathbb{P}_2(B) = 0$ or $P_{\Lambda,\Theta}(E) = (0,\infty)^{\mathfrak{I}}$.

Before proving Theorem 6.1 we shall prove the following lemma, where Definition 2.1 is used.

LEMMA 6.1. *For any measurable $B \subseteq P_{R,\Theta}$ and any $n \in \mathfrak{I}$, set*

$$Q_{n,E} := \{\Lambda \in B : \exists u_\Lambda \in D(H_\Lambda) \setminus \{0\}, H_\Lambda u_\Lambda = E u_\Lambda \text{ and } [(u_\Lambda(x_n-), u'_\Lambda(x_n-))^T] \neq [(\cos\theta_n, -\sin\theta_n)^T]\}.$$

Then $Q_{n,E}$ is measurable and $\mathbb{P}_1(Q_{n,E}) = 0$.

Proof. Let

$$\chi_B(\Lambda) = \begin{cases} 1 & \text{if } \Lambda \in B, \\ 0 & \text{if } \Lambda \notin B. \end{cases}$$

If $\Lambda \in Q_{n,E}$, then from the definition of $Q_{n,E}$ it follows that $\chi_B(\Lambda) = 1$.
Let $f : \mathbb{R}^{\mathfrak{I}\setminus\{n\}} \to [0,\infty)$ be defined by

$$f(\tilde{\Lambda}) := \int_{\mathbb{R}} \chi_B(\Lambda)\, dp_n(\Lambda(n)),$$

where $\tilde{\Lambda} = \sum_{k \in \mathfrak{I}\setminus\{n\}} \Lambda(k)e(k)$. Here $e(k) = (e_m)_{m\in\mathfrak{I}}$ are the canonical vectors with entries $e_m = 0$ if $k \neq m$ and $e_k = 1$. The measurability of f follows from Fubini's Theorem. (See [13, Theorem 7.8].)

If $\Lambda = \sum_{k\in\mathfrak{I}} \Lambda(k)e(k) \in Q_{n,E}$, then $f(\tilde{\Lambda}) = 0$, where $\tilde{\Lambda} = \sum_{k\in\mathfrak{I}\setminus\{n\}} \Lambda(k)e(k)$. This follows from Remark 5.3 since p_n is continuous.

Hence $Q_{n,E} \subseteq [f^{-1}(\{0\}) \times \mathbb{R}] \cap B$.

Now, using Fubini,

$$\begin{aligned} \int_{f^{-1}(\{0\})\times\mathbb{R}} \chi_B(\Lambda)\, d\mathbb{P}_1 &= \int_{f^{-1}(\{0\})} d\mathbb{P}_1(\tilde{\Lambda}) \int_{\mathbb{R}} \chi_B(\Lambda)\, dp_n(\Lambda(n)) \\ &= \int_{f^{-1}(\{0\})} f(\tilde{\Lambda})\, d\mathbb{P}_1(\tilde{\Lambda}) = 0. \end{aligned}$$

Then,

$$\int_{[f^{-1}(\{0\})\times\mathbb{R}]\cap B} \chi_B(\Lambda)\, d\mathbb{P}_1 = 0,$$

and since $\chi_B(\Lambda) = 1$ in B, we get $\mathbb{P}_1([f^{-1}(\{0\}) \times \mathbb{R}] \cap B) = 0$.

Since the measure $d\mathbb{P}_1$ is complete, any subset of a measurable set of measure zero is measurable with measure zero. Therefore $Q_{n,E}$ is measurable. □

Proof of Theorem 6.1. It will be enough to prove that if $ii)$ doesn't hold, then $i)$ must hold.

Assume that there exists $\Lambda_0 \in \mathbb{R}^{\mathfrak{I}}$ such that E is not an eigenvalue of H_{Λ_0}.

If E is not an eigenvalue of H_Λ for every $\Lambda \in \mathbb{R}^{\mathfrak{I}}$, then $\mathbb{P}_1(B) = 0$ and the result follows.

Suppose now $\Lambda \in B$, then $E \in \sigma_p(H_\Lambda)$, i.e. there exist $u_\Lambda \in D(H_\Lambda) \setminus \{0\}$ such that $H_\Lambda u_\Lambda = E u_\Lambda$. Then $\Lambda \in Q_{n,E}$ for some $n \in \mathfrak{I}$.

This follows because if $[(u_\Lambda(x_n-), u'_\Lambda(x_n-))^T] = [(\cos\theta_n, -\sin\theta_n)^T]$ for every $n \in \mathfrak{I}$, then there exist $c_n \in \mathbb{R}$ such that $(u(x_n-), u'(x_n-))^T = c_n(\cos\theta, -\sin\theta)$, hence

$$A_{\Lambda(n),r_n,\theta_n} \begin{pmatrix} u(x_n-) \\ u'(x_n-) \end{pmatrix} = A_{\Lambda(n),r_n,\theta_n} c_n \begin{pmatrix} \cos\theta_n \\ -\sin\theta_n \end{pmatrix} = c_n r_n \begin{pmatrix} 1 & \Lambda(n) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_n r_n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Since the right hand side does not depend on Λ, from the definition of H_Λ, E must be an eigenvalue of H_Λ for all $\Lambda \in \mathbb{R}^{\mathfrak{I}}$, in particular E is an eigenvalue of H_{Λ_0}, cf. proof Theorem 4.1 $c)$, which is not possible by our initial assumption. Therefore

$$B \subset \bigcup_{n \in I} Q_{n,E},$$

where $Q_{n,E}$ was defined in Lemma 6.1. Using that lemma we obtain $\mathbb{P}_1(\bigcup_{n\in\mathfrak{I}} Q_n) = 0$. Therefore the result follows. □

REMARK 6.3. If $\tau_{\Lambda,R,\Theta}$ is in lcc at b we need to assume it is regular at b, otherwise we cannot assure that the function u_Λ in the proof of Theorem 6.1 satisfies the boundary conditions at b for every $\Lambda(n)$, and the same for the endpoint a.

THEOREM 6.2. *Assume* $\mathbb{P}_3 = \times_{n\in\mathfrak{I}} q_n$ *is such that* q_{n_0} *is a continuous measure for some* $n_0 \in \mathfrak{I}$. *Let* $E \in \mathbb{R}$ *be fixed and let* B *be any measurable subset of* $P_{\Lambda,R}(E)$. *Then* $\mathbb{P}_3(B) = 0$.

Proof. Let

$$\chi_B(\Theta) = \begin{cases} 1 & \text{if } \Theta \in B, \\ 0 & \text{if } \Theta \notin B, \end{cases}$$

and define $f : \mathbb{R}^{\mathfrak{I}\setminus\{n_0\}} \to [0,\infty)$ as

$$f(\tilde{\Theta}) := \int_{\mathbb{R}} \chi_B(\Theta)\, dq_{n_0}(\Theta(n_0)),$$

where $\tilde{\Theta} = \sum_{k\in\mathfrak{I}\setminus\{n_0\}} \Theta(k)e(k)$. Here $e(k) = (e_m)_{m\in\mathfrak{I}}$ are the canonical vectors with entries $e_m = 0$ if $k \neq m$ and $e_k = 1$. The measurability of f follows from Fubini's Theorem. (See [13, Theorem 7.8].)

If $\Theta = \sum_{k\in\mathfrak{I}} \Theta(k)e(k) \in B$, then $f(\tilde{\Theta}) = 0$, where $\tilde{\Theta} = \sum_{k\in\mathfrak{I}\setminus\{n\}} \Theta(k)e(k)$. This follows from Theorem 5.1 since q_{n_0} is continuous.

Hence $B \subseteq [f^{-1}(\{0\}) \times \mathbb{R}]$.

Now, using Fubini,

$$\int_{f^{-1}(\{0\})\times\mathbb{R}} \chi_B(\Theta)\, d\mathbb{P}_3 = \int_{f^{-1}(\{0\})} d\mathbb{P}_3(\tilde{\Theta}) \int_{\mathbb{R}} \chi_B(\Theta)\, dq_{n_0}(\Theta(n_0))$$
$$= \int_{f^{-1}(\{0\})} f(\tilde{\Theta})\, d\mathbb{P}_3(\tilde{\Theta}) = 0.$$

Then, $\mathbb{P}_3([f^{-1}(\{0\}) \times \mathbb{R}]) = 0$. Therefore $\mathbb{P}_3(B) = 0$. □

DEFINITON 6.2. For any $E \in \mathbb{R}$, we define

$$P(E) := \{(\Lambda, R, \Theta) \in \Omega_1 \times \Omega_2 \times \Omega_3 : E \in \sigma_p(H_{\Lambda,R,\Theta})\}.$$

In the following Theorems the hypothesis of measurability of the set $P(E)$ is crucial. This assumption can be satisfied for example, if we assume the operators $H_{\Lambda,R,\Theta}$ are measurable. See Theorem 4.6 [4] which states this fact for any family of measurable operators defined in a separable Hilbert space.

THEOREM 6.3. *Assume $\mathbb{P}_3 = \times_{n\in\mathfrak{I}} q_n$ is such that q_{n_0} is a continuous measure for some $n_0 \in \mathfrak{I}$. Let $E \in \mathbb{R}$ be fixed and suppose that $P(E)$ is measurable. Let $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$. Then,*

$$\mathbb{P}(P(E)) = 0.$$

Proof. Let

$$\chi_{P(E)}(\Lambda, R, \Theta) = \begin{cases} 1 & \text{if } (\Lambda, R, \Theta) \in P(E), \\ 0 & \text{if } (\Lambda, R, \Theta) \notin P(E). \end{cases}$$

Then,

$$\mathbb{P}(P(E)) = \int_{\Omega_1\times\Omega_2\times\Omega_3} \chi_{P(E)}(\Lambda, R, \Theta)\, d\mathbb{P}.$$

Using Fubini we have

$$\int_{\Omega_1\times\Omega_2\times\Omega_3} \chi_{P(E)}(\Lambda, R, \Theta)\, d\mathbb{P} = \int_{\Omega_1\times\Omega_2} d\mathbb{P}_1 \times d\mathbb{P}_2 \int_{\Omega_3} \chi_{P_{\Lambda,R}(E)}(\Theta)\, d\mathbb{P}_3(\Theta),$$

where $P_{\Lambda,R}(E)$ is as in Definition 6.1.

Note that

$$\int_{\Omega_3} \chi_{P_{\Lambda,R}(E)}(\Theta)\, d\mathbb{P}_3(\Theta) = \mathbb{P}_3(P_{\Lambda,R}(E)),$$

and that Theorem 6.2 gives $\mathbb{P}_3(P_{R,\Theta}(E)) = 0$. Thus, the theorem follows. □

THEOREM 6.4. *Assume Ω_1 is complete and $\mathbb{P}_1 = \times_{n\in\mathfrak{I}} p_n$ is such that p_n are continuous measures for all $n \in \mathfrak{I}$. Let $E \in \mathbb{R}$ be fixed and suppose that $P(E)$ is measurable. Let $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$. Then one of the following options holds:*

i) $\mathbb{P}(P(E)) = 0$,

ii) $\mathbb{P}(P(E)) = 1$.

Proof. Let

$$\chi_{P(E)}(\Lambda, R, \Theta) = \begin{cases} 1 & \text{if } (\Lambda, R, \Theta) \in P(E), \\ 0 & \text{if } (\Lambda, R, \Theta) \notin P(E). \end{cases}$$

Then,

$$\mathbb{P}(P(E)) = \int_{\Omega_1 \times \Omega_2 \times \Omega_3} \chi_{P(E)}(\Lambda, R, \Theta)\, d\mathbb{P}.$$

Using Fubini we have

$$\int_{\Omega_1 \times \Omega_2 \times \Omega_3} \chi_{P(E)}(\Lambda, R, \Theta)\, d\mathbb{P} = \int_{\Omega_2 \times \Omega_3} d\mathbb{P}_2 \times d\mathbb{P}_3 \int_{\Omega_1} \chi_{P_{R,\Theta}(E)}(\Lambda)\, d\mathbb{P}_1(\Lambda),$$

where $P_{R,\Theta}(E)$ is as in Definition 6.1. Since

$$\int_{\Omega} \chi_{P_{R,\Theta}(E)}(\Lambda)\, d\mathbb{P}(\Lambda) = \mathbb{P}(P_{R,\Theta}(E)),$$

using Theorem 6.1 we conclude that either $\mathbb{P}(P_{R,\Theta}(E)) = 0$ or $\mathbb{P}(P_{R,\Theta}(E)) = 1$. Therefore the theorem follows. □

REMARK 6.4. An analogous result holds if we assume that Ω_2 satisfies the hypothesis of the theorem instead of Ω_1.

6.1. Oscillation of solutions

The next result, Theorem 6.5, shows that it is always possible to construct a set of point interactions M such that option *ii)* in Theorem 6.1 occurs.

Let $H = H_{\Lambda,R,\Theta}$ with $\Lambda = \{0\}_{n\in\mathfrak{I}}$, $R = \{1\}_{n\in\mathfrak{I}}$ and $\Theta = \{0\}_{n\in\mathfrak{I}}$ be the unperturbed operator. This operator does not depend on M and $\mathfrak{I}$, and it is the selfadjoint operator without interactions. The matrices introduced in Definition 5.1 are now the identity, that is

$$A_{\alpha_n, r_n, \theta_n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

DEFINITON 6.3. (See Section XI.6 in [7]) The equation

$$(\tau - E)u = 0$$

is said to be oscillatory on an interval J if every solution has infinitely many zeros on J.

If $t = b$ is a (possibly infinite) endpoint of J which does not belong to J, then the equation is said to be oscillatory at $t = b$ if every solution has an infinite number of zeros in J accumulating at b.

Define

$$\varphi(x) := arg(u'(x) + iu(x)) \qquad x \in (a,b).$$

The zeros of the solution u are given by the values of x such that $\varphi(x) = k\pi$ for some integer k. $(\tau - E)u = 0$ is oscillatory at b if and only if $\varphi(x) \to \infty$ as $x \to b$; see [8, p. 9].

LEMMA 6.2. *Given two consecutive zeros $t_1, t_2 \in (a,b)$ of a solution u of $(\tau - E)u = 0$ and given a vector $v = (v_1, v_2)^T \in \mathbb{R}^2$, there exists a point $x_0 \in [t_1, t_2)$ such that*

$$\left[\begin{pmatrix} u(x_0) \\ u'(x_0) \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right].$$

Proof. Since t_1 and t_2 are zeros of the solution u, there exist $k_1, k_2 \in \mathbb{Z}$ such that $\varphi(t_1) = k_1\pi$ and $\varphi(t_2) = k_2\pi$. Since φ cannot tend to a multiple of π from above, see [2, Theorem 8.4.3 *ii)*], we have $k_2 = k_1 + 1$. Since φ is continuous, there exists $x_0 \in [t_1, t_2)$ such that

$$\arg(u'(x_0) + iu(x_0)) = \arg(v_2 + iv_1).$$

Therefore, by Remark 2.1,

$$\left[\begin{pmatrix} u(x_0) \\ u'(x_0) \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right]. \qquad \square$$

THEOREM 6.5. *Let $(\tau - E)u = 0$ be oscillatory on (a,b), $-\infty \leqslant a < b \leqslant \infty$, $E \in \sigma_p(H)$ and the zeros of u does not accumulate at any interior point of (a,b). Fix $R = \{r_n\}_{n\in\mathfrak{I}}$ and $\Theta = \{\theta_n\}_{n\in\mathfrak{I}}$, where $\mathfrak{I}$ is finite or $\mathfrak{I} = \mathbb{N}$. Then there exists $M \subset \mathbb{R}$ discrete such that $P_{R,\Theta}(E) = \mathbb{R}^I$.*

Proof. Assume $Hu = Eu$.

Suppose $\mathfrak{I}$ is finite, $\mathfrak{I} = \{n_1, n_2, \ldots, n_r\}$, and $\Theta = \{\theta_{n_1}, \ldots, \theta_{n_r}\}$. Let $t_0, t_1, \ldots, t_r$ be $r+1$ consecutive zeros of u. For $n_i \in \mathfrak{I}$, let x_{n_i} be such that $x_{n_i} \in [t_{i-1}, t_i)$ and

$$\left[\begin{pmatrix} u(x_{n_i}) \\ u'(x_{n_i}) \end{pmatrix}\right] = \left[\begin{pmatrix} \cos\theta_{n_i} \\ -\sin\theta_{n_i} \end{pmatrix}\right].$$

Due to Lemma 6.2, such an x_{n_i} exists. Let $M = \{x_{n_i}\}_{i=1}^r$ and take $A_{\alpha_{n_i}, r_{n_i}, \theta_{n_i}} = P_{\alpha_{n_i}} H_{r_{n_i}} E_{\theta_{n_i}}$ as in Definition 5.1, for all $n_i \in \mathfrak{I}$. Then,

$$A_{\alpha_{n_i}, r_{n_i}, \theta_{n_i}} \begin{pmatrix} u(x_{n_i}-) \\ u'(x_{n_i}-) \end{pmatrix} = A_{\alpha_{n_i}, r_{n_i}, \theta_{n_i}} \begin{pmatrix} \cos\theta_{n_i} \\ -\sin\theta_{n_i} \end{pmatrix} = r_{n_i} \begin{pmatrix} 1 & \alpha_{n_i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = r_{n_i} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From the definition of H_Λ, E must be an eigenvalue of H_Λ for all $\Lambda \in \mathbb{R}^{\mathfrak{I}}$, and therefore $P_{R,\Theta}(E) = \mathbb{R}^I$.

Suppose $\mathfrak{I} = \mathbb{N}$. Let us assume that there are infinitely many zeros of u, increasingly enumerated by $t_0, t_1, \ldots$. Let $\Theta = \{\theta_n\}_{n\in\mathfrak{I}}$. Let x_1 be such that $x_1 \in [t_0, t_1)$ and

$$\left[\begin{pmatrix} u(x_1) \\ u'(x_1) \end{pmatrix}\right] = \left[\begin{pmatrix} \cos\theta_1 \\ -\sin\theta_1 \end{pmatrix}\right].$$

As above let x_2 be such that $x_2 \in [t_1, t_2)$ and

$$\left[\begin{pmatrix} u(x_2) \\ u'(x_2) \end{pmatrix}\right] = \left[\begin{pmatrix} \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}\right].$$

In this way we get a sequence $M := \{x_n\}_{n\in\mathfrak{I}}$. Take $A_{\alpha_n, r_n, \theta_n} = P_{\alpha_n} H_r E_{\theta_n}$ as in Definition 5.1. Then,

$$A_{\alpha_n, r_n, \theta_n} \begin{pmatrix} u(x_n-) \\ u'(x_n-) \end{pmatrix} = A_{\alpha_n, r_n, \theta_n} \begin{pmatrix} \cos\theta_n \\ -\sin\theta_n \end{pmatrix} = r_n \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = r_n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From the definition of H_Λ, E must be an eigenvalue of H_Λ for all $\Lambda \in \mathbb{R}^{\mathfrak{I}}$, and therefore $P_{R,\Theta}(E) = \mathbb{R}^{\mathfrak{I}}$. $\square$

REFERENCES

[1] S. ALBEVERIO, L. DĄBROWSKI AND P. KURASOV, *Symmetries of Schrödinger operators with point interactions*, Lett. Math. Phys., 45 (1): 33–47, 1998.

[2] F. V. ATKINSON,, *Discrete and continuous boundary problems*, Mathematics in Science and Engineering, Vol. 8, Academic Press, New York-London, 1964.

[3] D. BUSCHMANN, G. STOLZ AND J. WEIDMANN, *One-dimensional Schrödinger operators with local point interactions*, J. Reine Angew. Math., 467: 169–186, 1995.

[4] R. DEL RIO AND A. L. FRANCO, *Random Sturm-Liouville Operators with Point Interactions*, arXiv e-prints, page arXiv:1903.02714, Mar 2019.

[5] W. N. EVERITT, C. SHUBIN, G. STOLZ AND A. ZETTL, *Sturm-Liouville problems with an infinite number of interior singularities*, In Spectral theory and computational methods of Sturm-Liouville problems (Knoxville, TN, 1996), vol. 191 of Lecture Notes in Pure and Appl. Math., pages 211–249, Dekker, New York, 1997.

[6] F. GESZTESY AND W. KIRSCH,, *One-dimensional Schrödinger operators with interactions singular on a discrete set*, J. Reine Angew. Math., 362: 28–50, 1985.

[7] PHILIP HARTMAN, *Ordinary differential equations*, Birkhäuser, Boston, Mass., second edition, 1982.

[8] DON HINTON, *Sturm's 1836 oscillation results evolution of the theory*, In Sturm-Liouville theory, pages 1–27, Birkhäuser, Basel, 2005.

[9] SERGE LANG, $SL_2(\mathbf{R})$, volume 105 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1985, reprint of the 1975 edition.

[10] KONSTANTIN PANKRASHKIN, *Resolvents of self-adjoint extensions with mixed boundary conditions*, Rep. Math. Phys., 58 (2): 207–221, 2006.

[11] L. A. PASTUR, *Spectral properties of disordered systems in the one-body approximation*, Comm. Math. Phys., 75 (2): 179–196, 1980.

[12] LEONID PASTUR AND ALEXANDER FIGOTIN, *Spectra of random and almost-periodic operators*, vol. 297 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1992.

[13] WALTER RUDIN, *Real and complex analysis*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

[14] P. ŠEBA, *The generalized point interaction in one dimension*, Czechoslovak J. Phys. B, 36 (6): 667–673, 1986.

[15] AIPING WANG AND ANTON ZETTL, *Eigenvalues of Sturm-Liouville problems with discontinuous boundary conditions*, Electron. J. Differential Equations, pages Paper No. 127, 27, 2017.
[16] ANTON ZETTL, *Sturm-Liouville theory*, vol. 121 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2005.
[17] L. A. PASTUR, *Spectral properties of disordered systems in the one-body approximation*, Comm. Math. Phys., 75 (2): 179–196, 1980.
[18] JOACHIM WEIDMANN, *Linear operators in Hilbert spaces*, Graduate Texts in Mathematics, vol. 68, Springer-Verlag, New York-Berlin, 1980.

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