Operators and Matrices
Volume 15, Number 1 (2021), 1–15
doi:10.7153/oam-2021-15-01

INTEGRAL REPRESENTATIONS OF SOME FAMILIES OF OPERATOR MONOTONE FUNCTIONS

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(*Communicated by D. R. Farenick*)

Abstract. We obtain an integral representation of holomorphic function $P_\alpha(z)$ which is real on the positive part of the real axis and formed

$$P_\alpha(x) = \left(\frac{x^\alpha + 1}{2}\right)^{\frac{1}{\alpha}} \quad (x \geqslant 0).$$

For this purpose we define a two variable function which is substituted for an argument θ, and also find an explicit real and imaginary part of $P_\alpha(x+iy)$.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, and $\mathscr{B}(\mathscr{H})$ be the set of all bounded linear operators on $\mathscr{H}$. An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be positive if and only if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. We denote a positive operator A by $A \geqslant 0$. For self-adjoint operators $A, B \in \mathscr{B}(\mathscr{H})$, $A \leqslant B$ means $B - A$ is positive. A continuous function $f(x)$ defined on an interval I in $\mathbb{R}$ is called an operator monotone function if $A \leqslant B \implies f(A) \leqslant f(B)$ holds for every pair $A, B \in \mathscr{B}(\mathscr{H})$ whose spectra $\sigma(A)$ and $\sigma(B)$ lie in I. A typical example of it is x^α for $\alpha \in (0,1)$, this claims $0 < A \leqslant B \implies A^\alpha \leqslant B^\alpha$ for $0 < \alpha < 1$ ([4], [5]). This inequality is so famous and called the Löwner-Heinz inequality. This inequality also asserts that

$$\frac{A^\alpha - I}{\alpha} \leqslant \frac{B^\alpha - I}{\alpha}$$

holds for $\alpha \in (0,1)$, and by tending $\alpha \searrow 0$, both sides of the above inequality converge to $\log A$ and $\log B$ in the norm topology, respectively. From this fact, we can conclude that the logarithmic function $\log x$ is operator monotone too.

We call $f(z)$ a Pick function if $f(z)$ is holomorphic on $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \Im z > 0\}$ and satisfies $f(\mathbb{C}^+) \subset \mathbb{C}^+$. By Löwner's results ([1], [5]), a real function $f(x)$ is operator monotone if and only if a complex function $f(z)$ is a Pick function. Strictly speaking, an operator monotone function $f(x)$ defined on an interval (a,b) has an analytic continuation to the upper half plane as a Pick function, and, conversely, if a

Mathematics subject classification (2010): 15A60, 47A64.
Keywords and phrases: Operator mean, operator monotone function, power mean, Pick function.

Paper OaM-15-01

Pick function $f(z)$ satisfies $f((a,b)) \subset \mathbb{R}$ for an interval (a,b), then the restriction of $f(z)$ to (a,b) is operator monotone. For example, we have confirmed that $\log x$ is operator monotone on $(0,\infty)$, and, indeed, the logarithmic function has an analytic continuation to the cut plane $\mathbb{C} \setminus (-\infty, 0]$ as a Pick function

$$\mathrm{Log} z := \log r + i\theta,$$

where $z := re^{i\theta}(r > 0,\ -\pi < \theta < \pi)$. Moreover, it is well-known that a Pick function $f(z)$ has an integral representation

$$f(z) = \boldsymbol{\alpha} z + \boldsymbol{\beta} + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) d\mu(\lambda) \tag{$\star$}$$

where $\boldsymbol{\alpha} \geqslant 0$, $\boldsymbol{\beta} \in \mathbb{R}$ and $\mu(\lambda)$ is a nonnegative Borel measure on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} d\mu(\lambda) < \infty.$$

The measure μ in $(\star)$ is called representing measure of f. We remark that if f satisfies $f\big((a,b)\big) \subset \mathbb{R}$ for an interval (a,b), namely, f is an operator monotone function on (a,b), then the measure μ has no mass on (a,b). In particular, if f is an operator monotone function on $[a,b)$, then the measure μ has no mass on $[a,b)$. Recently F. Hansen showed interesting results about the representing measure of an operator monotone function on $(0,\infty)$;

THEOREM A. (Hansen [2]) *Let $g : (0,\infty) \to \mathbb{R}$ be an operator monotone function which has an integral representation*

$$g(x) = \boldsymbol{\alpha} x + \boldsymbol{\beta} + \int_{0}^{\infty} \left(\frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + x} \right) d\nu(\lambda)$$

where $\nu(\lambda)$ is a positive measure on the closed positive half-line $[0,\infty)$ with

$$\int_{0}^{\infty} \frac{1}{\lambda^2 + 1} d\nu(\lambda) < \infty.$$

Let $\tilde{\nu}$ be the measure obtained from ν by removing a possible atom in zero. Then

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \Im g(-t + i\varepsilon) h(t) dt = \frac{h(0)}{2} \nu(\{0\}) + \int_{0}^{\infty} h(\lambda) d\tilde{\nu}(\lambda)$$

for every continuous, bounded and integrable function h defined in $[0,\infty)$.

It is also known that constants $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and measure $\mu(\lambda)$ which appear in the above integral representation $(\star)$ is found as

$$\boldsymbol{\alpha} = \lim_{y \to \infty} \frac{f(iy)}{iy},\ \boldsymbol{\beta} = \Re f(i),\ \pi d\mu(\lambda) = \lim_{y \searrow 0} \Im f(\lambda + iy) d\lambda,$$

where the last limit is in the vague topology. Following this method, we can easily get $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. But it is little harder to obtain measure $\mu(\lambda)$ than previous case, because we need to find not only a limit of a family of functions but also convergence in the vague topology. For this purpose, it is usually required for us to show that a convergence theorem is applicable to $\Im f(\lambda+iy)$. However, there are some functions such that we can confirm the validity of its integral representation only using a simple computation, for instance

$$DL(z) := \frac{z\mathrm{Log}z}{z-1} = \frac{\pi}{4} + \int_{-\infty}^{0} \left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2+1} \right) \frac{\lambda}{\lambda - 1} d\lambda.$$

Note that a "real" function $DL(x)$ can be extended continuously to $[0,\infty)$ by $DL(1)=1$ and $DL(0)=0$. This function is also known as the representing function of the dual of the logarithmic mean ([6]). In $[6]$, we proved that the imaginary part of $DL(z)$ satisfies

$$0 < \Im DL(re^{i\theta}) < \theta$$

for $z = re^{i\theta} \in \mathbb{C}^+$, where θ is an "argument" of z. Hence $\exp\{DL(x)\}$ is operator monotone on $(0,\infty)$.

The 1-parameter family of functions $\{P_\alpha(x)\}_{\alpha\in[-1,1]}$;

$$P_\alpha(x) = \left(\frac{x^\alpha + 1}{2} \right)^{\frac{1}{\alpha}} \quad (-1 \leqslant \alpha \leqslant 1)$$

is one of the most famous family of operator monotone functions, and also known as the representing function of the Power mean [7]. When we confirm operator monotonicity of $P_\alpha(x)$, we usually show that $P_\alpha(x)$ has a holomorphic branch, which maps the upper half plane into itself, by checking their "argument" θ. This technique is very simple and useful, but, in its proof, there is no information about an explicit form of holomorphic branch $P_\alpha(z)$. If we want to find an integral representation of $P_\alpha(x+iy)$ by the above way, then we have to describe its real part $\Re P_\alpha(x+iy)$ and imaginary part $\Im P_\alpha(x+iy)$ concretely. In Section 2, we give a "device" to express this real and imaginary parts, and we obtain an explicit form of $P_\alpha(x+iy)$ in Section 3. Lastly, in Section 4, we obtain an integral representation of $P_\alpha(z)$.

2. $\mathrm{Tan}^{-1}(x,y)$

As mentioned in the Section 1, it is well-known that a real function $g_\alpha(x) = x^\alpha$, which is continuous and increasing on $[0,\infty)$, is operator monotone for $\alpha \in (0,1]$. $g_\alpha(x)$ has a holomorphic branch

$$g_\alpha(z) := r^\alpha e^{i\alpha\theta},$$

where $z = re^{i\theta}$ $(r > 0,\ -\pi < \theta < \pi)$, and is also known as a Pick function. This form is described by an "argument" θ, and thus it is difficult to express like $g_\alpha(x+iy) = u(x,y) + iv(x,y)$. We remark that

$$P_\alpha(x) = g_{\frac{1}{\alpha}}\left(\frac{g_\alpha(x)+1}{2} \right) \quad (x > 0).$$

In [3] F. Hansen gave imaginary part and real part of $g_{\frac{1}{\alpha}}\big(g_\alpha(z)+1\big) = (z^\alpha+1)^{\frac{1}{\alpha}}$ by using "argument" θ, but their form was not explicit. So we consider introducing a two variable function which is substituted for an argument θ to express concrete real and imaginary part of $g_\alpha(x+iy)$.

DEFINITION 1. Let $\mathbb{A} := \mathbb{R}^2 \setminus \{(a,b)\in\mathbb{R}^2 \mid -\infty < a \leqslant 0, b=0\}$. We define the two variable function $\mathrm{Tan}^{-1} : \mathbb{A} \to (-\pi,\pi) \in \mathbb{R}$ as the following;

$$\mathrm{Tan}^{-1}(x,y) := \begin{cases} \tan^{-1}\left(\frac{y}{x}\right)+\pi & (x<0, y>0) \\ \frac{\pi}{2} & (x=0, y>0) \\ \tan^{-1}\left(\frac{y}{x}\right) & (x>0) \\ -\frac{\pi}{2} & (x=0, y<0) \\ \tan^{-1}\left(\frac{y}{x}\right)-\pi & (x<0, y<0). \end{cases}$$

Clearly, $\mathrm{Tan}^{-1}(x,y)$ is continuous on $\mathbb{A}$. On the other hand, next proposition determines how to treat $\mathrm{Tan}^{-1}(x,y)$ for the case $y=0$.

PROPOSITION 1. (1) $\lim\limits_{x<0, y\searrow 0} \mathrm{Tan}^{-1}(x,y) = \lim\limits_{y>0, x\to-\infty} \mathrm{Tan}^{-1}(x,y) = \pi$,

(2) $\lim\limits_{x<0, y\nearrow 0} \mathrm{Tan}^{-1}(x,y) = \lim\limits_{y<0, x\to-\infty} \mathrm{Tan}^{-1}(x,y) = -\pi$,

(3) $\lim\limits_{x\to\infty} \mathrm{Tan}^{-1}(x,y) = 0$.

Proof. (1) When $x<0$ and $y>0$, $\mathrm{Tan}^{-1}(x,y) = \tan^{-1}\left(\frac{y}{x}\right)+\pi$. So we have

$$\lim_{x<0, y\searrow 0} \mathrm{Tan}^{-1}(x,y) = \lim_{y\to 0} \tan^{-1}\left(\frac{y}{x}\right)+\pi = \pi,$$

$$\lim_{y>0, x\to-\infty} \mathrm{Tan}^{-1}(x,y) = \lim_{x\to-\infty} \tan^{-1}\left(\frac{y}{x}\right)+\pi = \pi.$$

(2) We can prove similar to the case (1).
(3) For $x>0$,

$$\lim_{x\to\infty} \mathrm{Tan}^{-1}(x,y) = \lim_{x\to\infty} \tan^{-1}\left(\frac{y}{x}\right) = 0. \quad \square$$

From the Definition 1, we can easily find that a two variable function $\mathrm{Tan}^{-1}(x,y)$ defined above has many properties that an argument θ satisfies. We introduce some of these without proof in the following. These properties will often appear as useful tools.

PROPOSITION 2. (1) *For* $a>0$, $\mathrm{Tan}^{-1}(ax,ay)=\mathrm{Tan}^{-1}(x,y)$,
(2) *For* $b<0$, (i) $y>0 \Longrightarrow \mathrm{Tan}^{-1}(bx,by)=\mathrm{Tan}^{-1}(x,y)-\pi$,
(ii) $y<0 \Longrightarrow \mathrm{Tan}^{-1}(bx,by)=\mathrm{Tan}^{-1}(x,y)+\pi$

LEMMA 1. (1) *For* $y>0$ *and* $x_1>x_2$, $\mathrm{Tan}^{-1}(x_1,y)<\mathrm{Tan}^{-1}(x_2,y)$,
(2) *For* $y<0$ *and* $x_1>x_2$, $\mathrm{Tan}^{-1}(x_1,y)>\mathrm{Tan}^{-1}(x_2,y)$,
(3) *For* $x>0$ *and* $y_1>y_2$, $\mathrm{Tan}^{-1}(x,y_1)>\mathrm{Tan}^{-1}(x,y_2)$,
(4) *For* $x<0$ *and* $y_1>y_2>0>y_3>y_4$,

$$\mathrm{Tan}^{-1}(x,y_2)>\mathrm{Tan}^{-1}(x,y_1)>\mathrm{Tan}^{-1}(x,y_4)>\mathrm{Tan}^{-1}(x,y_3).$$

LEMMA 2. (1) *Let* $x>0$ *and* $y>0$. *Then*
(i) $\mathrm{Tan}^{-1}(-x,y)=-\mathrm{Tan}^{-1}(x,y)+\pi$,
(ii) $\mathrm{Tan}^{-1}(x,-y)=-\mathrm{Tan}^{-1}(x,y)$,
(iii) $\mathrm{Tan}^{-1}(-x,-y)=\mathrm{Tan}^{-1}(x,y)-\pi$,
(2) $\mathrm{Tan}^{-1}(x,-y)=-\mathrm{Tan}^{-1}(x,y)$,
(3) (i) $y>0$ *implies* $\mathrm{Tan}^{-1}(-x,y)=-\mathrm{Tan}^{-1}(x,y)+\pi$,
(ii) $y<0$ *implies* $\mathrm{Tan}^{-1}(-x,y)=-\mathrm{Tan}^{-1}(x,y)-\pi$.

Next proposition asserts that $\mathrm{Tan}^{-1}(x,y)$ can be substituted for an argument θ.

PROPOSITION 3.

$$\sin\big(\mathrm{Tan}^{-1}(x,y)\big)=\frac{y}{\sqrt{x^2+y^2}},\ \cos\big(\mathrm{Tan}^{-1}(x,y)\big)=\frac{x}{\sqrt{x^2+y^2}}.$$

Proof. When $x<0$ and $y>0$,

$$\begin{aligned}\frac{\sin\big(\mathrm{Tan}^{-1}(x,y)\big)}{\cos\big(\mathrm{Tan}^{-1}(x,y)\big)}&=\tan\Big(\mathrm{Tan}^{-1}(x,y)\big)\Big)\\&=\tan\left(\tan^{-1}\left(\frac{y}{x}\right)+\pi\right)\\&=\tan\left(\tan^{-1}\left(\frac{y}{x}\right)\right)=\frac{y}{x}.\end{aligned}\tag{$*$}$$

From this, we have

$$x^2\sin^2\big(\mathrm{Tan}^{-1}(x,y)\big)=y^2\cos^2\big(\mathrm{Tan}^{-1}(x,y)\big)=y^2\Big(1-\sin^2\big(\mathrm{Tan}^{-1}(x,y)\big)\Big),$$

and therefore

$$(x^2+y^2)\sin^2\big(\mathrm{Tan}^{-1}(x,y)\big)=y^2.$$

Since $y>0$ and $\sin\big(\mathrm{Tan}^{-1}(x,y)\big)>0$,

$$\sqrt{x^2+y^2}\sin\big(\mathrm{Tan}^{-1}(x,y)\big)=y.$$

By this fact and $(*)$ we obtain

$$\sin\bigl(\mathrm{Tan}^{-1}(x,y)\bigr) = \frac{y}{\sqrt{x^2+y^2}},\ \cos\bigl(\mathrm{Tan}^{-1}(x,y)\bigr) = \frac{x}{\sqrt{x^2+y^2}}. \quad \square$$

By Proposition 3, we can immediately have

$$x+iy = \sqrt{x^2+y^2}\Bigl(\cos\bigl(\mathrm{Tan}^{-1}(x,y)\bigr) + i\sin\bigl(\mathrm{Tan}^{-1}(x,y)\bigr)\Bigr).$$

From this, it is expected that $\mathrm{Tan}^{-1}(x,y)$ will be able to express real and imaginary part of $g_\alpha(x+iy)$ as an explicit form, instead of θ.

PROPOSITION 4. *Define the function* $\mathbb{G}_\alpha : \mathbb{R}\times\mathbb{C}\setminus(-\infty,0] \to \mathbb{C}$ *by*

$$\mathbb{G}_\alpha(x+iy) := \bigl(x^2+y^2\bigr)^{\frac{\alpha}{2}}\Bigl\{\cos\bigl(\alpha\mathrm{Tan}^{-1}(x,y)\bigr) + i\sin\bigl(\alpha\mathrm{Tan}^{-1}(x,y)\bigr)\Bigr\}.$$

Then the following hold;

(1) $\mathbb{G}_\alpha(x+iy)$ *is holomorphic on* $\mathbb{C}\setminus(-\infty,0]$,

(2) *If* $\alpha\in(0,1)$, *then* $\mathbb{G}_\alpha(\mathbb{C}^+)\subset\mathbb{C}^+$, *where* $\mathbb{C}^+ = \{z\in\mathbb{C} \mid \Im z>0\}$.

Proof. (1) We verify that $\mathbb{G}_\alpha$ satisfies the Cauchy-Riemann equations. Put $\Re\mathbb{G}_\alpha(x+iy) = R(x,y), \Im\mathbb{G}_\alpha(x+iy) = I(x,y)$. Since

$$\frac{\partial}{\partial x}\mathrm{Tan}^{-1}(x,y) = -\frac{y}{x^2+y^2},\ \frac{\partial}{\partial y}\mathrm{Tan}^{-1}(x,y) = \frac{x}{x^2+y^2},$$

we obtain

$$\frac{\partial}{\partial x}R(x,y) = \frac{\partial}{\partial y}I(x,y),\ \frac{\partial}{\partial y}R(x,y) = -\frac{\partial}{\partial x}I(x,y).$$

(2) Take $y>0$. Then $\mathrm{Tan}^{-1}(x,y)\in(0,\pi)$, and we have $\alpha\mathrm{Tan}^{-1}(x,y)\in(0,\alpha\pi)\subset(0,\pi)$. Accordingly,

$$\sin\bigl(\alpha\mathrm{Tan}^{-1}(x,y)\bigr) > 0$$

holds and it implies $I(x,y)>0$. $\square$

For $x+iy\in\mathbb{C}^+$, it is clear that $\mathbb{G}_\alpha(x+iy)\to x^\alpha$ as $y\searrow 0$. From this we find that $\mathbb{G}_\alpha(x+iy)$ is an analytic continuation of $g_\alpha(x)$ such that $\mathbb{G}_\alpha(\mathbb{C}^+)\subset\mathbb{C}^+$. Furthermore, this holomorphic branch will play an important role when we construct an explicit form of $P_\alpha(x+iy)$. Also, we find that $\mathbb{G}_\alpha$ has some properties which a real power function satisfies.

LEMMA 3. (1) *For* $\alpha\in(0,1)$,

$$\mathrm{Tan}^{-1}\Bigl(\cos\bigl(\alpha\mathrm{Tan}^{-1}(x,y)\bigr), \sin\bigl(\alpha\mathrm{Tan}^{-1}(x,y)\bigr)\Bigr) = \alpha\mathrm{Tan}^{-1}(x,y).$$

(2) $$\mathrm{Tan}^{-1}\Bigl(\cos\bigl(\mathrm{Tan}^{-1}(x,y)\bigr), -\sin\bigl(\mathrm{Tan}^{-1}(x,y)\bigr)\Bigr) = -\mathrm{Tan}^{-1}(x,y).$$

Proof. If $x<0$, $y>0$, then $\dfrac{\pi}{2}<\alpha\mathrm{Tan}^{-1}(x,y)<\pi$. So

$$\mathrm{Tan}^{-1}\Big(\cos\big(\alpha\mathrm{Tan}^{-1}(x,y)\big),\sin\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)\Big)=\alpha\mathrm{Tan}^{-1}(x,y),$$

$$\mathrm{Tan}^{-1}\Big(\cos\big(\mathrm{Tan}^{-1}(x,y)\big),-\sin\big(\mathrm{Tan}^{-1}(x,y)\big)\Big)=-\mathrm{Tan}^{-1}(x,y). \quad \square$$

PROPOSITION 5. $\mathbb{G}_{-\alpha}(z)=\dfrac{1}{\mathbb{G}_\alpha(z)}=\mathbb{G}_\alpha\left(z^{-1}\right)$

PROPOSITION 6. (1) $\mathbb{G}_\alpha(z)\mathbb{G}_\beta(z)=\mathbb{G}_\beta(z)\mathbb{G}_\alpha(z)=\mathbb{G}_{\alpha+\beta}(z)$,
(2) *If* $\alpha,\beta\in(-1,1)$, *then* $\mathbb{G}_\alpha\big(\mathbb{G}_\beta(z)\big)=\mathbb{G}_\beta\big(\mathbb{G}_\alpha(z)\big)=\mathbb{G}_{\alpha\beta}(z)$.

REMARK 1. Proposition 6.(2) doesn't hold for if $|\alpha|>1$ or $|\beta|>1$. For example, we put $z=-1+i$, $\alpha=2$ and $\beta=\dfrac{1}{2}$. Then

$$\mathbb{G}_2\big(\mathbb{G}_{\frac{1}{2}}(z)\big)=-1+i=z,\ \mathbb{G}_{\frac{1}{2}}\big(\mathbb{G}_2(z)\big)=1-i=-z.$$

3. An explicit form of P_α

In this section, we define an explicit form of P_α anew by applying $\mathbb{G}_\alpha$ which is determined in the previous section.

Firstly, let $\alpha\in(0,1)$. For a “real ” $x>0$, $P_\alpha(x)$ is described as

$$P_\alpha(x)=\left(\frac{x^\alpha+1}{2}\right)^{\frac{1}{\alpha}}=g_{\frac{1}{\alpha}}\left(\frac{g_\alpha(x)+1}{2}\right)$$

by “real function” $g_\alpha(x)=x^\alpha$. From this relation, we define a “complex function” $\mathbb{P}_\alpha$ as

$$\mathbb{P}_\alpha(z)=\mathbb{G}_{\frac{1}{\alpha}}\left(\frac{\mathbb{G}_\alpha(z)+1}{2}\right).$$

Then $\lim_{x>0,y\searrow 0}\mathbb{P}_\alpha(x+iy)=\left(\dfrac{x^\alpha+1}{2}\right)^{\frac{1}{\alpha}}$ is clear. By Proposition 4, $\mathbb{G}_\alpha$ is holomorphic on $\mathbb{C}\setminus(-\infty,0]$. Since the set of all holomorphic functions is closed under composition, $\mathbb{P}_\alpha$ is also holomorphic. For $z=x+iy$ $(y>0)$,

$$\mathbb{P}_\alpha(x+iy)=\Big(R_\alpha(x,y)^2+I_\alpha(x,y)^2\Big)^{\frac{1}{2\alpha}}$$
$$\times\left\{\cos\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,y),I_\alpha(x,y)\big)\right)+i\sin\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,y),I_\alpha(x,y)\big)\right)\right\},$$

where

$$R_\alpha(x,y)=\frac{\left(x^2+y^2\right)^{\frac{\alpha}{2}}\cos\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)+1}{2},I_\alpha(x,y)=\frac{\left(x^2+y^2\right)^{\frac{\alpha}{2}}\sin\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)}{2}.$$

Since $y>0$, $\alpha\mathrm{Tan}^{-1}(x,y)\in(0,\pi)$. Hence $\mathrm{Tan}^{-1}\big(R_\alpha(x,y),I_\alpha(x,y)\big)>0$. By Proposition 2, Lemma 1 and Lemma 3,

$$\begin{aligned}
0&<\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,y),I_\alpha(x,y)\big)\\
&=\frac{1}{\alpha}\mathrm{Tan}^{-1}\Big(\big(x^2+y^2\big)^{\frac{\alpha}{2}}\cos\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)+1,\big(x^2+y^2\big)^{\frac{\alpha}{2}}\sin\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)\Big)\\
&<\frac{1}{\alpha}\mathrm{Tan}^{-1}\Big(\cos\big(\alpha\mathrm{Tan}^{-1}(x,y)\big),\sin\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)\Big)\\
&=\frac{1}{\alpha}\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)=\mathrm{Tan}^{-1}(x,y)<\pi.
\end{aligned}$$

Since

$$\Im\mathbb{P}_\alpha(x+iy)=\Big(R_\alpha(x,y)^2+I_\alpha(x,y)^2\Big)^{\frac{1}{2\alpha}}\sin\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,y),I_\alpha(x,y)\big)\right),$$

we see that $\mathbb{P}_\alpha$ is a Pick function for any $\alpha\in(0,1)$.

Next we consider the case $\alpha\in(-1,0)$. For real function $P_\alpha(x)$,

$$P_\alpha(x)=\left(\frac{x^\alpha+1}{2}\right)^{\frac{1}{\alpha}}=\left(\frac{2x^{|\alpha|}}{x^{|\alpha|}+1}\right)^{\frac{1}{|\alpha|}}=g_{\frac{1}{|\alpha|}}\left(2-\frac{2}{g_{|\alpha|}(x)+1}\right)$$

holds, and we determine a complex function $\mathbb{Q}_\alpha$, similar to the case $\alpha\in(0,1)$, as the following;

$$\mathbb{Q}_\alpha(z)=\mathbb{G}_{\frac{1}{\alpha}}\left(2-\frac{2}{\mathbb{G}_\alpha(z)+1}\right)\ \big(\alpha\in(0,1)\big).$$

Clearly, $\displaystyle\lim_{x>0,y\searrow0}\mathbb{Q}_\alpha(x+iy)=\left(\frac{2x^\alpha}{x^\alpha+1}\right)^{\frac{1}{\alpha}}$. For $z=x+iy\in\mathbb{C}^+$

$$\begin{aligned}
\mathbb{Q}_\alpha(x+iy)&=\Big(S_\alpha(x,y)^2+J_\alpha(x,y)^2\Big)^{\frac{1}{2\alpha}}\\
&\times\left\{\cos\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(S_\alpha(x,y),J_\alpha(x,y)\big)\right)+i\sin\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(S_\alpha(x,y),J_\alpha(x,y)\big)\right)\right\},
\end{aligned}$$

where

$$S_\alpha(x,y)=\frac{2\Big\{\big(x^2+y^2\big)^\alpha+\big(x^2+y^2\big)^{\frac{\alpha}{2}}\cos\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)\Big\}}{\big(x^2+y^2\big)^\alpha+2\big(x^2+y^2\big)^{\frac{\alpha}{2}}\cos\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)+1},$$

$$J_\alpha(x,y)=\frac{2\big(x^2+y^2\big)^{\frac{\alpha}{2}}\sin\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)}{\big(x^2+y^2\big)^\alpha+2\big(x^2+y^2\big)^{\frac{\alpha}{2}}\cos\big(\alpha\mathrm{Tan}^{-1}(x,y)\big)+1}.$$

Similarly to $\mathbb{P}_\alpha$, we can easily obtain

$$0<\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(S_\alpha(x,y),J_\alpha(x,y)\big)<\pi.$$

Consequently, we have $\mathbb{Q}_\alpha(\mathbb{C}^+) \subset \mathbb{C}^+$.

From the above, we have obtained next two theorems;

THEOREM 1. *Define the function* $\mathbb{P}_\alpha : \mathbb{R}\times\mathbb{C}\setminus(-\infty,0] \to \mathbb{C}$ *by*

$$\mathbb{P}_\alpha(z) = \mathbb{G}_{\frac{1}{\alpha}}\left(\frac{\mathbb{G}_\alpha(z)+1}{2}\right).$$

Then $\mathbb{P}_\alpha(z)$ *is a Pick function for* $\alpha \in (0,1)$, *and for* $x+iy \in \mathbb{C}^+$

$$\lim_{x>0, y\searrow 0} \mathbb{P}_\alpha(x+iy) = P_\alpha(x).$$

THEOREM 2. *Define the function* $\mathbb{Q}_\alpha : \mathbb{R}\times\mathbb{C}\setminus(-\infty,0] \to \mathbb{C}$ *by*

$$\mathbb{Q}_\alpha(z) = \mathbb{G}_{\frac{1}{\alpha}}\left(2-\frac{2}{\mathbb{G}_\alpha(z)+1}\right).$$

Then $\mathbb{Q}_\alpha(z)$ *is a Pick function for* $\alpha \in (0,1)$, *and for* $x+iy \in \mathbb{C}^+$

$$\lim_{x>0, y\searrow 0} \mathbb{Q}_\alpha(x+iy) = P_{-\alpha}(x).$$

REMARK 2. $P_\alpha(x)$ can be extended naturally to $[0,\infty)$ for $\alpha \in (0,\infty)$, and so $\mathbb{P}_\alpha$ and $\mathbb{Q}_\alpha$ can be extended naturally to $\mathbb{C}\setminus(0,\infty)$. Thus the representing measure of them have no mass on $[0,\infty)$.

REMARK 3. It follows from their definitions that both $\mathbb{P}_\alpha$ and $\mathbb{Q}_\alpha$ are continuous in $\alpha \in (0,1)$. Namely, for fixed $z \in \mathbb{C}\setminus(0,\infty]$ and any sequence δ_n which converges to δ, we can confirm that the following equations

$$\lim_{n\to\infty} \mathbb{P}_{\delta_n}(z) = \mathbb{P}_\delta(z), \ \lim_{n\to\infty} \mathbb{Q}_{\delta_n}(z) = \mathbb{Q}_\delta(z)$$

are satisfied.

From Theorem 1, Theorem 2 and the identity theorem, we could obtain an explicit form of $P_\alpha(z)$ for $\alpha \in (-1,0)\cup(0,1)$. But we have left a question that how $\mathbb{P}_\alpha$ (or $\mathbb{Q}_\alpha$) is treated for $\alpha = 0$. Thus we haven't complete to get an explicit form of $P_\alpha(z)$, and we must solve this question. For the case of a "real function", $P_\alpha(x)$ converges pointwise to $x^{\frac{1}{2}}$ as $\alpha \to 0$. We shall show that this relation is satisfied for "complex functions" $\mathbb{P}_\alpha(x+iy)$ and $\mathbb{Q}_\alpha(x+iy)$ from definitions of them, and we will treat $\mathbb{P}_0(x+iy)$ and $\mathbb{Q}_0(x+iy)$ as these results.

LEMMA 4.

$$\mathbb{G}_\alpha(\bar{z}) = \overline{\mathbb{G}_\alpha(z)}, \ \mathbb{P}_\alpha(\bar{z}) = \overline{\mathbb{P}_\alpha(z)}, \ \mathbb{Q}_\alpha(\bar{z}) = \overline{\mathbb{Q}_\alpha(z)}.$$

Proof. For $z = x+iy$, we have $\overline{z} = x-iy$.Firstly we consider $\mathbb{G}_\alpha$. By Lemma 2, $\mathrm{Tan}^{-1}(x,-y) = -\mathrm{Tan}^{-1}(x,y)$. This yields

$$\cos\big(\alpha \mathrm{Tan}^{-1}(x,-y)\big) = \cos\big(\alpha \mathrm{Tan}^{-1}(x,y)\big),\ \sin\big(\alpha \mathrm{Tan}^{-1}(x,-y)\big) = -\sin\big(\alpha \mathrm{Tan}^{-1}(x,y)\big).$$

Accordingly,

$$\begin{aligned}\mathbb{G}_\alpha(\overline{x+iy}) &= \big(x^2+(-y)^2\big)^{\frac{\alpha}{2}}\Big\{\cos\big(\alpha \mathrm{Tan}^{-1}(x,-y)\big) + i\sin\big(\alpha \mathrm{Tan}^{-1}(x,-y)\big)\Big\}\\ &= \big(x^2+y^2\big)^{\frac{\alpha}{2}}\Big\{\cos\big(\alpha \mathrm{Tan}^{-1}(x,y)\big) - i\sin\big(\alpha \mathrm{Tan}^{-1}(x,y)\big)\Big\} = \overline{\mathbb{G}_\alpha(x+iy)}.\end{aligned}$$

Next we consider $\mathbb{P}_\alpha$. From the definition of $\mathbb{P}_\alpha$,

$$\begin{aligned}&\mathbb{P}_\alpha(x-iy) = \Big(R_\alpha(x,-y)^2 + I_\alpha(x,-y)^2\Big)^{\frac{1}{2\alpha}}\\ &\times\left\{\cos\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,-y), I_\alpha(x,-y)\big)\right) + i\sin\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,-y), I_\alpha(x,-y)\big)\right)\right\}.\end{aligned}$$

Applying the above relations, we have $R_\alpha(x,-y) = R_\alpha(x,y)$, $I_\alpha(x,-y) = -I_\alpha(x,y)$. This fact and Lemma 2 yield

$$\mathrm{Tan}^{-1}\big(R_\alpha(x,-y), I_\alpha(x,-y)\big) = -\mathrm{Tan}^{-1}\big(R_\alpha(x,y), I_\alpha(x,y)\big).$$

Therefore,

$$\cos\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,-y), I_\alpha(x,-y)\big)\right) = \cos\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,y), I_\alpha(x,y)\big)\right),$$

$$\sin\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,-y), I_\alpha(x,-y)\big)\right) = -\sin\left(\frac{1}{\alpha}\mathrm{Tan}^{-1}\big(R_\alpha(x,y), I_\alpha(x,y)\big)\right).$$

From the above

$$\mathbb{P}_\alpha(x-iy) = \overline{\mathbb{P}_\alpha(x+iy)}.$$

For $\mathbb{Q}_\alpha$, we can also obtain $S_\alpha(x,-y) = S_\alpha(x,y), J_\alpha(x,-y) = -J_\alpha(x,y)$ and hence get a desired assertion. □

THEOREM 3. *For families of functions* $\{\mathbb{P}_\alpha(z)\}_{\alpha\in(0,1)}$ *and* $\{\mathbb{Q}_\alpha(z)\}_{\alpha\in(0,1)}$

$$\lim_{\alpha\searrow 0}\mathbb{P}_\alpha(z) = \lim_{\alpha\searrow 0}\mathbb{Q}_\alpha(z) = \mathbb{G}_{\frac{1}{2}}(z) \quad \big(z\in\mathbb{C}\setminus(-\infty,0]\big)$$

holds, namely, $\{\mathbb{P}_\alpha(z)\}_{\alpha\in(0,1)}$ *and* $\{\mathbb{Q}_\alpha(z)\}_{\alpha\in(0,1)}$ *converge pointwise to* $\mathbb{G}_{\frac{1}{2}}(z)$ *when* $\alpha\searrow 0$.

Proof. Firstly we consider $\mathbb{P}_\alpha(x+iy)$. It is clear for the case $z\in(0,\infty)$ from Theorem 1. It is sufficient to show the case $z\in\mathbb{C}^+$, because if we can prove that $\lim_{\alpha\searrow 0}\mathbb{P}_\alpha(z) = \mathbb{G}_{\frac{1}{2}}(z)$ $\big(z\in\mathbb{C}^+\big)$, then

$$\lim_{\alpha\searrow 0}\mathbb{P}_\alpha(\overline{z}) = \lim_{\alpha\searrow 0}\overline{\mathbb{P}_\alpha(z)} = \overline{\mathbb{G}_{\frac{1}{2}}(z)} = \mathbb{G}_{\frac{1}{2}}(\overline{z})$$

by Lemma 4. Put $z=x+iy\in\mathbb{C}^+$. For any $x+iy$ there exists a sufficiently small $\alpha>0$ such that $\alpha\mathrm{Tan}^{-1}(x,y)\in\left(0,\dfrac{\pi}{2}\right)$. Therefore we can assume that $\cos\left(\alpha\mathrm{Tan}^{-1}(x,y)\right)>0$. We easily get

$$\left(R_\alpha(x,y)^2+I_\alpha(x,y)^2\right)^{\frac{1}{2\alpha}}=\left(\frac{\left(x^2+y^2\right)^\alpha+2\left(x^2+y^2\right)^{\frac{\alpha}{2}}\cos\left(\alpha\mathrm{Tan}^{-1}(x,y)\right)+1}{4}\right)^{\frac{1}{2\alpha}}$$

Applying l'Hospital's theorem, we have

$$\lim_{\alpha\searrow 0}\log\left(R_\alpha(x,y)^2+I_\alpha(x,y)^2\right)^{\frac{1}{2\alpha}}=\frac{\log\left(x^2+y^2\right)}{4}.$$

Accordingly, $\lim\limits_{\alpha\searrow 0}\left(R_\alpha(x,y)^2+I_\alpha(x,y)^2\right)^{\frac{1}{2\alpha}}=\left(x^2+y^2\right)^{\frac{1}{4}}$. We apply l'Hospital's theorem again and get

$$\lim_{\alpha\searrow 0}\frac{\mathrm{Tan}^{-1}\left(R_\alpha(x,y),I_\alpha(x,y)\right)}{\alpha}=\frac{1}{2}\mathrm{Tan}^{-1}(x,y).$$

From the above, $\lim\limits_{\alpha\searrow 0}\mathbb{P}_\alpha(x+iy)=\mathbb{G}_{\frac{1}{2}}(x+iy)$ holds for $x+iy\in\mathbb{C}^+$. Next we consider $\mathbb{Q}_\alpha(x+iy)$. We easily obtain

$$\left(S_\alpha(x,y)^2+J_\alpha(x,y)^2\right)^{\frac{1}{2\alpha}}=\left(\frac{4\left(x^2+y^2\right)^\alpha}{\left(x^2+y^2\right)^\alpha+2\left(x^2+y^2\right)^{\frac{\alpha}{2}}\cos\left(\alpha\mathrm{Tan}^{-1}(x,y)\right)+1}\right)^{\frac{1}{2\alpha}}.$$

So we can similarly get

$$\left(S_\alpha(x,y)^2+J_\alpha(x,y)^2\right)^{\frac{1}{2\alpha}}\to\left(x^2+y^2\right)^{\frac{1}{4}},\ \frac{\mathrm{Tan}^{-1}\left(S_\alpha(x,y),J_\alpha(x,y)\right)}{\alpha}\to\frac{1}{2}\mathrm{Tan}^{-1}(x,y)$$

when $\alpha\searrow 0$. Therefore $\lim\limits_{\alpha\searrow 0}\mathbb{Q}_\alpha(x+iy)=\mathbb{G}_{\frac{1}{2}}(x+iy)$. $\square$

4. Integral representations of $P_\alpha(z)$

In this section, we shall find an integral representation of $P_\alpha(z)$. $P_\alpha(z)$ is treated by divided it into three parts, namely $\mathbb{P}_\alpha(z)$ when $\alpha\in(0,1)$, $\mathbb{Q}_\alpha(z)$ when $\alpha\in(-1,0)$ and $\mathbb{G}_{\frac{1}{2}}(z)$ when $\alpha=0$, as before. But, we have already known that $\mathbb{G}_{\frac{1}{2}}$ has an integral representation

$$\mathbb{G}_{\frac{1}{2}}(z)=\frac{1}{\sqrt{2}}+\int_{-\infty}^{0}\left(\frac{1}{\lambda-z}-\frac{\lambda}{\lambda^2+1}\right)\frac{\sqrt{|\lambda|}}{\pi}d\lambda$$

(see [1, p. 27]). Therefore we only have to consider $\mathbb{P}_\alpha$ and $\mathbb{Q}_\alpha$.

THEOREM 4. *Let $0<\alpha<1$. Then $\mathbb{P}_\alpha(z)$ has an integral representation*

$$\left(\frac{1}{2}\right)^{\frac{1}{\alpha}} z+\left(\frac{\cos\left(\frac{\alpha}{2}\pi\right)+1}{2}\right)^{\frac{1}{2\alpha}} \cos\left(\frac{1}{\alpha}\tan^{-1}\left(\frac{\sin\left(\frac{\alpha}{2}\pi\right)}{\cos\left(\frac{\alpha}{2}\pi\right)+1}\right)\right) + \int_{-\infty}^{0}\left(\frac{1}{\lambda-z}-\frac{\lambda}{\lambda^2+1}\right)p_\alpha(\lambda)d\lambda,$$

where

$$p_\alpha(\lambda)=\frac{1}{\pi}\left(\frac{|\lambda|^{2\alpha}+2|\lambda|^\alpha\cos\alpha\pi+1}{4}\right)^{\frac{1}{2\alpha}}\sin\left(\frac{\mathrm{Tan}^{-1}\left(|\lambda|^\alpha\cos\alpha\pi+1,|\lambda|^\alpha\sin\alpha\pi\right)}{\alpha}\right).$$

Proof. From Theorem 1, we know that $\mathbb{P}_\alpha(z)$ is a Pick function for $0<\alpha<1$. Thus $\mathbb{P}_\alpha$ has an integral representation

$$\mathbb{P}_\alpha(z)=\boldsymbol{\alpha}_\alpha z+\boldsymbol{\beta}_\alpha+\int_{-\infty}^{\infty}\left(\frac{1}{\lambda-z}-\frac{\lambda}{\lambda^2+1}\right)d\mu_\alpha(\lambda),$$

where $\boldsymbol{\alpha}_\alpha,\boldsymbol{\beta}_\alpha$ and $\mu_\alpha(\lambda)$ are constants and measure which depend on α, respectively. In the following we find $\boldsymbol{\alpha}_\alpha,\boldsymbol{\beta}_\alpha$ and $\mu_\alpha(\lambda)$. Put $z=\lambda+iy\in\mathbb{C}^+$.

$$\frac{\mathbb{P}_\alpha(iy)}{iy}=\left(\frac{y^{2\alpha}+2y^\alpha\cos\left(\frac{\alpha}{2}\pi\right)+1}{4y^{2\alpha}}\right)^{\frac{1}{2\alpha}} \times\left\{\sin\left(\frac{\mathrm{Tan}^{-1}\left(R_\alpha(0,y),I_\alpha(0,y)\right)}{\alpha}\right)-i\cos\left(\frac{\mathrm{Tan}^{-1}\left(R_\alpha(0,y),I_\alpha(0,y)\right)}{\alpha}\right)\right\}.$$

From Definition 1, $R_\alpha(0,y)=\dfrac{y^\alpha\cos\left(\frac{\alpha}{2}\pi\right)+1}{2}$, $I_\alpha(0,y)=\dfrac{y^\alpha\sin\left(\frac{\alpha}{2}\pi\right)}{2}$. Since $0<\dfrac{\alpha}{2}\pi<\dfrac{\pi}{2}$, we have $\cos\left(\dfrac{\alpha}{2}\pi\right),\sin\left(\dfrac{\alpha}{2}\pi\right)\in(0,1)$ and then $R_\alpha(0,y),I_\alpha(0,y)>0$. Therefore

$$\lim_{y\to\infty}\mathrm{Tan}^{-1}\left(R_\alpha(0,y),I_\alpha(0,y)\right)=\lim_{y\to\infty}\tan^{-1}\left(\frac{\sin\left(\frac{\alpha}{2}\pi\right)}{\cos\left(\frac{\alpha}{2}\pi\right)+y^{-\alpha}}\right)=\frac{\alpha}{2}\pi.$$

By this fact,

$$\lim_{y\to\infty}\sin\left(\frac{\mathrm{Tan}^{-1}\left(R_\alpha(0,y),I_\alpha(0,y)\right)}{\alpha}\right)=1,\ \lim_{y\to\infty}\cos\left(\frac{\mathrm{Tan}^{-1}\left(R_\alpha(0,y),I_\alpha(0,y)\right)}{\alpha}\right)=0.$$

We thus find $\displaystyle\lim_{y\to\infty}\frac{\mathbb{P}_\alpha(iy)}{iy}=\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}$. By putting $\lambda=0,y=1$ we also find

$$\Re\{\mathbb{P}_\alpha(i)\}=\left(\frac{\cos\left(\frac{\alpha}{2}\pi\right)+1}{2}\right)^{\frac{1}{2\alpha}}\cos\left(\frac{1}{\alpha}\tan^{-1}\left(\frac{\sin\left(\frac{\alpha}{2}\pi\right)}{\cos\left(\frac{\alpha}{2}\pi\right)+1}\right)\right).$$

Lastly, we find $\mu_\alpha(\lambda)$. We have already known that

$$\Im\{\mathbb{P}_\alpha(\lambda+iy)\} = \left(R_\alpha(\lambda,y)^2+I_\alpha(\lambda,y)^2\right)^{\frac{1}{2\alpha}}\sin\left(\frac{\operatorname{Tan}^{-1}\left(R_\alpha(\lambda,y),I_\alpha(\lambda,y)\right)}{\alpha}\right).$$

From Theorem 1, $\mathbb{P}_\alpha(\lambda)\in\mathbb{R}$ when $\lambda\geqslant 0$. Therefore $\Im\left\{\mathbb{P}_\alpha(\lambda+iy)\right\}\to 0\ (\lambda\geqslant 0, y\searrow 0)$. Since $\lim\limits_{\lambda<0,y\searrow 0}\operatorname{Tan}^{-1}(\lambda,y)=\pi$, $R_\alpha(\lambda,y)\to\dfrac{|\lambda|^\alpha\cos\alpha\pi+1}{2}$ and $I_\alpha(\lambda,y)\to\dfrac{|\lambda|^\alpha\sin\alpha\pi}{2}$ hold when $\lambda<0, y\searrow 0$. By Proposition 2, we get

$$\lim_{\lambda<0,y\searrow 0}\sin\left(\frac{\operatorname{Tan}^{-1}\left(R_\alpha(\lambda,y),I_\alpha(\lambda,y)\right)}{\alpha}\right)=\sin\left(\frac{\operatorname{Tan}^{-1}\left(|\lambda|^\alpha\cos\alpha\pi+1,|\lambda|^\alpha\sin\alpha\pi\right)}{\alpha}\right),$$

and also have $\left(R_\alpha(\lambda,y)^2+I_\alpha(\lambda,y)^2\right)^{\frac{1}{2\alpha}}\to\left(\dfrac{|\lambda|^{2\alpha}+2|\lambda|^\alpha\cos\alpha\pi+1}{4}\right)^{\frac{1}{2\alpha}}$ when $\lambda<0, y\searrow 0$. Accordingly, we can find that $\Im\{\mathbb{P}_\alpha(\lambda+iy)\}$ converges pointwise to $\pi p_\alpha(\lambda)$ as $\lambda<0, y\searrow 0$. Moreover, when we put $y=\dfrac{1}{n}$ $(n\in\mathbb{N})$,

$$\frac{\left(\lambda^2+\frac{1}{n^2}\right)^\alpha+2\left(\lambda^2+\frac{1}{n^2}\right)^{\frac{\alpha}{2}}\cos\left(\alpha\operatorname{Tan}^{-1}(\lambda,\frac{1}{n})\right)+1}{4}\leqslant\frac{\left(\lambda^2+1\right)^\alpha+2\left(\lambda^2+1\right)^{\frac{\alpha}{2}}+1}{4}$$

holds. Thus we get

$$\Im\left\{\mathbb{P}_\alpha\left(\lambda+i\frac{1}{n}\right)\right\}\leqslant\left(\frac{(\lambda^2+1)^{\frac{\alpha}{2}}+1}{2}\right)^{\frac{1}{\alpha}}\leqslant\frac{(\lambda^2+1)^{\frac{1}{2}}+1}{2}\leqslant\frac{\lambda^2+4}{4}.$$

Since $\dfrac{\lambda^2+4}{4}$ is integrable on $(-\infty,0)$, we see that dominated convergence theorem is applicable. Let $\phi(\lambda)$ be a nonnegative continuous function and assume that its support is compact. From the assumption, support is contained in closed interval $[-K,K]$ for $K>0$. By dominated convergence theorem,

$$\int_{-K}^{K}\phi(\lambda)\Im\left\{\mathbb{P}_\alpha\left(\lambda+i\frac{1}{n}\right)\right\}d\lambda\longrightarrow\int_{-K}^{K}\phi(\lambda)\pi p_\alpha(\lambda)d\lambda\ (n\to\infty).$$

Therefore, we conclude that $\Im\left\{\mathbb{P}_\alpha\left(\lambda+i\frac{1}{n}\right)\right\}$ converges $\pi p_\alpha(\lambda)$ in the vague topology, and thus $d\mu_\alpha(\lambda)=p_\alpha(\lambda)d\lambda$. □

THEOREM 5. *Let $0<\alpha<1$. Then $\mathbb{Q}_\alpha(z)$ has an integral representation*

$$\left(\frac{2}{1+\cos(\frac{\alpha}{2}\pi)}\right)^{\frac{1}{2\alpha}}\cos\left(\frac{1}{\alpha}\tan^{-1}\left(\frac{\sin\left(\frac{\alpha}{2}\pi\right)}{\cos\left(\frac{\alpha}{2}\pi\right)+1}\right)\right)$$
$$+\int_{-\infty}^{0}\left(\frac{1}{\lambda-z}-\frac{\lambda}{\lambda^2+1}\right)q_\alpha(\lambda)d\lambda,$$

where

$$q_\alpha(\lambda) = \frac{1}{\pi}\left(\frac{4|\lambda|^{2\alpha}}{|\lambda|^{2\alpha}+2|\lambda|^\alpha\cos\alpha\pi+1}\right)^{\frac{1}{2\alpha}}\sin\left(\frac{\mathrm{Tan}^{-1}\left(|\lambda|^\alpha+\cos\alpha\pi,\sin\alpha\pi\right)}{\alpha}\right).$$

Proof. We find $\boldsymbol{\alpha}_\alpha, \boldsymbol{\beta}_\alpha$ and $d\mu_\alpha(\lambda)$ similar to a proof of Theorem 4. For $z = \lambda + iy \in \mathbb{C}^+$,

$$\frac{\mathbb{Q}_\alpha(iy)}{iy} = \left(\frac{4}{y^{2\alpha}+2y^\alpha\cos\left(\frac{\alpha}{2}\pi\right)+1}\right)^{\frac{1}{2\alpha}} \times\left\{\sin\left(\frac{\mathrm{Tan}^{-1}\left(S_\alpha(0,y),J_\alpha(0,y)\right)}{\alpha}\right) - i\cos\left(\frac{\mathrm{Tan}^{-1}\left(S_\alpha(0,y),J_\alpha(0,y)\right)}{\alpha}\right)\right\}.$$

It is easy to find

$$S_\alpha(0,y) = \frac{2\left(y^{2\alpha}+y^\alpha\cos\left(\frac{\alpha}{2}\pi\right)\right)}{y^{2\alpha}+2y^\alpha\cos\left(\frac{\alpha}{2}\pi\right)+1} > 0, \quad J_\alpha(0,y) = \frac{2y^\alpha\sin\left(\frac{\alpha}{2}\pi\right)}{y^{2\alpha}+2y^\alpha\cos\left(\frac{\alpha}{2}\pi\right)+1} > 0$$

since $\cos\left(\dfrac{\alpha}{2}\pi\right), \sin\left(\dfrac{\alpha}{2}\pi\right) \in (0,1)$. From this relation, we obtain

$$\lim_{y\to\infty}\mathrm{Tan}^{-1}\left(S_\alpha(0,y),J_\alpha(0,y)\right) = \frac{\alpha}{2}\pi.$$

Therefore $\displaystyle\lim_{y\to\infty}\frac{\mathbb{Q}_\alpha(iy)}{iy} = 0$. Putting $\lambda = 0, y = 1$, we also have

$$\Re\{\mathbb{Q}_\alpha(i)\} = \left(\frac{2}{\cos\left(\frac{\alpha}{2}\pi\right)+1}\right)^{\frac{1}{2\alpha}}\cos\left(\frac{1}{\alpha}\tan^{-1}\left(\frac{\sin\left(\frac{\alpha}{2}\pi\right)}{\cos\left(\frac{\alpha}{2}\pi\right)+1}\right)\right).$$

Lastly we find $d\mu_\alpha(\lambda)$. We can assume that $\lambda < 0$ and $y > 0$ similar to a proof of Theorem 4. Then

$$\lim_{y\searrow 0} S_\alpha(\lambda,y) = \frac{2\left(|\lambda|^{2\alpha}+|\lambda|^\alpha\cos\alpha\pi\right)}{|\lambda|^{2\alpha}+2|\lambda|^\alpha\cos\alpha\pi+1}, \quad \lim_{y\searrow 0} J_\alpha(\lambda,y) = \frac{2|\lambda|^\alpha\cos\alpha\pi}{|\lambda|^{2\alpha}+2|\lambda|^\alpha\cos\alpha\pi+1}.$$

By Proposition 2,

$$\lim_{\lambda<0,y\searrow 0}\sin\left(\frac{\mathrm{Tan}^{-1}\left(S_\alpha(\lambda,y),J_\alpha(\lambda,y)\right)}{\alpha}\right) = \sin\left(\frac{\mathrm{Tan}^{-1}\left(|\lambda|^\alpha+\cos\alpha\pi,\sin\alpha\pi\right)}{\alpha}\right).$$

It follows from this fact that $\Im\{\mathbb{Q}_\alpha(\lambda+iy)\}$ converges pointwise to $\pi q_\alpha(\lambda)$ as $\lambda < 0, y \searrow 0$. In the following we show that dominated convergence theorem is applicable to $\Im\{\mathbb{Q}_\alpha(\lambda+iy)\}$. Since $0 < \alpha\mathrm{Tan}^{-1}(\lambda,y) < \pi$, $-1 < \cos\left(\alpha\mathrm{Tan}^{-1}(\lambda,y)\right) < 1$. Thus $0 < \cos^2\left(\alpha\mathrm{Tan}^{-1}(\lambda,y)\right) < 1$. From this fact we can choose a constant $C_\alpha > 4$, which depends on only α, such that $\cos^2\left(\alpha \mathrm{Tan}^{-1}(\lambda, y)\right) < \dfrac{C_\alpha - 4}{C_\alpha} < 1$. For this constant $C_\alpha > 4$

$$\frac{4\left(\lambda^2+y^2\right)^\alpha}{\left(\lambda^2+y^2\right)^\alpha + 2\left(\lambda^2+y^2\right)^{\frac{\alpha}{2}}\cos\left(\alpha \mathrm{Tan}^{-1}(\lambda, y)\right)+1} < C_\alpha$$

holds. Accordingly, $\Im\left\{\mathbb{Q}_\alpha\left(\lambda+\frac{i}{n}\right)\right\} < C_\alpha^{\frac{1}{2\alpha}}$ for any $n \in \mathbb{N}$. From the above, we see that dominated convergence theorem is applicable. □

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(Received May 27, 2019)

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