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A NOTE ON THE STRUCTURE OF MATRIX *–SUBALGEBRAS WITH SCALAR DIAGONALS

G. MacDonald[1], L. W. Marcoux[1], M. Mastnak[1], M. Omladič[2] and H. Radjavi

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Abstract. We characterize those unital, self-adjoint algebras of complex $n \times n$ matrices that are simultaneously unitarily similar to algebras in which every member has a scalar diagonal.

1. Introduction

This short paper is concerned with the structure of the set of diagonal entries of complex matrices in an algebra, group or semigroup. More specifically, we are interested in characterising those *-subalgebras $\mathscr{A}$ of $\mathbb{M}_n(\mathbb{C})$ for which there exists an orthonormal basis with respect to which each element $A \in \mathscr{A}$ admits a diagonal, all of whose entries are equal. We shall say that such an algebra has *scalar diagonals*.

The question "What can we say about the diagonal of a matrix up to unitary similarity?" has a rich history; below we only mention a few highlights. This short note is, as far as we know, the first to explore a related question for collections of matrices. In our setting we restrict ourselves exclusively to the finite-dimensional setting. When we say a matrix, we always mean a matrix of finite dimensions with complex entries.

A well-known fact about *any* single matrix A is that it is unitarily similar to a matrix with scalar diagonal (see, for example, page 109 of [3]); i.e, for some unitary matrix U, the matrix U^*AU has all of its diagonal entries equal, or $\mathrm{Diag}(U^*AU) = \lambda I$ (with $\lambda = \frac{1}{n}\mathrm{tr}(A)$, where n is the size of A). This result also has infinite-dimensional generalizations [1]. If general similarities are allowed, then every sequence $(x_1, x_2, \ldots, x_n)$ satisfying $\mathrm{tr}(A) = \sum_{k=1}^{n} x_k$ occurs as the sequence of diagonal entries in some matrix similar to A. (See, e.g., [5] or [2], where the statement is proved for matrices over an arbitrary field.)

We also mention the interesting characterisation of diagonals of various types of matrices. For example, Horn [4] shows that the real sequence $(d_1, d_2, \ldots, d_n)$ is the diagonal of a rotation matrix of order n (i.e., a real orthogonal $n \times n$ matrix of determinant 1) if and only if it is in the convex hull of those sequences $(\pm 1, \pm 1, \ldots, \pm 1)$

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in which -1 occurs an even number of times (possibly zero times). The same paper shows that if $d_i \geqslant 0$ for all i, then $(d_1, d_2, \ldots, d_n)$ is the diagonal of a rotation matrix if and only if it is the diagonal of a doubly stochastic matrix: whence the well-known Schur-Horn Theorem follows. (That theorem states that there exists a complex $n \times n$ hermitian matrix with eigenvalues $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ and diagonal entries $d_1 \leqslant \ldots \leqslant d_n$ if and only if for each $i = 1, \ldots, n-1$ we have that $d_1 + \cdots + d_i \leqslant \lambda_1 + \cdots + \lambda_i$ and $d_1 + \cdots + d_n = \lambda_1 + \cdots + \lambda_n$. Analogous result also holds for real orthogonal matrices.)

We are interested in collections of matrices with scalar diagonals. Perhaps the simplest example of a (non-trivial, multiplicative) group of matrices with this property is that generated by a cyclic permutation matrix, where the diagonal of each member consists of either zeros or ones. With a little care, one can actually show that if A is any invertible normal matrix, there is a unitary matrix U such that the group generated by U^*AU consists of matrices with scalar diagonals. In fact, it is a special case of our main result that every commutative, self-adjoint group of matrices (or the algebra generated by such a group) has the scalar-diagonal property after a suitable unitary similarity.

We wish to characterise, up to unitary similarity, those self-adjoint subalgebras (equivalently, semigroups) of $\mathbb{M}_n(\mathbb{C})$, every member of which has a scalar diagonal. Such an algebra $\mathscr{A}$ will necessarily have non-trivial invariant subspaces (if $n > 1$). This is a consequence of Burnside's Theorem, which states that the only subalgebra of $\mathbb{M}_n(\mathbb{C})$ without non-trivial invariant subspaces is $\mathbb{M}_n(\mathbb{C})$ itself. Clearly $\mathbb{M}_n(\mathbb{C})$ does not admit scalar diagonals.

One can in fact show that, under the hypothesis, the number of mutually orthogonal invariant subspaces for $\mathscr{A}$ is no less than $\sqrt{n}$. But our characterisation will imply more: $\mathscr{A}$ is unitarily similar to a direct sum of algebras $\mathbb{M}_{m_i}(\mathbb{C}) \otimes I_{k_i}$, where $m_i \leqslant k_i$ for each i.

Since a unital, self-adjoint semigroup $\mathscr{S}$, e.g. a group of unitary matrices, has the scalar-diagonal property if and only if the algebra generated by it does, the main result has an easy corollary for such semigroups.

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2. Notation and preliminary observations

We use the convention whereby *we start counting at* 0. In particular, we enumerate rows and columns of an $n \times n$ matrix by the integers $0, 1, \ldots, (n-1)$ with the convention that the row and column indices *are taken modulo* n. For example, if $n = 3$ and

$$A = (a_{i,j})_{i,j=0}^{n-1} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in \mathscr{M}_3(\mathbb{C}),$$

then $a_{0,0} = a_{3,0} = 1$, $a_{1,2} = a_{7,-1} = 6$, the 0-row of A is $(1\ 2\ 3)$, and the 1-column of A is $(2,5,8) := \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$. For an integer $n \geqslant 1$, we consider elements of the vector space $\mathbb{C}^n$ to be column vectors; we often use the usual comma-delimited row notation for them (as at the end of the preceding sentence).

For a strictly positive integer n, we use $\omega_n = e^{2\pi i/n}$ to denote a fixed primitive n-th root of 1 and we use F_n to denote the $n \times n$ discrete Fourier transform (DFT) matrix

$$F_n = \frac{1}{\sqrt{n}} \left(\omega_n^{jk}\right)_{j,k=0}^{n-1} \in \mathscr{M}_n(\mathbb{C}).$$

For example

$$F_1 = (1), \quad F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega_3 & \omega_3^2 \\ 1 & \omega_3^2 & \omega_3 \end{pmatrix}.$$

It is well known (and easy to see) that F_n is a unitary matrix.

The space of all $n \times n$ matrices with scalar diagonal is denoted by $\mathscr{SD}_n$, i.e.,

$$\mathscr{SD}_n = \left\{ \left(a_{j,k}\right)_{j,k=0}^{n-1} \in \mathscr{M}_n(\mathbb{C}) : a_{0,0} = a_{1,1} = \ldots = a_{n-1,n-1} \right\} \subseteq \mathscr{M}_n(\mathbb{C}).$$

The space of all $n \times n$ matrices whose non-main-diagonal circulant sums are 0 (we call the sum in the definition below a *circulant sum*) is denoted by $\mathscr{ZCS}_n$, i.e.,

$$\mathscr{ZCS}_n = \left\{ \left(a_{j,k}\right)_{j,k=0}^{n-1} \in \mathscr{M}_n(\mathbb{C}) : \forall m \in \{1, \ldots, n-1\}, \sum_{j=0}^{n-1} a_{j,j+m} = 0 \right\} \subseteq \mathscr{M}_n(\mathbb{C}).$$

We use $\mathscr{ZDS}_n$ to denote the space of all matrices whose non-main-diagonal diagonal sums are 0, i.e.,

$$\mathscr{ZDS}_n = \left\{ \left(a_{j,k}\right)_{j,k=0}^{n-1} : \forall m \in \{1, \ldots, n-1\}, \sum_{j=0}^{n-m} a_{j,j+m} = 0 = \sum_{j=0}^{n-m} a_{j+m,j} \right\} \subseteq \mathscr{M}_n(\mathbb{C}).$$

In the proof of the main result of the paper we will use the fact that $\mathscr{ZDS}_n \subseteq \mathscr{ZCS}_n$ and the following lemma.

LEMMA 2.1. $F_n(\mathscr{ZCS}_n)F_n^* = \mathscr{SD}_n$.

Proof. Since each of the spaces $\mathscr{ZCS}_n, F_n(\mathscr{ZCS}_n)F_n^*$ and $\mathscr{SD}_n$ has dimension $n^2 - n + 1$, it is sufficient to prove that $F_n(\mathscr{ZCS}_n)F_n^* \subseteq \mathscr{SD}_n$. Let $A = (a_{i,j})_{i,j=0}^{n-1} \in \mathscr{ZCS}_n$. That is, for all $k \in \{1, \ldots, n-1\}$ we have that $\sum_{i=0}^{n-1} a_{i,i+k} = 0$. We now compute $b_{i,i}$, the (i,i)-entry of $B = F_n A F_n^*$. If $\zeta = \omega_n^i$, then (recall that we are taking indices modulo n):

$$\begin{aligned} nb_{i,i} &= \left(1 \ \zeta \ \ldots \ \zeta^{n-1}\right) A \begin{pmatrix} 1 \\ \overline{\zeta} \\ \vdots \\ \overline{\zeta}^{n-1} \end{pmatrix} \\ &= \left(\sum_{j=0}^{n-1} \zeta^j a_{j,0} \ \sum_{j=0}^{n-1} \zeta^j a_{j,1} \ \ldots \ \sum_{j=0}^{n-1} \zeta^j a_{j,n-1}\right) \begin{pmatrix} 1 \\ \overline{\zeta} \\ \vdots \\ \overline{\zeta}^{n-1} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\zeta^j a_{j,k}\overline{\zeta}^k = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\zeta^{j-k}a_{j,k} \\
&\overset{r=k-j}{=} \sum_{j=0}^{n-1}\sum_{r=0}^{n-1}\zeta^{-r}a_{j,j+r} = \sum_{r=0}^{n-1}\sum_{j=0}^{n-1}\zeta^{-r}a_{j,j+r} \\
&= \sum_{r=0}^{n-1}\zeta^{-r}\sum_{j=0}^{n-1}a_{j,j+r} = \sum_{j=0}^{n-1}a_{j,j} + \sum_{r=1}^{n-1}\zeta^{-r}\sum_{j=0}^{n-1}a_{j,j+r} \\
&= \sum_{j=0}^{n-1}a_{j,j} + \sum_{r=1}^{n-1}\zeta^{-r}\cdot 0 = \sum_{j=0}^{n-1}a_{j,j} = \operatorname{tr}(A). \quad \square
\end{aligned}
$$

Throughout the paper we use the canonical isomorphism $\mathscr{M}_{n_1}(\mathbb{C})\otimes\mathscr{M}_{n_2}(\mathbb{C}) \xrightarrow{\sim} \mathscr{M}_{n_1 n_2}(\mathbb{C})$ given by identifying $A\otimes B$ with the $n_2\times n_2$ block matrix whose (i,j)-block is the $n_1\times n_1$ matrix $b_{i,j}A$. In particular, we identify $A\otimes I_n$ with the block diagonal matrix $\operatorname{diag}(A,\dots,A)$.

We use $\langle -,-\rangle$ to denote the standard inner product on $\mathbb{C}^n$, i.e., for column vectors $x=(x_0,\dots,x_{n-1})$ and $y=(y_0,\dots,y_{n-1})$, here written in the comma-delimited row notation, we have $\langle x,y\rangle = \sum_{i=0}^{n-1} x_i\overline{y}_i$. If $X\subseteq\mathbb{C}^n$, then $X^\perp = \{y\in\mathbb{C}^n : \forall x\in X, \langle x,y\rangle = 0\}$ denotes its orthogonal complement. We employ the same notations $\langle -,-\rangle$ and $(-)^\perp$ to denote the standard inner product and orthogonal complement on $\mathscr{M}_n(\mathbb{C})$. That is, if $A=(a_{i,j})_{i,j=0}^{n-1}$ and $B=(b_{i,j})_{i,j=0}^{n-1}$, then

$$
\langle A,B\rangle = \operatorname{tr}(AB^*) = \sum_{i,j=0}^{n-1} a_{i,j}\overline{b_{i,j}}.
$$

If $\mathscr{L}\subseteq\mathscr{M}_n(\mathbb{C})$, then $\mathscr{L}^\perp = \{A\in\mathscr{M}_n(\mathbb{C}) : \forall X\in\mathscr{M}_n(\mathbb{C}), \langle A,X\rangle = 0\}$.

We use $e_0,\dots,e_{n-1}$ to denote the standard basis of $\mathbb{C}^n$. We use $C_i\colon \mathscr{M}_n(\mathbb{C})\to\mathbb{C}^n$, $i\in\{0,\dots,n-1\}$ to denote the projection to column i, i.e., if $A\in\mathscr{M}_n(\mathbb{C})$, then $C_i(A)=Ae_i$. We use $Q_i\colon\mathbb{C}^n\to\mathscr{M}_n(\mathbb{C})$, $i=0,\dots,n-1$ to denote the injection into the column i, i.e., for a column vector $x\in\mathbb{C}^n$ we have $Q_i(x)=xe_i^*$.

3. Matrix $*$-algebras that have scalar diagonal

Lemma 3.1. *Let D be a diagonal matrix of trace 0 and let $\mathscr{A}\subseteq\mathscr{M}_n(\mathbb{C})$ be a unital $*$-algebra. If $\mathscr{A}\subseteq\mathscr{S}\mathscr{D}_n$, then $\mathscr{A}D\subseteq\mathscr{A}^\perp$.*

Proof. Let $A,X\in\mathscr{A}$. Since $\mathscr{A}$ is a $*$-algebra we have that $A^*X\in\mathscr{A}\subseteq\mathscr{S}\mathscr{D}_n$, hence A^*X has a scalar diagonal, and therefore $0=\operatorname{tr}(D^*(A^*X)) = \langle X,AD\rangle$. $\square$

Lemma 3.2. *Let $\mathscr{A}\subseteq\mathscr{M}_n(\mathbb{C})$ be a $*$-algebra. If $\mathscr{A}\subseteq\mathscr{S}\mathscr{D}_n$, then for each $i=0,\dots,n-1$ we have that $\dim\mathscr{A} = \dim C_i(\mathscr{A})$.*

Proof. We will prove that $\dim\mathscr{A} = \dim C_0(\mathscr{A})$. The claim of the lemma then follows from the fact that the condition $\mathscr{A}\subseteq\mathscr{S}\mathscr{D}_n$ is invariant under any permutational similarity. Indeed, if $\mathscr{A} \subseteq \mathscr{S}\mathscr{D}_n$ and P is the permutation matrix corresponding to the transposition that switches 0 and j, then $P\mathscr{A}P \subseteq \mathscr{S}\mathscr{D}_n$ and $C_0(P\mathscr{A}P) = P(C_j(\mathscr{A}))$.

For $i = 0,\dots,n-1$ let $\mathscr{C}_i = C_i(\mathscr{A})$, let $k_i = \dim \mathscr{C}_i$, and let $\mathscr{Q}_i = Q_i(\mathscr{C}_i^{\perp})$. It is clear that for all i we have that $\mathscr{Q}_i \subseteq \mathscr{A}^{\perp}$.

Let

$$\mathscr{X} = \operatorname{Span}\{AD : A \in \mathscr{A}, D \text{ diagonal}, \operatorname{tr}(D) = 0\}.$$

By Lemma 3.1 we have that $\mathscr{X} \subseteq \mathscr{A}^{\perp}$. We claim that $\dim \mathscr{X} \geqslant \sum_{i=1}^{n-1} k_i$. To this end let, for $i = 1,\dots,n-1$, $\widetilde{C}_i \colon \mathscr{C}_i \to \mathscr{A}$ denote a fixed right linear inverse of the surjective linear map $C_i|_{\mathscr{A}} \colon \mathscr{A} \to \mathscr{C}_i$ (e.g., find C_i-pre-images of a basis of $\mathscr{C}_i$ and extend by linearity). Also note that for $D_i = -e_0e_0^* + e_ie_i^*$, we have that M_{D_i}, the right multiplication by D_i, is equal to $-Q_0C_0 + Q_iC_i$. Using this fact it is easy to see that the linear map

$$F \colon \mathscr{C}_1 \times \ldots \times \mathscr{C}_{n-1} \to \mathscr{X}$$

given by

$$F(x_1,\dots,x_{n-1}) := \sum_{i=1}^{n-1} M_{D_i}(\widetilde{C}_i(x_i)) = \left(-\textstyle\sum_{i=1}^{n-1} Q_0C_0\widetilde{C}_i(x_i)\ x_1\ \ldots\ x_{n-1}\right) \in \mathscr{X},$$

is a well-defined linear injection; so $\sum_{i=1}^{n-1} k_i = \dim(\mathscr{C}_1 \times \ldots \times \mathscr{C}_{n-1}) \leqslant \dim \mathscr{X}$.

Since the spaces $\mathscr{X}, \mathscr{Q}_0,\dots,\mathscr{Q}_{n-1}$, are pairwise orthogonal we therefore have that

$$\begin{aligned}
\dim \mathscr{A}^{\perp} &\geqslant \dim(\mathscr{X} + \mathscr{Q}_0 + \ldots + \mathscr{Q}_{n-1})\\
&= \dim \mathscr{X} + \sum_{i=0}^{n-1} \dim \mathscr{Q}_i\\
&\geqslant \sum_{i=1}^{n-1} k_i + \sum_{i=0}^{n-1} (n - k_i) = n^2 - k_0.
\end{aligned}$$

Hence

$$\dim \mathscr{A} = n^2 - \dim \mathscr{A}^{\perp} \leqslant n^2 - (n^2 - k_0) = k_0 = \dim C_0(\mathscr{A}) \leqslant \dim \mathscr{A},$$

and thus $\dim \mathscr{A} = \dim C_0(\mathscr{A})$. $\square$

LEMMA 3.3. *Let $\mathscr{A} = \mathscr{M}_m \otimes I_n \oplus 0_p \subseteq \mathscr{M}_{mn+p}$. If $\mathscr{A}$ is unitarily similar to a subset of $\mathscr{S}\mathscr{D}_{mn+p}$, then $m \leqslant n$.*

Proof. Let U be a unitary matrix such that $U\mathscr{A}U^* \subseteq \mathscr{S}\mathscr{D}_{mn+p}$.

Since $\mathscr{A}$ has a common range of of dimension mn, we have that $\dim C_0(U\mathscr{A}U^*) \leqslant mn$. In more detail: let $\mathscr{X} \subseteq \mathbb{C}^{mn+p}$ be the subspace spanned by the first mn basis vectors $e_0,\dots,e_{mn-1}$. Then for each $f \in \mathbb{C}^{mn+p}$, the space $\mathscr{A}f$ is contained in $\mathscr{X}$. Therefore $C_0(U\mathscr{A}U^*) = U(\mathscr{A}(U^*e_0))$ is contained in $U(\mathscr{X})$.

Hence by Lemma 3.2 we have that

$$m^2 = \dim \mathscr{A} = \dim(U\mathscr{A}U^*) = \dim C_0(U\mathscr{A}U^*) \leqslant mn,$$

and therefore $m \leqslant n$. $\square$

LEMMA 3.4. *Let* $\mathscr{A} = \mathscr{M}_m(\mathbb{C}) \otimes I_n \subseteq \mathscr{M}_{mn}(\mathbb{C})$ *with* $m \leqslant n$. *Let* $D_{m,n} = \operatorname{diag}(1, \omega_n, \ldots, \omega_n^{m-1}) \in \mathscr{M}_m(\mathbb{C})$ *and let* $\widetilde{D}_{m,n} = \operatorname{diag}\left(I_m, D_{m,n}, \ldots, D_{m,n}^{n-1}\right) \in \mathscr{M}_{mn}$. *Then* $\widetilde{D}_{m,n}\mathscr{A}\widetilde{D}_{m,n}^* \subseteq \mathscr{Z}\mathscr{D}\mathscr{S}_{mn}$.

Proof. Let $A = (a_{i,j})_{i,j=0}^{m-1} \in \mathscr{M}_m(\mathbb{C})$, let $B = (b_{i,j})_{i,j=0}^{mn-1} = \widetilde{D}_{m,n}(A \otimes I_n)\widetilde{D}_{m,n}^*$. We will prove that all strictly upper diagonal sums $\sum_{i=0}^{mn-1-k} b_{i,i+k}$, $k \in \{1, \ldots, mn-1\}$ are zero (the proof of the fact that all strictly lower diagonal sums are zero is symmetric). Let $k \in \{1, \ldots, mn-1\}$. Note that for $p \in \{0, \ldots, m-1\}$, and $q \in \{0, \ldots, n-1\}$, we have that

$$b_{mq+p,mq+p+k} = \begin{cases} \omega_n^{-kq} a_{p,p+k} & \text{if } k \leqslant m-1-p \\ 0 & \text{if } k > m-1-p \end{cases}.$$

Hence for $k > m-1-p$ we clearly have that $\sum_{i=0}^{mn-1-k} b_{i,i+k} = 0$. Now assume that $k \leqslant m-1-p$. Since $m \leqslant n$ we have $\omega_n^k \neq 1$ and thus $\sum_{q=0}^{n-1} \omega^{-kq} = 0$. Therefore

$$\begin{aligned} \sum_{i=0}^{mn-1-k} b_{i,i+k} &= \sum_{p=0}^{m-1-k} \sum_{q=0}^{n-1} b_{mq+p,mq+p+k} \\ &= \sum_{p=0}^{m-1-k} \sum_{q=0}^{n-1} \omega_n^{-kq} a_{p,p+k} \\ &= \sum_{p=0}^{m-1-k} a_{p,p+k} \sum_{q=0}^{n-1} \omega_n^{-kq} = 0. \quad \square \end{aligned}$$

THEOREM 3.5. *Let* $\mathscr{A}$ *be a unital* $*$*-subalgebra of* $\mathscr{M}_n(\mathbb{C})$. *Then* $\mathscr{A}$ *is unitarily similar to a subspace of* $\mathscr{S}\mathscr{D}_n$ *if and only if up to unitary similarity we have that* $\mathscr{A} = \bigoplus_{i=1}^k \mathscr{M}_{m_i} \otimes I_{n_i}$ *with* $m_i \leqslant n_i$ *for* $i = 1, \ldots, k$.

Proof. We assume, with no loss of generality, that $\mathscr{A} = \bigoplus_{i=1}^k \mathscr{M}_{m_i} \otimes I_{n_i}$ for some natural numbers k, $m_1, \ldots, m_k$, $n_1, \ldots, n_k$.

($\Longrightarrow$): Assume that $\mathscr{A}$ is unitarily similar to a subspace of $\mathscr{S}\mathscr{D}_n$. For each $i \in \{1, \ldots, k\}$, $\mathscr{A}$ has a subalgebra $\mathscr{A}_i$ unitarily similar to $\mathscr{M}_{m_i} \otimes I_{n_i} \oplus 0_{n-m_in_i}$. Since each $\mathscr{A}_i$ is also unitarily similar to a subspace of $\mathscr{S}\mathscr{D}_n$ we have, by Lemma 3.3, that $m_i \leqslant n_i$ for all i.

($\Longleftarrow$): Assume that for all $i = 1, \ldots, k$ we have that $m_i \leqslant n_i$. Let $D = \operatorname{diag}(\widetilde{D}_{m_1,n_1}, \ldots, \widetilde{D}_{m_k,n_k})$. By Lemma 3.4 we have that $D\mathscr{A}D^* \subseteq \mathscr{Z}\mathscr{D}\mathscr{S}_n$. Since $\mathscr{Z}\mathscr{D}\mathscr{S}_n \subseteq \mathscr{Z}\mathscr{C}\mathscr{S}_n$ we therefore have, by Lemma 2.1, that $F_nD\mathscr{A}D^*F_n^* \subseteq \mathscr{S}\mathscr{D}_n$. $\square$

COROLLARY 3.6. *Every unital, commutative, self-adjoint semigroup of matrices – in particular, every commutative group of unitaries – is unitarily similar to a semigroup of matrices with scalar diagonal.*

Proof. Let $\mathscr{S}$ be a unital, commutative, self-adjoint semigroup of matrices. We can assume with no loss of generality that $\mathscr{S}$ consists of diagonal matrices. Then so does the algebra $\mathscr{A}$ generated by $\mathscr{S}$. Just note that $\mathscr{A}$ is a direct sum of algebras of the form $\mathbb{M}_{m_i} \otimes I_{n_i}$ with $m_i = 1$ for all i. $\square$

The corollary above also follows from Lemma 2.1 which gives an explicit formula for the unitary similarity in question.

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G. MacDonald
School of Mathematical and Computational Sciences
University of Prince Edward Island
Charlottetwon, PE, Canada C1A 4P3
e-mail: gmacdonald@upei.ca

L. W. Marcoux
Department of Pure Mathematics
University of Waterloo
Waterloo, ON, Canada N2L 3G1
e-mail: LWMarcoux@uwaterloo.ca

M. Mastnak
Department of Mathematics and Computing Science
Saint Mary's University
923 Robie St, Halifax, Nova Scotia, Canada B3N 1Z9
e-mail: mmastnak@cs.smu.ca

M. Omladič
Institute of Mathematics, Physics and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia
e-mail: matjaz@omladic.net

H. Radjavi
Department of Pure Mathematics
University of Waterloo
Waterloo, ON, Canada N2L 3G1
e-mail: hradjavi@uwaterloo.ca