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THE NORM OF AN INFINITE L–MATRIX

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Abstract. Evaluating the norm of infinite matrices, as operators acting on the sequence space ℓ^2, is not an easy task. For a few celebrated matrices, e.g., the Hilbert matrix and the Cesàro matrix, the precise value of the norm is known. But, for many other important cases we use estimated values of norm. In this note, we study the norm of L-matrices $A=[a_n]$, which appear in studying Hadamard multipliers of function spaces. We provide some necessary and sufficient conditions for the finiteness of norm and study the sharpness of these conditions. In particular, for the decay rate $a_n = O(1/n^{\alpha})$, our characterization is complete. Finally, parallel to the above classical results of Hilbert and Cesàro, we succeed to show that $\|A_s\| = 4$ for the family of L-matrices $A_s = [1/(n+s)]$, irrelevant of the parameter s which runs over $[1/2,\infty)$.

1. Introduction

Infinite matrices appear in studying bounded linear operators on infinite dimensional Hilbert spaces. In particular, we consider the sequence Hilbert space $\ell^2 = \{(x_0,x_1,\dots) : \|x\| < \infty\}$, where $\|x\| := \left(\sum_{n=0}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$, and the operators $A : \ell^2 \to \ell^2$ equipped with the operator norm $\|A\|_{\ell^2\to\ell^2} = \sup_{x\in\ell^2\setminus\{0\}} \frac{\|Ax\|}{\|x\|}$. Each such operator has the canonical representation $A=[a_{ij}]$, where $a_{ij} = \langle Ae_j, e_i\rangle_{\ell^2}$, with respect to the standard orthonormal basis $(e_n)_{n\geqslant 0}$ of ℓ^2. Due to connections to function theory, the index is started from zero. However, without loss of generality, it can equally start from one. The precise determination of $\|A\|_{\ell^2\to\ell^2}$ is usually a difficult task. Except for some special cases, we are mostly content with upper estimations of the norm. The Schur test is an effective method to obtain such upper bounds [13].

Let us mention two celebrated examples. Upon studying some questions in approximation theory, the matrix

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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was introduced by D. Hilbert in 1894 [7]. He obtained exact formula for the determinant of finite Hilbert matrices and investigated their asymptotics. We also know that $\|H\| = \pi$ [3, 5]. The Cesàro matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is related to the simplest Cesàro summation method which appears in studying divergent series [6]. We know that $\|C\| = 2$ [2]. As a byproduct of our main results, we show that

$$A_s = \begin{pmatrix} \frac{1}{s} & \frac{1}{s+1} & \frac{1}{s+2} & \cdots \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+2} & \cdots \\ \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{s+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a bounded operator on ℓ^2 and, more importantly, we have $\|A_s\| = 4$ for all $s \geqslant 1/2$.

Infinite matrices have been the center of several recent studies. It is not possible to address them all here. We mention just a few which actually reveals authors' research preferences. Bozkurt [1], Solak [14], Solak–Bozkurt [15] and Orr [12] studied the norm of infinite matrices. van de Mee–Seatzu [16] gave a very interesting algorithm to generate infinite multi-index positive self-adjoint Toeplitz matrices. Ismail–Štampach [9] and Dai–Ismail–Wang [4] provided a complete spectral analysis of self-adjoint operators action on $\ell^2(\mathbb{Z})$ and studied their connections to difference equations. See also N. Hindman [8].

2. The origin of L-matrices and main results

We encountered these matrices in studying the Hadamard multipliers in function spaces [10, 11]. Characterizing $\mathrm{Mult}(X)$, the multipliers of a Banach space of analytic functions on the open unit disc $\mathbb{D}$, is essential in various studies of function spaces, e.g., zero sets, invariant subspaces, cyclic elements, etc. In [11], we observed that $h(z) = \sum_{n=0}^{\infty} c_n z^n$ is a Hadamard multiplier for the Dirichlet Space D_ω if and only if the infinite matrix

$$T_h = \begin{pmatrix} c_1 - c_0 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \cdots \\ 0 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \cdots \\ 0 & 0 & c_3 - c_2 & c_4 - c_3 & \cdots \\ 0 & 0 & 0 & c_4 - c_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

acts as a bonded operator on ℓ^2. This essential observation gave birth to the study of L-matrices, which is an interesting subject by itself. Let $(a_n)_{n \geqslant 0}$ be a sequence of complex numbers. Then the infinite matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

is called an *L-matrix*. Abusing the notation, we will write $A = [a_n]$. However, despite being slightly confusing, a general element of A will also be denoted by a_{ij}, where i and j run through $\{0,1,2,\dots\}$. This concept should not be mixed with another family of matrices, which is also called L-matrices, in the theory of large linear systems [17, Page 42].

In this note, our main goal is to evaluate the norm of an L-matrix. We start with the necessary condition

$$a_n = O\Big(\frac{1}{\sqrt{n}}\Big), \qquad (n \to \infty),$$

in Section 3 and study its sharpness. Then, in Section 4, we study positive decreasing sequences. In Section 5, we study the general case and present a sufficient condition. Section 6 contains two definitive results. First, the general theorem leads to a complete description of sequences which satisfy the decay rate $1/n^{\alpha}$. Second, it also enables us to detect a very interesting phenomenon for a special family of L-matrices which depend on a parameter. Surprisingly enough, the norm does not depend on the parameter and, moreover, we can precisely determine the norm.

3. A necessary condition and its sharpness

If A is bounded on ℓ^2, then each column should be an element of ℓ^2. Therefore, by considering the norm of n-th column

$$(n+1)|a_n|^2 + |a_{n+1}|^2 + |a_{n+2}|^2 + \cdots < \infty,$$

we see that a necessary condition is

$$a_n = O\Big(\frac{1}{\sqrt{n}}\Big), \qquad (n \to \infty). \tag{1}$$

We provide two examples; one to show that this condition is not sufficient, the other to show that the rate $1/\sqrt{n}$ is sharp.

EXAMPLE 1. Let

$$a_{4^n} = \frac{1}{n2^n}, \qquad (n \geqslant 1),$$

and $a_j = 0$ for other values of index. This is a sparse matrix for which we have

$$\sum_{i,j=0}^{\infty} |a_{ij}|^2 = \sum_{n=1}^{\infty} \frac{2\cdot 4^n + 1}{n^2 4^n} < \infty.$$

Therefore, A is a bounded (indeed, Hilbert–Schmidt) operator on ℓ^2. We see that

$$\sqrt{m}\,a_m = \begin{cases} \frac{\log 4}{\log m} & \text{if} \quad m = 4^n, \\ 0 & \text{otherwise.} \end{cases}$$

However, with a similar technique, the decay rate $1/\log m$ can be decreased as much as required. As a matter of fact, let $\varphi(n)$ be any sequence of positive number with $\varphi(n) \to 0$, as $n \to \infty$. Note that there is no restriction of the rate of decay of $\varphi(n)$ (in the previous concrete example, we have $\varphi(n) = 1/n$). Pick a subsequence n_k such that

$$\sum_{k=1}^{\infty} \varphi^2(n_k) < \infty.$$

E.g., we can choose n_k such that $\varphi(n_k) < 1/k$. Then put

$$a_{4^{n_k}} = \frac{\varphi(n_k)}{2^{n_k}}, \qquad (k \geqslant 1),$$

and $a_j = 0$ for other values of index. Then, as in the above calculation, we easily verify that A is a Hilbert–Schmidt operator on ℓ^2 and, moreover,

$$\sqrt{n}\,a_n = O\big(\varphi(n)\big), \qquad (\text{as } n \to \infty).$$

This example shows that the decay rate $1/\sqrt{n}$ in the necessary condition (1) is optimal.

EXAMPLE 2. To show that the condition (1) is not sufficient consider

$$a_n = \frac{1}{(n+1)^{\alpha}}, \qquad (n \geqslant 0),$$

where $\alpha < 1$ is fixed. Even though we just need the case $\alpha = 1/2$ at this stage, we treat a slightly more general case to provide the motivation for an upcoming sufficient condition. Now consider the vector

$$x = (1^{\alpha}, 2^{\alpha}, \ldots, n^{\alpha}, 0, 0, 0, \ldots)^{tr}.$$

Then

$$\|x\|^2 = 1^{2\alpha} + 2^{2\alpha} + \cdots + n^{2\alpha} \asymp n^{2\alpha+1},$$

while

$$Ax = \begin{pmatrix} 1 \\ * \\ * \\ \vdots \\ * \\ * \\ \vdots \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ * \\ \vdots \\ * \\ * \\ \vdots \end{pmatrix} + \cdots + \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ * \\ \vdots \end{pmatrix},$$

where $*$s represent some positive numbers, and thus

$$\|Ax\|^2 = \sum_{j=0}^{\infty} |(Ax)_j|^2 \geqslant \sum_{j=1}^{n} j^2 \asymp n^3.$$

Here, by $X(n) \asymp Y(n)$ we mean that there are two positive constants c_1 and c_2, independent of the index n, such that the inequalities $c_1 X(n) \leqslant Y(n) \leqslant c_2 X(n)$ uniformly hold for all values of n. Therefore,

$$\frac{\|Ax\|}{\|x\|} \geqslant C n^{1-\alpha} \to \infty.$$

This observation shows that the condition

$$a_n = O\Big(\frac{1}{n^\alpha}\Big), \qquad (\alpha < 1),$$

is not sufficient to ensure that A is a bounded operator on ℓ^2. This example also raises the following question: is the condition

$$a_n = O\Big(\frac{1}{n}\Big), \qquad (n \to \infty), \tag{2}$$

sufficient to ensure the boundedness of A on ℓ^2? We will shortly see that the answer is affirmative, and thus we conclude that the exponent $\alpha = 1$ in the expression (2) is also sharp.

4. The sufficient condition – decreasing sequences

According to a special case of Schur's test, if $T = [t_{ij}]$ is a symmetric matrix with positive entries and there are $p_i > 0$ and $\alpha > 0$ such that

$$\sum_i p_i t_{ij} \leqslant \alpha p_j$$

for all j, then T is a bounded operator on ℓ^2 with $\|T\|_{\ell^2 \to \ell^2} \leqslant \alpha$. This criteria is applied below to obtain a sufficient condition for the boundedness of L-matrices. We will see that the condition provides a complete characterization in some particular cases. We start with the special case of decreasing sequences. Then we present the general situation.

THEOREM 1. *Let $A = [a_n]$ be an L-matrix such that*

$$a_0 > a_1 > a_2 > \cdots > 0$$

and that

$$\Delta := \sup_{n \geqslant 1} \frac{2a_n(a_n + a_{n-1})}{a_{n-1} - a_n} < \infty.$$

Then $A \in \mathscr{L}(\ell^2)$ and, moreover,

$$\|A\|_{\ell^2 \to \ell^2} \leqslant \max\{2a_0, \Delta\}.$$

Proof. Let $p_0 := 1$ and

$$p_n = \frac{a_{n-1} - a_n}{2a_n a_{n-1}} S_{n-1}, \qquad (n \geqslant 1), \tag{3}$$

where $S_0 := a_0$ and, for $n \geqslant 1$,

$$S_n := \frac{a_0}{2^n}\left(1+\frac{a_1}{a_0}\right)\left(1+\frac{a_2}{a_1}\right)\left(1+\frac{a_3}{a_2}\right)\cdots\left(1+\frac{a_n}{a_{n-1}}\right). \tag{4}$$

Since each factor

$$\frac{1}{2}\left(1+\frac{a_n}{a_{n-1}}\right) < 1,$$

the limit $S_\infty := \lim_{n\to\infty} S_n$ exists and $S_\infty \geqslant 0$. By induction, S_n satisfies

$$S_n = a_n \sum_{i=0}^{n} p_i, \qquad (n \geqslant 0), \tag{5}$$

and thus $S_{n-1} - S_n = a_n p_n$, $n \geqslant 1$. Hence,

$$\sum_{i=n+1}^{\infty} a_i p_i = S_n - S_\infty \leqslant S_n, \qquad (n \geqslant 0). \tag{6}$$

Therefore, by (5) and (6), for a fixed $j \geqslant 0$,

$$\sum_{i=0}^{\infty} p_i a_{ij} = a_j \sum_{i=0}^{j} p_i + \sum_{i=j+1}^{\infty} p_i a_i \leqslant 2S_j.$$

For $j = 0$, this becomes

$$\sum_{i=0}^{\infty} p_i a_{i0} \leqslant 2a_0 p_0,$$

and, for $j \geqslant 1$, we get

$$\sum_{i=0}^{\infty} p_i a_{ij} \leqslant 2S_j = \left(1+\frac{a_j}{a_{j-1}}\right) S_{j-1} = \frac{2a_j(a_{j-1}+a_j)}{a_{j-1}-a_j} p_j \leqslant \Delta p_j,$$

which gives

$$\frac{1}{p_j}\sum_{i=0}^{\infty} p_i a_{ij} \leqslant \max\{2a_0, \Delta\}, \qquad (j \geqslant 0).$$

Therefore, by Schur's test, $A \in \mathscr{L}(\ell^2)$ with $\|A\| \leqslant \max\{2a_0, \Delta\}$. □

5. The sufficient condition – general case

As the combination $a_{n-1} - a_n$ in the denominator of expression for Δ shows, that the sequence (a_n) is strictly decreasing was heavily used in the proof of Theorem 1. For the general case, we need to find a remedy. This is done in the following, where the role is played by the sequence δ_n.

THEOREM 2. *Let $A=[a_n]$ be an L-matrix. Suppose that there is a sequence of strictly decreasing positive numbers δ_n, $n\geqslant 0$, such that*

$$\Delta := \sup_{n\geqslant 1} \frac{(|a_n|+\delta_{n-1})(|a_n|+\delta_n)}{\delta_{n-1}-\delta_n} < \infty.$$

Then $A\in\mathscr{L}(\ell^2)$ and, moreover,

$$\|A\|_{\ell^2\to\ell^2} \leqslant \max\{\delta_0+|a_0|,\Delta\}.$$

Proof. Since $\big\|[a_n]\big\|_{\ell^2\to\ell^2} \leqslant \big\|[|a_n|]\big\|_{\ell^2\to\ell^2}$, without loss of generality, we assume that $a_n\geqslant 0$, for all $n\geqslant 0$. Let $p_0:=1$ and

$$p_n := \frac{\delta_{n-1}-\delta_n}{\delta_n+a_n}\sum_{i=0}^{n-1} p_i, \qquad (n\geqslant 1).$$

By induction, it is straightforward to see that

$$p_n = \frac{\delta_{n-1}-\delta_n}{(\delta_n+a_n)\delta_{n-1}} S_{n-1}, \qquad (n\geqslant 1),$$

where $S_0:=\delta_0$ and

$$S_n := \left(\frac{\delta_0+a_1}{\delta_1+a_1}\right)\left(\frac{\delta_1+a_2}{\delta_2+a_2}\right)\cdots\left(\frac{\delta_{n-1}+a_n}{\delta_n+a_n}\right)\delta_n, \qquad (n\geqslant 1).$$

Equivalently, S_n satisfies

$$S_n = \delta_n\sum_{i=0}^{n} p_i, \qquad (n\geqslant 0). \tag{7}$$

The sequence S_n also satisfies the recurrence relation

$$S_{n-1}-S_n = a_n p_n, \qquad (n\geqslant 1). \tag{8}$$

Hence, as in the previous case, the sequence S_n is positive decreasing and

$$S_\infty := \lim_{n\to\infty} S_n$$

exists and $S_\infty \geqslant 0$. As a matter of fact, we can show that under some mild conditions $S_\infty = 0$, e.g., $\delta_n = O(a_n)$ suffices. But, this is not needed below. The difference equation (8) also implies

$$\sum_{i=n+1}^{\infty} a_i p_i = S_n - S_\infty \leqslant S_n, \qquad (n\geqslant 0). \tag{9}$$

Now, we are ready to apply Schur's test. Hence, by (7) and (9), for a fixed $j \geqslant 0$,

$$\sum_{i=0}^{\infty} p_i a_{ij} = a_j \sum_{i=0}^{j} p_i + \sum_{i=j+1}^{\infty} p_i a_i \leqslant \frac{a_j}{\delta_j} S_j + S_j$$
$$= (\delta_j + a_j) \frac{S_j}{\delta_j}.$$

For $j = 0$, this becomes

$$\sum_{i=0}^{\infty} p_i a_{i0} \leqslant (\delta_0 + a_0) p_0.$$

For $j \geqslant 1$, we get

$$\sum_{i=0}^{\infty} p_i a_{ij} \leqslant (\delta_j + a_j) \frac{S_j}{\delta_j} = (\delta_j + a_j) \left(\frac{\delta_{j-1} + a_j}{\delta_j + a_j} \right) \frac{S_{j-1}}{\delta_{j-1}}$$
$$= \frac{(\delta_{j-1} + a_j)(\delta_j + a_j)}{\delta_{j-1} - \delta_j} p_j \leqslant \Delta p_j.$$

In Short,

$$\frac{1}{p_j} \sum_{i=0}^{\infty} p_i a_{ij} \leqslant \max\{\delta_0 + a_0, \Delta\}, \qquad (j \geqslant 0).$$

Therefore, by Schur's test, $A \in \mathscr{L}(\ell^2)$ and $\|A\| \leqslant \max\{\delta_0 + a_0, \Delta\}$. □

6. The decay $1/n^\alpha$

In section 3, we started the discussion on the condition $a_n = O(1/n^\alpha)$. Using Theorem 2 we can complete the picture as follows. For the boundedness of L-matrix $A = [a_n]$, the condition $a_n = O(1/n^\alpha)$ is

$$\begin{cases} \text{necessary} & \text{if } \alpha = \frac{1}{2}, \\ \text{neither necessary nor sufficient} & \text{if } \frac{1}{2} < \alpha < 1, \\ \text{sufficient} & \text{if } \alpha = 1. \end{cases}$$

The necessary condition was shown at the beginning of section 3. Moreover, Examples 1, and 2 reveal that the condition $a_n = O(1/n^\alpha)$, $\frac{1}{2} < \alpha < 1$, is neither necessary nor sufficient. It remains to verify the last part. We state it as a simple corollary of Theorem 2.

COROLLARY 1. *Let $A = [a_n]$ be an L-matrix, such that*

$$a_n = O\left(\frac{1}{n}\right), \qquad (n \to \infty).$$

Then $A \in \mathscr{L}(\ell^2)$.

Proof. By assumption, there is a constant $M > 0$ such that

$$|a_n| \leqslant \frac{M}{n+1}, \qquad (n \geqslant 0).$$

Put

$$\delta_n = \frac{M}{n+1}, \qquad (n \geqslant 0).$$

Then, for $n \geqslant 1$,

$$\frac{(|a_n| + \delta_{n-1})(|a_n| + \delta_n)}{\delta_{n-1} - \delta_n} \leqslant \frac{(\frac{M}{n+1} + \frac{M}{n})(\frac{M}{n+1} + \frac{M}{n+1})}{\frac{M}{n} - \frac{M}{n+1}} = \frac{2(2n+1)M}{n+1} \leqslant 4M < \infty.$$

Hence, by Theorem 2, A is a bounded operator on ℓ^2. $\square$

In particular, Corollary 1 ensures that

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathscr{L}(\ell^2),$$

which is a known result. However, in this special case, we can precisely determine the norm.

COROLLARY 2. *For the L-matrix* $A_s = [\frac{1}{n+s}]$, *where* $s \geqslant \frac{1}{2}$, *we have*

$$\|A\|_{\ell^2 \to \ell^2} = 4.$$

Proof. Upper bound: by Theorem 1, we have

$$\|A_s\| \leqslant \max\left\{\frac{2}{s}, 4 - \frac{2}{n+s}\,(n \geqslant 1)\right\} = 4.$$

Lower bound: we use the inequality

$$\|A_s\| \geqslant \frac{\|Ax\|}{\|x\|},$$

where x is properly chosen. In fact, using the notations in the proof of Theorem 1, we set

$$x = x_m := (p_0, p_1, p_2, \cdots, p_m, 0, 0, \cdots)^{tr},$$

and then let $m \to \infty$. Write

$$(y_0, y_1, y_2, \cdots, \cdots)^{tr} := A_s x_m.$$

Then, for $0 \leqslant n \leqslant m-1$, we have

$$y_n = a_n \sum_{j=0}^{n} p_j + \sum_{j=n+1}^{m} a_j p_j$$

while, for $n \geqslant m$,

$$y_n = a_n \sum_{j=0}^{m} p_j.$$

Therefore, according to (5), we can simplify y_n as

$$y_n = \begin{cases} 2S_n - S_m & \text{if } \ 0 \leqslant n \leqslant m-1, \\ \\ \frac{a_n}{a_m} S_m & \text{if } \qquad n \geqslant m. \end{cases}$$

This observation implies

$$\|A_s x_m\|^2 = \sum_{n=0}^{m-1} (2S_n - S_m)^2 + \frac{S_m^2}{a_m^2} \sum_{n=m}^{\infty} a_n^2.$$

and thus

$$\|A_s x_m\|^2 \geqslant 4 \sum_{n=0}^{m-1} S_n^2 - 4 S_m \sum_{n=0}^{m-1} S_n. \tag{10}$$

To effectively use (10), we need to find S_n, $n \geqslant 0$. For the matrix A, $S_0 = \frac{1}{s}$ and for $n \geqslant 1$, by (4),

$$\begin{aligned} S_n &= \frac{a_0}{2^n} \left(1 + \frac{a_1}{a_0}\right) \left(1 + \frac{a_2}{a_1}\right) \left(1 + \frac{a_3}{a_2}\right) \cdots \left(1 + \frac{a_n}{a_{n-1}}\right) \\ &= \frac{1}{2^n s} \left(1 + \frac{s}{s+1}\right) \left(1 + \frac{s+1}{s+2}\right) \cdots \left(1 + \frac{s+n-1}{s+n}\right) \\ &= \frac{(s+\frac{1}{2})(s+\frac{3}{2}) \cdots (s+\frac{2n-1}{2})}{s(s+1)(s+2) \cdots (s+n)} \\ &= \frac{\Gamma(s)\,\Gamma(s+n+\frac{1}{2})}{\Gamma(s+\frac{1}{2})\,\Gamma(s+n+1)}. \end{aligned}$$

Hence, by (3),

$$p_n = \frac{1}{2} S_{n-1} = \frac{\Gamma(s)\,\Gamma(s+n-\frac{1}{2})}{2\Gamma(s+\frac{1}{2})\,\Gamma(s+n)} \asymp \frac{1}{\sqrt{n}}, \qquad (n \geqslant 1), \tag{11}$$

where Stirling's formula was used. By a combinatorial identity

$$\sum_{n=0}^{m-1} S_n = 2(m+s) S_m - 2.$$

As a matter of fact, such a precise identity is no needed. According to (11), $S_n = O(1/\sqrt{n})$ and thus $\sum_{n=0}^{m-1} S_n = O(\sqrt{m})$. This is enough for us. Note that in the precise identity above, we also have $2(m+s)S_m - 2 = O(\sqrt{m})$.

Therefore, we can write (10) as

$$\|A_s x_m\|^2 \geqslant 16\|x_m\|^2 - 8(m+s)S_m^2 - 16 = 16\|x_m\|^2 + O(1).$$

As the last observation, by (11),

$$\|x_m\|^2 = \sum_{j=0}^{m} p_j \asymp \sum_{j=1}^{m} \frac{1}{\sqrt{j}} \asymp \sqrt{m} \to \infty.$$

Finally, since $\|x_m\| \to \infty$, we conclude that

$$\|A_s\| \geqslant \lim_{m\to\infty} \frac{\|Ax_m\|}{\|x_m\|} = 4. \quad \square$$

7. Concluding remarks

1. For the L-matrix $A_s = [\frac{1}{n+s}]$, since $a_0 = \frac{1}{s}$, we certainly have

$$\|A_s\|_{\ell^2\to\ell^2} > 4, \qquad (0 < s < \frac{1}{4}).$$

On the other hand, Corollary 2, says

$$\|A_s\|_{\ell^2\to\ell^2} = 4, \qquad (s \geqslant \frac{1}{2}).$$

We have not being able to determine the behavior of $\|A_s\|_{\ell^2\to\ell^2}$ for small values of s, in particular in between $1/4$ and $1/2$. An interesting question is to determine the constant s_0, where s_0 is defined by

$$s_0 := \inf\{s : \|A_s\|_{\ell^2\to\ell^2} = 4\}.$$

At this stage, we just know that $\frac{1}{4} \leqslant s_0 \leqslant \frac{1}{2}$, even though with some more accurate calculations it is possible to slightly modify the end points. However, it seems that the current technics are not powerful enough to detect s_0.

2. In this note, we just considered the ℓ^2 norm. What happens if we consider A as a mapping between different ℓ^p spaces?

3. Does the norm of A_s still remain constant, e.g., for an interval $[s_{pq}, \infty)$, when we treat A_s as an operator mapping ℓ^p to ℓ^q? How does s_{pq} depend on the parameters p and q?

4. Lacunary L-matrices were considered in Example 1. In general case, is it possible to estimate their norm at least as mappings on ℓ^2?

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