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RANKS OF COMMUTATORS OF TRUNCATED TOEPLITZ OPERATORS ON FINITE DIMENSIONAL SPACES

YONG CHEN, KEI JI IZUCHI AND YOUNG JOO LEE

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Abstract. We study the rank of commutator $[A_\eta, A_\eta^*]$ of truncated Toeplitz operators A_η and A_η^* with several type of inner symbols η on the model space $\mathscr{H}_\theta$ with finite Blaschke product θ.

1. Introduction

Let $\mathbb{D}$ be the unit disk and $\mathbb{T}$ be the unit circle. We let H^2 be the classical Hardy space on $\mathbb{D}$ which can be identified with a closed subspace of L^2. Here, $L^p := L^p(\mathbb{T}, \sigma)$ denotes the usual Lebesgue space on $\mathbb{T}$ where σ is the normalized Lebesgue measure on $\mathbb{T}$. A function $\theta \in H^2$ is said to be *inner* if $|\theta(z)| = 1$ a.e. on $\mathbb{T}$. To each nonconstant inner function θ, we associate the model space $\mathscr{H}_\theta$ defined by

$$\mathscr{H}_\theta = H^2 \ominus \theta H^2$$

which is a nontrivial invariant subspace for the backward shift operator on H^2. When θ is a finite Blaschke product (see Section 2 for its definition) with order N, that is, $\operatorname{ord}\theta = N$, then $\dim \mathscr{H}_\theta = N$ (see Lemma 4), so in this case, $\mathscr{H}_\theta$ is a finite dimensional space. Let P_θ be the Hilbert space orthogonal projection from L^2 to $\mathscr{H}_\theta$. Given a function $\varphi \in L^\infty$, the truncated Toeplitz operator (briefly, TTO) A_φ with symbol φ is defined on $\mathscr{H}_\theta$ by

$$A_\varphi f = P_\theta(\varphi f)$$

for functions $f \in \mathscr{H}_\theta$. Then A_φ is a bounded linear operator on $\mathscr{H}_\theta$ and clearly $A_\varphi^* = A_{\overline{\varphi}}$.

Truncated Toeplitz operators are compressions of multiplication operators to model subspaces of the Hardy space H^2; they represent a far reaching generalization of classical Toeplitz matrices. Although particular case had appeared before in the literature, the general theory has been initiated in the seminal paper [11]. Since then, truncated Toeplitz operators have constituted an active area of research. We mention only a few papers: [1, 2, 3, 5, 6, 12, 13]; see also the recent survey [9] and the references within.

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In a recent paper [4], the rank of complex skew symmetric operators has been studied and, as a consequence, the following result about the rank of a commutator of two TTOs has been known. For two bounded operators S and T on a Hilbert space, we let $[S,T] = ST - TS$ be the commutator of S and T.

THEOREM 1. *Let θ be a non-constant inner function and $\varphi, \psi \in L^\infty$. If $[A_\varphi, A_\psi]$ has finite rank on $\mathscr{H}_\theta$, then the rank of $[A_\varphi, A_\psi]$ must be even.*

In view of this result, one might ask whether for any non-constant inner function θ and integer $N \geqslant 1$, there is a commutator of two TTOs on $\mathscr{H}_\theta$ whose rank is $2N$ exactly. At the same paper, it has been proved that this is true on model spaces corresponding to monomials by showing $[A_{z^N}, A^*_{z^N}]$ has rank exactly $2N$ on $\mathscr{H}_{z^n}$ when $2N \leqslant n$; see Proposition 7 of [4]. Motivated by this result, it is natural to ask the following question.

QUESTION 2. For finite Blaschke product θ and inner function η, what is the rank of the commutator $[A_\eta, A^*_\eta]$ on finite dimensional space $\mathscr{H}_\theta$?

In this paper, we consider the model space corresponding to general finite Blaschke product θ and then study the rank of the commutator $[A_\eta, A^*_\eta]$ induced by several types of inner functions η.

Suppose $\dim \mathscr{H}_\theta = N$. Note that $[A_\eta, A^*_\eta]$ is self adjoint. If the dimension of $\ker[A_\eta, A^*_\eta] = L$ is known, then the rank of $[A_\eta, A^*_\eta]$ is $N - L$. So it is important to characterize the kernel of $[A_\eta, A^*_\eta]$. Along this idea, we show that when $2\operatorname{ord}\eta \leqslant \operatorname{ord}\theta$, the rank of $[A_\eta, A^*_\eta]$ equals to $2\operatorname{ord}\eta$; see Theorem 7 in Section 3. This result extends Proposition 7 of [4] mentioned above, and also is closely related to kernels of Toeplitz operators on the Hardy space and the multipliers between certain model spaces; see Remark 8.

It is difficult to characterize the rank of $[A_\eta, A^*_\eta]$ when $2\operatorname{ord}\eta > \operatorname{ord}\theta$. So we consider some special cases. We first consider certain inner symbols η which has a nontrivial common inner divisor with θ; see Theorems 11 and 12 in Section 3. We also consider finite Blaschke product η which has no nontrivial common inner divisor with θ and obtain a rank inequality; see Theorem 15 in Section 3.

The paper is organized as follows. In Section 2 we just give several lemmas which will be used in the proofs of the main results. In Section 3 we present the main results and their proofs, also some corollaries are given. At Section 4, we give two examples relating to results obtained in Sections 3.

2. Preliminaries

Given $\psi \in L^\infty$, we recall the classical Toeplitz operator T_ψ with symbol ψ defined on H^2 by $T_\psi f = P(\psi f)$ for $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2 which can be given by

$$Pg(w) = \int_{\mathbb{T}} \frac{g(\zeta)}{1 - w\overline{\zeta}}\, d\sigma(\zeta), \qquad w \in \mathbb{D}$$

for functions $g \in L^2$. Note that $T_\psi^* = T_{\overline{\psi}}$. For $\varphi \in H^\infty$ and inner θ, it is easy to check that $T_\varphi^* \mathscr{H}_\theta \subset \mathscr{H}_\theta$ and hence

$$A_\varphi^* f = T_\varphi^* f \tag{1}$$

for functions $f \in \mathscr{H}_\theta$. Also, it is easy to verify that for $\varphi, \psi \in H^\infty$, $A_\varphi A_\psi = A_{\varphi\psi}$ on $\mathscr{H}_\theta$.

Given an inner function θ, it is well known that the orthogonal projection P_θ admits the following integral representation

$$P_\theta f(w) = \int_{\mathbb{T}} f(\zeta) \frac{1-\theta(w)\overline{\theta(\zeta)}}{1-w\overline{\zeta}}\, d\sigma(\zeta), \qquad w \in \mathbb{D}$$

and hence $P_\theta f = Pf - \theta P(\overline{\theta} f)$ for functions $f \in L^2$. In particular, we have

$$T_\theta^* f = \frac{f - P_\theta f}{\theta} \tag{2}$$

for every $f \in H^2$. See Chapter 5 of [8] for details and related facts.

We start with the following kernel description of a certain commutator of TTOs which will be useful. In the following, $\operatorname{rank} T$ and $\ker T$ denote the rank and kernel respectively of a bounded operator T on a Hilbert space.

LEMMA 3. *Let θ, η be two inner functions and $f \in \mathscr{H}_\theta$. If A_η is the TTO defined on $\mathscr{H}_\theta$, then $f \in \ker[A_\eta, A_\eta^*]$ if and only if*

$$\eta f - \eta P_\eta f - P_\theta(\eta f) + P_\eta P_\theta(\eta f) \in \eta\theta H^2.$$

Proof. By (1) and (2), we see

$$A_\eta A_\eta^* f = A_\eta T_\eta^* f = A_\eta \frac{f - P_\eta f}{\eta} = P_\theta(f - P_\eta f)$$

and similarly

$$A_\eta^* A_\eta f = A_\eta^* P_\theta(\eta f) = T_\eta^* P_\theta(\eta f) = \frac{P_\theta(\eta f) - P_\eta P_\theta(\eta f)}{\eta}$$

for functions $f \in \mathscr{H}_\theta$. Thus, $f \in \ker[A_\eta, A_\eta^*]$ if and only if

$$f - P_\eta f - \frac{P_\theta(\eta f) - P_\eta P_\theta(\eta f)}{\eta} \in \theta H^2,$$

which gives the desired assertion. The proof is complete. $\square$

Given $\lambda \in \mathbb{D}$, let

$$b_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}$$

be the Möbius transformation of $\mathbb{D}$. For any finite points $\lambda_1, \cdots, \lambda_N$ in $\mathbb{D}$, the inner function B defined by $B := \prod_{n=1}^N b_{\lambda_n}$ is called a finite Blaschke product of order N and we write $\operatorname{ord} B = N$. For finite Blaschke products, we have the following explicit description of the corresponding model space which is taken from Corollary 5.18 of [8].

LEMMA 4. *Let $a_1,\cdots,a_m \in \mathbb{D}$ be distinct points and $k_1,\cdots,k_m$ be positive integers. Put $B=\prod_{j=1}^m b_{a_j}^{k_j}$ and $N=\sum_{j=1}^m k_j$. Then we have*

$$\mathscr{H}_B = \sum_{j=0}^{N-1} \mathbb{C}\cdot \frac{z^j}{\prod_{n=1}^m (1-\overline{a_n}z)^{k_n}}.$$

In particular, we have $\dim \mathscr{H}_B = N$.

The following lemma is also useful in our study.

LEMMA 5. *Let θ,η be finite Blaschke products of order N,L respectively. Write $\theta=\prod_{n=1}^N b_{\alpha_n}$ and $\eta=\prod_{n=1}^L b_{\beta_n}$. Put*

$$M=\{f\in\mathscr{H}_\theta : \eta f\in\mathscr{H}_\theta\}, \qquad K=\{f\in M:\eta f\in M\}.$$

Then the following statements hold.

(a) $M\neq\{0\}$ if and only if $L<N$. In which case, we have

$$M=\frac{\prod_{n=1}^L(1-\overline{\beta_n}z)}{\prod_{n=1}^N(1-\overline{\alpha_n}z)}\sum_{j=0}^{N-L-1}\mathbb{C}\cdot z^j. \tag{3}$$

(b) $K\neq\{0\}$ if and only if $2L<N$. In which case, we have

$$K=\frac{\prod_{n=1}^L(1-\overline{\beta_n}z)^2}{\prod_{n=1}^N(1-\overline{\alpha_n}z)}\sum_{j=0}^{N-2L-1}\mathbb{C}\cdot z^j.$$

Proof. Since the proof of (b) is similar to that of (a), we only prove (a). First suppose $M\neq\{0\}$ and denote E as the set on the right side of (3). Let $g\in\mathscr{H}_\theta$ be nonzero for which $\eta g\in\mathscr{H}_\theta$. By Lemma 4, we may write

$$\eta g=\frac{\sum_{j=0}^{N-1}c_jz^j}{\prod_{n=1}^N(1-\overline{\alpha_n}z)}\in\mathscr{H}_\theta$$

for some constants c_j and hence

$$\frac{1}{\prod_{n=1}^N(1-\overline{\alpha_n}z)}\frac{\prod_{n=1}^L(1-\overline{\beta_n}z)}{\prod_{n=1}^L(\beta_n-z)}\sum_{j=0}^{N-1}c_jz^j=g\in\frac{\sum_{j=0}^{N-1}\mathbb{C}\cdot z^j}{\prod_{n=1}^N(1-\overline{\alpha_n}z)}.$$

Hence

$$\frac{\prod_{n=1}^L(1-\overline{\beta_n}z)}{\prod_{n=1}^L(\beta_n-z)}\sum_{j=0}^{N-1}c_jz^j\in\sum_{j=0}^{N-1}\mathbb{C}\cdot z^j$$

and then

$$\frac{\sum_{j=0}^{N-1}c_jz^j}{\prod_{n=1}^L(\beta_n-z)}$$

must be a polynomial. Since $\sum_{j=0}^{N-1}c_jz^j\neq 0$, we have $N-1\geqslant L$ and hence $N>L$, as desired. In this case, we have

$$\sum_{j=0}^{N-1}c_jz^j=\Big(\prod_{n=1}^L(\beta_n-z)\Big)\sum_{j=0}^{N-1-L}d_jz^j$$

for some constants d_j and hence

$$g = \frac{\prod_{n=1}^{L}(1-\overline{\beta}_n z)}{\prod_{n=1}^{N}(1-\overline{\alpha}_n z)} \sum_{j=0}^{N-L-1} d_j z^j \in E.$$

Thus $M \subset E$. On the other hand, if $N > L$, we see $E \subset \mathscr{H}_\theta$ and

$$\eta E = \frac{\prod_{n=1}^{L}(\beta_n - z)}{\prod_{n=1}^{N}(1-\overline{\alpha}_n z)} \sum_{j=0}^{N-L-1} \mathbb{C}\cdot z^j \subset \mathscr{H}_\theta$$

by Lemma 4, so $E \subset M$ holds. Consequently, we have $M \neq \{0\}$ and (3). The proof is complete. $\square$

For the set K introduced in Lemma 5, it is easy to see that

$$K = \{f \in \mathscr{H}_\theta : \eta^2 f \in \mathscr{H}_\theta\}. \tag{4}$$

For two inner functions u and v, we have

$$\mathscr{H}_{uv} = \mathscr{H}_u \oplus u\mathscr{H}_v; \tag{5}$$

see Lemma 5.10 of [8] for example. We say that two inner functions are *relatively prime* if they have no nontrivial common inner divisors.

LEMMA 6. *Let θ, η be finite Blaschke products which are relatively prime. If A_η is the TTO defined on $\mathscr{H}_\theta$, then the following statements hold.*

(a) *A_η is invertible on $\mathscr{H}_\theta$.*

(b) $\dim P_\theta \mathscr{H}_\eta = \min\{\operatorname{ord}\theta, \operatorname{ord}\eta\}$.

(c) $\dim P_{\theta\mathscr{H}_\eta} \eta\mathscr{H}_\theta = \min\{\operatorname{ord}\theta, \operatorname{ord}\eta\}$.

(d) *$P_{\eta\mathscr{H}_\theta} : \mathscr{H}_\theta \to \eta\mathscr{H}_\theta$ is one-to-one and onto.*

Proof. Let $f \in \mathscr{H}_\theta$. If $A_\eta f = 0$, then $\eta f = \theta h$ for some $h \in H^2$. Since θ and η are relatively prime, we see $f \in \theta H^2$ and hence $f = 0$. So A_η is one-to-one and then onto because $\dim \mathscr{H}_\theta$ is finite by Lemma 4.

To prove (b), we first study the case when $\operatorname{ord}\eta \leqslant \operatorname{ord}\theta$. Suppose $g \in \mathscr{H}_\eta$ be nonzero such that $P_\theta g = 0$. Then $g = \theta g_1$ for some $g_1 \in H^2$. Since $g \in \mathscr{H}_\eta$, the number of zeros of g counting multiplicity in $\mathbb{D}$ is less than or equal to $\operatorname{ord}\eta - 1$. But the total number of zeros of $g = \theta g_1$ counting multiplicity in $\mathbb{D}$ is greater than or equal to $\operatorname{ord}\theta$. Then $\operatorname{ord}\theta \leqslant \operatorname{ord}\eta - 1$, which is a contradiction because $\operatorname{ord}\eta \leqslant \operatorname{ord}\theta$. Therefore $P_\theta : \mathscr{H}_\eta \to \mathscr{H}_\theta$ is one-to-one and $\dim P_\theta\mathscr{H}_\eta = \dim \mathscr{H}_\eta = \operatorname{ord}\eta$.

Next, we study the case $\operatorname{ord}\theta < \operatorname{ord}\eta$. Let $M = \{f \in \mathscr{H}_\eta : \theta f \in \mathscr{H}_\eta\}$. Then $\mathscr{H}_\eta = (\mathscr{H}_\eta \ominus \theta M) \oplus \theta M$ and $\dim M = \operatorname{ord}\eta - \operatorname{ord}\theta$ by Lemma 5. If $g \in \mathscr{H}_\eta \ominus \theta M$ satisfy $P_\theta g = 0$, then $g = \theta g_1$ for some $g_1 \in H^2$ and $g_1 = T_\theta^* g \in \mathscr{H}_\eta$. Thus $g_1 \in M$, $g \in \theta M$ and $g = 0$. Thus $P_\theta(\mathscr{H}_\eta \ominus \theta M)$ and $\mathscr{H}_\eta \ominus \theta M$ have the same dimensions. Therefore

$$\dim P_\theta \mathscr{H}_\eta = \dim(\mathscr{H}_\eta \ominus \theta M) = \dim \mathscr{H}_\eta - \dim M = \operatorname{ord}\theta,$$

thus (b) follows.

Now, to prove (c), we first see the case ord $\theta \leqslant$ ord η. Let $f \in \mathscr{H}_\theta$ be nonzero satisfying $P_{\theta\mathscr{H}_\eta}\eta f = 0$. Then $\eta f \in \mathscr{H}_{\theta\eta} = \mathscr{H}_\theta \oplus \theta\mathscr{H}_\eta$ and then $\eta f \in \mathscr{H}_\theta$. Note that the total number of zeros of ηf in $\mathbb{D}$ counting multiplicity is greater than or equal to ord η. Since $\eta f \in \mathscr{H}_\theta$, the total number of zeros of ηf in $\mathbb{D}$ counting multiplicity is less than or equal to ord $\theta - 1$. Hence ord $\eta \leqslant$ ord $\theta - 1$. This contradiction shows that $P_{\theta\mathscr{H}_\eta} : \eta\mathscr{H}_\theta \to \theta\mathscr{H}_\eta$ is one-to-one and hence dim $P_{\theta\mathscr{H}_\eta}\eta\mathscr{H}_\theta =$ dim $\eta\mathscr{H}_\theta =$ ord θ.

Next, we study the case when ord $\theta >$ ord η. Let $M = \{f \in \mathscr{H}_\theta : \eta f \in \mathscr{H}_\theta\}$. Then $\mathscr{H}_\theta = M \oplus (\mathscr{H}_\theta \ominus M)$ and dim $M =$ ord $\theta -$ ord η by Lemma 5. Since $\eta M \subset \mathscr{H}_\theta$, we have $P_{\theta\mathscr{H}_\eta}\eta\mathscr{H}_\theta = P_{\theta\mathscr{H}_\eta}\eta(\mathscr{H}_\theta \ominus M)$. Let $h \in \mathscr{H}_\theta \ominus M$ such that $P_{\theta\mathscr{H}_\eta}\eta h = 0$. Since $\eta h \in \mathscr{H}_{\theta\eta}$ and $\mathscr{H}_{\theta\eta} = \mathscr{H}_\theta \oplus \theta\mathscr{H}_\eta$, we have $\eta h \in \mathscr{H}_\theta$ and so $h \in M$. As a result, $h = 0$ and $P_{\theta\mathscr{H}_\eta} : \eta(\mathscr{H}_\theta \ominus M) \to \theta\mathscr{H}_\eta$ is one-to-one. Therefore we see

$$\begin{aligned}\dim P_{\theta\mathscr{H}_\eta}\eta\mathscr{H}_\theta &= \dim P_{\theta\mathscr{H}_\eta}\eta(\mathscr{H}_\theta \ominus M)\\ &= \dim(\mathscr{H}_\theta \ominus M)\\ &= \operatorname{ord}\theta - \dim M = \operatorname{ord}\eta,\end{aligned}$$

so we have (c).

Finally, in order to prove (d), let $h \in \mathscr{H}_\theta$ satisfy $P_{\eta\mathscr{H}_\theta}h = 0$. Since $h \in \mathscr{H}_{\theta\eta} = \mathscr{H}_\eta \oplus \eta\mathscr{H}_\theta$, we have $h \in \mathscr{H}_\eta$. Since θ, η are relatively prime, $h \in \mathscr{H}_\theta \cap \mathscr{H}_\eta = \{0\}$, so $P_{\eta\mathscr{H}_\theta} : \mathscr{H}_\theta \to \eta\mathscr{H}_\theta$ is one-to-one. Since dim $\mathscr{H}_\theta =$ dim $\eta\mathscr{H}_\theta$, $P_{\eta\mathscr{H}_\theta} : \mathscr{H}_\theta \to \eta\mathscr{H}_\theta$ is onto. The proof is complete. $\square$

We remark in passing that (a) of Lemma 6 remains still valid for general inner functions η as long as η and θ are relatively prime.

3. Main results and the proofs

The following theorem shows that on a general finite dimensional model space, for any suitable even integer $2L$ and any finite Blaschke product η with order $2L$, the rank of $[A_\eta, A_\eta^*]$ is exactly $2L$.

THEOREM 7. *Let θ, η be finite Blaschke products of order N, L respectively. Write $\theta = \prod_{n=1}^N b_{\alpha_n}$ and $\eta = \prod_{n=1}^L b_{\beta_n}$. If $2L \leqslant N$, then*

$$\ker[A_\eta, A_\eta^*] = \frac{\prod_{n=1}^L(\beta_n - z)(1 - \overline{\beta_n}z)}{\prod_{n=1}^N(1 - \overline{\alpha_n}z)} \sum_{j=0}^{N-2L-1} \mathbb{C}\cdot z^j.$$

Moreover, the rank of $[A_\eta, A_\eta^*]$ is $2L$.

Proof. By Lemma 3, we first note that for $f \in \mathscr{H}_\theta$, $f \in \ker[A_\eta, A_\eta^*]$ if and only if

$$\eta f - P_\theta(\eta f) - \eta P_\eta f + P_\eta P_\theta(\eta f) \in \eta\theta H^2. \tag{6}$$

Let $f \in \ker[A_\eta, A_\eta^*]$. Since $\eta f - P_\theta(\eta f) \in \theta H^2$, (6) shows $\eta P_\eta f - P_\eta P_\theta(\eta f) \in \theta H^2$. We shall show that

$$\eta P_\eta f = P_\eta P_\theta(\eta f). \tag{7}$$

By Lemma 4, we may write

$$P_\eta f = \frac{\sum_{j=0}^{L-1} c_j z^j}{\prod_{n=1}^{L}(1-\overline{\beta}_n z)^n}, \qquad P_\eta P_\theta(\eta f) = \frac{\sum_{j=0}^{L-1} d_j z^j}{\prod_{n=1}^{L}(1-\overline{\beta}_n z)^n}$$

for some constants c_j and d_j. Note that

$$\begin{aligned}
&\eta P_\eta f - P_\eta P_\theta(\eta f) \\
&\quad = \frac{1}{\prod_{n=1}^{L}(1-\overline{\beta}_n z)^n}\Big[\Big(\prod_{n=1}^{L}\frac{(\beta_n - z)^n}{(1-\overline{\beta}_n z)^n}\Big)\sum_{j=0}^{N-1} c_j z^j - \sum_{j=0}^{N-1} d_j z^j\Big] \\
&\quad = \frac{p}{\prod_{n=1}^{L}(1-\overline{\beta}_n z)^{2n}}
\end{aligned}$$

where

$$p := \prod_{n=1}^{L}(\beta_n - z)^n \sum_{j=0}^{N-1} c_j z^j - \prod_{n=1}^{L}(1-\overline{\beta}_n z)^n \sum_{j=0}^{N-1} d_j z^j.$$

Since $\eta P_\eta f - P_\eta P_\theta(\eta f) \in \theta H^2$, we have $p \in \theta H^2$ either. Note $\deg p \leqslant 2L-1 < N$. Since $p/\theta \in H^2$, we have $p = 0$ and (7) follows from the observation above.

Now, by (6) and (7), we see $\eta f - P_\theta(\eta f) \in \eta\theta H^2$. Clearly, since $\eta f - P_\theta(\eta f) \perp \eta\theta H^2$, we have $\eta f - P_\theta(\eta f) = 0$ and hence $\eta f \in \mathscr{H}_\theta$. Since two functions in (7) are orthogonal each other, both are zero and hence $P_\eta f = 0$. On the other hand, one can see that a function $f \in \mathscr{H}_\theta$ satisfying $\eta f \in \mathscr{H}_\theta$ and $P_\eta f = 0$ satisfies (6). Thus, by an observation above, we see that for $f \in \mathscr{H}_\theta$, $f \in \ker[A_\eta, A_\eta^*]$ if and only if

$$\eta f \in \mathscr{H}_\theta \qquad \text{and} \qquad P_\eta f = 0. \tag{8}$$

By the above, it is easy to show that $\ker[A_\eta, A_\eta^*] = \eta K$, where K is defined by (4). Thus using Lemma 5 we obtain the desired kernel identity, which then gives the rank of $[A_\eta, A_\eta^*]$ is $N-(N-2L) = 2L$ since $[A_\eta, A_\eta^*]$ is self adjoint. The proof is complete. □

We remark in passing that Theorem 7 is closely related to kernels of Toeplitz operators on H^2 and the multipliers between certain model spaces as shown in the following remark. See [7] or [10] for details of multipliers between two model spaces.

REMARK 8. Recall K given by (4). Note $\eta^2 f \in \ker T_{\overline{\theta}}$ if and only if $f \in \ker T_{\overline{\theta}\eta^2}$. Since $\mathscr{H}_\theta = \ker T_{\overline{\theta}}$, we see $K = \ker T_{\overline{\theta}\eta^2}$. By Theorem 7 above and Theorem 4.2 of [10], we see that

$$\ker[A_\eta, A_\eta^*] = \eta \ker T_{\overline{\theta}\eta^2} = \eta \mathscr{M}(z\eta^2, \theta)$$

where $\mathscr{M}(z\eta^2, \theta)$ is the set of all multipliers from $\mathscr{H}_{z\eta^2}$ into $\mathscr{H}_\theta$. Therefore,

$$\operatorname{rank}[A_\eta, A_\eta^*] = N - \dim\ker T_{\overline{\theta}\eta^2} = N - \dim \mathscr{M}(z\eta^2, \theta).$$

For two inner functions η, θ, Proposition 6.3 of [5] shows that $[A_\eta, A_\eta^*] = 0$ on $\mathscr{H}_\theta$ if and only if $A_\eta = \lambda I$ on $\mathscr{H}_\theta$ for some $\lambda \in \mathbb{D}$, which is equivalent to that $\lambda - \eta = \theta h$ for some $h \in H^2$ by Theorem 3.1 of [11]. Noting

$$b_\lambda \circ \eta = \frac{\lambda - \eta}{1-\overline{\lambda}\eta} = \theta \frac{h}{1-\overline{\lambda}\eta},$$

we see $\eta = b_\lambda \circ (\theta\zeta)$, where $\zeta := h/(1-\overline{\lambda}\eta)$ is an inner function. Conversely, if $\eta = b_\lambda \circ (\theta\zeta)$ for some $\lambda \in \mathbb{D}$ and ζ inner, we see that A_η and A_η^* induce the rank zero commutator on $\mathscr{H}_\theta$.

COROLLARY 9. *Let $\lambda \in \mathbb{D} \setminus \{0\}$. Let θ and B be finite Blaschke products with $2\operatorname{ord} B \leqslant \operatorname{ord}\theta$ and $\eta = B \cdot b_\lambda \circ (\theta\zeta)$ or $\eta = B^{-1} \cdot b_\lambda \circ (\theta\zeta)$ for some inner function ζ. Then* $\operatorname{rank}[A_\eta, A_\eta^*] = 2\operatorname{ord} B$.

Proof. Since

$$b_\lambda \circ (\theta\zeta) - \lambda = \frac{\lambda - \theta\zeta}{1-\overline{\lambda}\theta\zeta} - \lambda = -\theta\zeta \frac{1-|\lambda|^2}{1-\overline{\lambda}\theta\zeta} \in \theta H^2,$$

we first have $A_{b_\lambda \circ (\theta\zeta)} = \lambda I$ on $\mathscr{H}_\theta$ by Theorem 3.1 of [11]. If $\eta = B \cdot b_\lambda \circ (\theta\zeta)$, we see

$$A_\eta = A_B A_{b_\lambda \circ (\theta\zeta)} = \lambda A_B$$

and thus $[A_\eta, A_\eta^*] = |\lambda|^2 [A_B, A_B^*]$, which has rank $2\operatorname{ord} B$ by Theorem 7. Also, if $\eta = B^{-1} \cdot b_\lambda \circ (\theta\zeta)$, then $B\eta = b_\lambda \circ (\theta\zeta)$ and

$$A_\eta A_B = A_B A_\eta = A_{b_\lambda \circ (\theta\zeta)} = \lambda I,$$

which means A_B is invertible on $\mathscr{H}_\theta$ and $A_\eta = \lambda A_B^{-1}$. Hence

$$\begin{aligned}
[A_\eta, A_\eta^*] &= |\lambda|^2 (A_B^{-1} A_B^{*-1} - A_B^{*-1} A_B^{-1}) \\
&= |\lambda|^2 A_B^{-1} A_B^{*-1} (A_B A_B^* - A_B^* A_B) A_B^{*-1} A_B^{-1} \\
&= |\lambda|^2 A_B^{-1} A_B^{*-1} [A_B, A_B^*] A_B^{*-1} A_B^{-1},
\end{aligned}$$

which has rank $2\operatorname{ord} B$ by Theorem 7 again. The proof is complete. □

The following corollary is also interesting.

COROLLARY 10. *Let η and ζ be two finite Blaschke products and $\theta = \eta\zeta$. If A_η is TTO defined on $\mathscr{H}_\theta$, then* $\operatorname{rank}[A_\eta, A_\eta^*] = 2\min\{\operatorname{ord}\eta, \operatorname{ord}\zeta\}$.

Proof. Since $-f + \eta P(\overline{\eta} f) + \zeta P(\overline{\zeta} f)$ is orthogonal to θH^2 for every $f \in \mathscr{H}_\theta$, it follows from the proof of Lemma 3 and applications of $P_\Theta g = Pg - \Theta P(\bar{\Theta} g)$ for inner function Θ and $g \in L^2$ that

$$\begin{aligned}
[A_\eta, A_\eta^*] f &= P_\theta\big(-f + \eta P(\overline{\eta} f) + \zeta P(\overline{\zeta} f)\big) \\
&= -f + \eta P(\overline{\eta} f) + \zeta P(\overline{\zeta} f) \\
&= (P_\theta - P_\eta - P_\zeta) f
\end{aligned} \tag{9}$$

for all $f \in \mathscr{H}_\theta$. Noting the above is symmetric with respect to η and ζ, we have $[A_\eta, A_\eta^*] = [A_\zeta, A_\zeta^*]$ on $\mathscr{H}_\theta$.

Without loss of generality we assume $\operatorname{ord}\zeta \leqslant \operatorname{ord}\eta$. Then we have $2\operatorname{ord}\zeta \leqslant \operatorname{ord}\theta$, so by Theorem 7 we get $\operatorname{rank}[A_\zeta, A_\zeta^*] = 2\operatorname{ord}\zeta$, to obtain the desired. The proof is complete. □

In the following, we want to care for the Question 2 when $2\operatorname{ord}\eta > \operatorname{ord}\theta$. For this end, suppose θ is a finite Blaschke product and η is an inner function such that θ,η have a nontrivial common inner divisor. We then study the rank of the commutator induced by A_η and A_η^* on $\mathscr{H}_\theta$ in certain special cases. We notice that Corollary 10 is also a case when $\operatorname{ord}\eta > \operatorname{ord}\zeta$.

In the proofs, we shall use several TTOs defined on different model spaces. So we redefine the notation of TTOs to avoid some confusion. Given two inner functions η,θ, we shall use the notation A_η^θ to denote the TTO defined on $\mathscr{H}_\theta$ by $A_\eta^\theta f = P_\theta(\eta f)$ for $f \in \mathscr{H}_\theta$. So the TTO A_η defined on $\mathscr{H}_\theta$ is just A_η^θ. Given three inner functions θ,η and ζ, we can easily see by using an application of (2)

$$A_{\eta\zeta}^{\theta\zeta} f = \zeta A_\eta^\theta f, \qquad f \in \mathscr{H}_\theta. \tag{10}$$

First we study a special case.

THEOREM 11. *Let η_1 be a finite Blaschke product and η_2 be an inner function which is relatively prime with η_1. Let $\theta = \eta_1^2$ and $\eta = \eta_1\eta_2$. If A_η is the TTO defined on $\mathscr{H}_\theta$, then* $\operatorname{rank}[A_\eta, A_\eta^*] = \operatorname{ord}\theta$.

Proof. Note that $\mathscr{H}_\theta = \mathscr{H}_{\eta_1} \oplus \eta_1 \mathscr{H}_{\eta_1}$ by (5). Since $A_\eta A_\eta^* = 0$ and

$$A_\eta^* A_\eta = A_\eta^* A_{\eta_1\eta_2}^{\eta_1\eta_1} = A_{\eta_2}^* A_{\eta_2}^{\eta_1}$$

on $\mathscr{H}_{\eta_1}$ by (10), we see $[A_\eta, A_\eta^*]\mathscr{H}_{\eta_1} = A_{\eta_2}^* A_{\eta_2}^{\eta_1} \mathscr{H}_{\eta_1}$. On the other hand, since η_1 and η_2 are relatively prime, we see $A_{\eta_2}^{\eta_1}\mathscr{H}_{\eta_1} = \mathscr{H}_{\eta_1}$ by Lemma 6(a) and

$$[A_\eta, A_\eta^*]\mathscr{H}_{\eta_1} = A_{\eta_2}^* \mathscr{H}_{\eta_1} = A_{\eta_2}^{\eta_1 *} \mathscr{H}_{\eta_1} = \mathscr{H}_{\eta_1}.$$

Also, we note that $A_\eta^* A_\eta \eta_1 f = A_\eta^* P_\theta(\eta_1^2\eta_2 f) = 0$ and $A_\eta A_\eta^* \eta_1 f = A_\eta A_{\eta_2}^* f$ for every $f \in \mathscr{H}_{\eta_1}$. Hence

$$A_\eta A_\eta^* \eta_1 \mathscr{H}_{\eta_1} = A_\eta A_{\eta_2}^* \mathscr{H}_{\eta_1} = A_\eta \mathscr{H}_{\eta_1} = A_{\eta_1\eta_2}^{\eta_1\eta_1}\mathscr{H}_{\eta_1} = \eta_1 A_{\eta_2}^{\eta_1}\mathscr{H}_{\eta_1} = \eta_1\mathscr{H}_{\eta_1}.$$

Therefore $[A_\eta, A_\eta^*]\eta_1\mathscr{H}_{\eta_1} = \eta_1\mathscr{H}_{\eta_1}$ and

$$[A_\eta, A_\eta^*]\mathscr{H}_\theta = [A_\eta, A_\eta^*]\mathscr{H}_{\eta_1} \oplus [A_\eta, A_\eta^*]\eta_1\mathscr{H}_{\eta_1} = \mathscr{H}_{\eta_1} \oplus \eta_1\mathscr{H}_{\eta_1} = \mathscr{H}_\theta,$$

which gives the desired result. The proof is complete. $\square$

When θ is a finite Blaschke product and the common inner divisor of θ,η is also a finite Blaschke product, we have the more exact result.

THEOREM 12. *Let θ_1,η_1 be finite Blaschke products which are relatively prime. Let η_2 be an inner function which is relatively prime with θ_1. Put $\theta = \theta_1\eta_1$ and $\eta = \eta_1\eta_2$. Let A_η be the TTO defined on $\mathscr{H}_\theta$. Then the following statements hold.*

(a) If $\operatorname{ord}\theta_1 > \operatorname{ord}\eta_1$, *then* $\operatorname{rank}[A_\eta, A_\eta^*] \geqslant 2\operatorname{ord}\eta_1$.

(b) If $\operatorname{ord}\theta_1 \leqslant \operatorname{ord}\eta_1$, *then* $\operatorname{rank}[A_\eta, A_\eta^*] = 2\operatorname{ord}\theta_1$.

(c) If $\eta_2 = b_\alpha(\theta_1\zeta)$ for some $\alpha \in \mathbb{D}\setminus\{0\}$ and ζ is inner, then we have

$$\operatorname{rank}[A_\eta, A_\eta^*] = 2\min\{\operatorname{ord}\theta_1, \operatorname{ord}\eta_1\}.$$

Proof. First note that $\mathscr{H}_\theta = \mathscr{H}_{\eta_1} \oplus \eta_1 \mathscr{H}_{\theta_1} = \mathscr{H}_{\theta_1} \oplus \theta_1 \mathscr{H}_{\eta_1}$ by (5). Fix $g \in \mathscr{H}_{\eta_1}$. Since $\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}}(\theta_1 g) \in \mathscr{H}_{\theta_1}$, we have by (10)

$$\begin{aligned}
A_\eta A_\eta^*(\theta_1 g) &= A_\eta A_\eta^* \left(P_{\eta_1}(\theta_1 g) \oplus P_{\eta_1 \mathscr{H}_{\theta_1}}(\theta_1 g)\right) \\
&= A_\eta A_\eta^* P_{\eta_1 \mathscr{H}_{\theta_1}}(\theta_1 g) \\
&= A_\eta P_\theta [\overline{\eta}_2 (\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}}(\theta_1 g))] \\
&= A_\eta A_{\eta_2}^{\theta_1 *} (\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}}(\theta_1 g)) \\
&= \eta_1 A_{\eta_2}^{\theta_1} A_{\eta_2}^{\theta_1 *} (\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}}(\theta_1 g)).
\end{aligned}$$

Since $A_\eta^* A_\eta \theta_1 g = 0$, it follows that

$$[A_\eta, A_\eta^*]\theta_1 \mathscr{H}_{\eta_1} = \eta_1 A_{\eta_2}^{\theta_1} A_{\eta_2}^{\theta_1 *} (\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}} \theta_1 \mathscr{H}_{\eta_1}) \subset \eta_1 \mathscr{H}_{\theta_1}. \tag{11}$$

On the other hand, we have $\dim \overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}} \theta_1 \mathscr{H}_{\eta_1} = \min\{\operatorname{ord}\theta_1, \operatorname{ord}\eta_1\}$ by Lemma 6(c) and then

$$\dim [A_\eta, A_\eta^*]\theta_1 \mathscr{H}_{\eta_1} = \min\{\operatorname{ord}\theta_1, \operatorname{ord}\eta_1\} \tag{12}$$

by Lemma 6(a).

Now fix $h \in \mathscr{H}_{\theta_1}$. Then by the similar argument above, we see

$$\begin{aligned}
A_\eta^* A_\eta h &= A_\eta^* P_\theta \eta_1 \eta_2 h = A_\eta^* P_\theta \eta_1 \left(P_{\theta_1} \eta_2 h \oplus P_{\theta_1 \mathscr{H}_{\eta_1}} \eta_2 h\right) \\
&= A_\eta^* P_\theta \eta_1 P_{\theta_1} \eta_2 h = A_\eta^* \eta_1 A_{\eta_2}^{\theta_1} h = A_{\eta_2}^{\theta_1 *} A_{\eta_2}^{\theta_1} h \in \mathscr{H}_{\theta_1}
\end{aligned}$$

and hence

$$A_\eta^* A_\eta \mathscr{H}_{\theta_1} = A_{\eta_2}^{\theta_1 *} A_{\eta_2}^{\theta_1} \mathscr{H}_{\theta_1} = \mathscr{H}_{\theta_1} \tag{13}$$

because $A_{\eta_2}^{\theta_1 *} A_{\eta_2}^{\theta_1}$ is invertible on $\mathscr{H}_{\theta_1}$ by Lemma 6(a). Also, since $\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}} h \in \mathscr{H}_{\theta_1}$, we have by (10)

$$\begin{aligned}
A_\eta A_\eta^* h &= A_\eta A_\eta^* \left(P_{\eta_1} h \oplus P_{\eta_1 \mathscr{H}_{\theta_1}} h\right) \\
&= A_\eta A_\eta^* P_{\eta_1 \mathscr{H}_{\theta_1}} h \\
&= A_\eta P_\theta \overline{\eta}_2 (\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}} h) \\
&= A_\eta A_{\eta_2}^{\theta_1 *} (\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}} h) \\
&= \eta_1 A_{\eta_2}^{\theta_1} A_{\eta_2}^{\theta_1 *} (\overline{\eta}_1 P_{\eta_1 \mathscr{H}_{\theta_1}} h).
\end{aligned}$$

It follows from (a) and (d) of Lemma 6 that

$$A_\eta A_\eta^* \mathscr{H}_{\theta_1} = \eta_1 \mathscr{H}_{\theta_1}. \tag{14}$$

Now we shall prove (a), (b) and (c). If $\operatorname{ord}\theta_1 > \operatorname{ord}\eta_1$, then by (11), (14), (12), (13) and we have

$$\begin{aligned}
\dim [A_\eta, A_\eta^*]\mathscr{H}_\theta &= \dim [A_\eta, A_\eta^*](\mathscr{H}_{\theta_1} \oplus \theta_1 \mathscr{H}_{\eta_1}) \\
&\geqslant \dim P_{\eta_1} [A_\eta, A_\eta^*]\mathscr{H}_{\theta_1} + \dim [A_\eta, A_\eta^*]\theta_1 \mathscr{H}_{\eta_1} \\
&= \dim P_{\eta_1} A_\eta^* A_\eta \mathscr{H}_{\theta_1} + \operatorname{ord}\eta_1 \\
&= \dim P_{\eta_1} \mathscr{H}_{\theta_1} + \operatorname{ord}\eta_1 \\
&= 2\operatorname{ord}\eta_1
\end{aligned}$$

where the last equality follows from Lemma 6 (b), which proves (a).

Also, if $\operatorname{ord}\theta_1 \leqslant \operatorname{ord}\eta_1$, we have $\dim[A_\eta, A_\eta^*]\theta_1\mathscr{H}_{\eta_1} = \operatorname{ord}\theta_1$ by (12). Thus (11) implies $[A_\eta, A_\eta^*]\theta_1\mathscr{H}_{\eta_1} = \eta_1\mathscr{H}_{\theta_1}$. In this case, we have $\mathscr{H}_{\theta_1} \cap \eta_1\mathscr{H}_{\theta_1} = \{0\}$ (see the proof of Theorem 10 (c)). It follows from (13) and (14) that

$$\begin{aligned}
\dim[A_\eta, A_\eta^*]\mathscr{H}_\theta &= \dim[A_\eta, A_\eta^*](\mathscr{H}_{\theta_1} \oplus \theta_1\mathscr{H}_{\eta_1}) \\
&= \dim(\mathscr{H}_{\theta_1} + \eta_1\mathscr{H}_{\theta_1}) \\
&= \dim\mathscr{H}_{\theta_1} + \dim\eta_1\mathscr{H}_{\theta_1} \\
&= 2\operatorname{ord}\theta_1,
\end{aligned}$$

which proves (b).

Finally, if $\eta_2 = b_\alpha(\theta_1\zeta)$ for some $\alpha \in \mathbb{D}\setminus\{0\}$ and ζ inner, we have by the remark mentioned just before Corollary 9, $A_{\eta_2}^{\theta_1} = \alpha I$ on $\mathscr{H}_{\theta_1}$. So, by the observation before the proof of (a), we see

$$[A_\eta, A_\eta^*]\theta_1\mathscr{H}_{\eta_1} = |\alpha|^2 P_{\eta_1\mathscr{H}_{\theta_1}}\theta_1\mathscr{H}_{\eta_1}$$

and

$$[A_\eta, A_\eta^*]\mathscr{H}_{\theta_1} = |\alpha|^2(P_{\eta_1\mathscr{H}_{\theta_1}} - I)\mathscr{H}_{\theta_1} = -|\alpha|^2 P_{\eta_1}\mathscr{H}_{\theta_1}.$$

It follows from (b) and (c) of Lemma 6 that

$$\dim[A_\eta, A_\eta^*]\mathscr{H}_\theta = \dim P_{\eta_1\mathscr{H}_{\theta_1}}\theta_1\mathscr{H}_{\eta_1} + \dim P_{\eta_1}\mathscr{H}_{\theta_1} = 2\min\{\operatorname{ord}\theta_1, \operatorname{ord}\eta_1\},$$

so (c) follows as desired. The proof is complete. $\square$

REMARK 13. Having Theorem 12, we have a few remarks in passing. We assume the same assumption as in Theorem 12.

(i) By the proof, it is not difficult to see that the equality holds in (a) of Theorem 12 if and only if

$$[A_\eta, A_\eta^*]\mathscr{H}_{\theta_1} \cap \eta_1\mathscr{H}_{\theta_1} \subset [A_\eta, A_\eta^*]\theta_1\mathscr{H}_{\eta_1}.$$

(ii) There is an example such that the inequality holds in (a) of Theorem 12. For example, suppose that η_2 is a Blaschke product and $\operatorname{ord}\theta_1 - \operatorname{ord}\eta_1 \geqslant 2\operatorname{ord}\eta_2$. Then we have

$$\begin{aligned}
2\operatorname{ord}\eta &= 2\operatorname{ord}\eta_1 + 2\operatorname{ord}\eta_2 \\
&\leqslant 2\operatorname{ord}\eta_1 + \operatorname{ord}\theta_1 - \operatorname{ord}\eta_1 \\
&= \operatorname{ord}\theta = \dim\mathscr{H}_\theta
\end{aligned}$$

and then by Theorem 7,

$$\operatorname{rank}[A_\eta, A_\eta^*] = 2\operatorname{ord}\eta > 2\operatorname{ord}\eta_1.$$

The inequality $\operatorname{rank}[A_\eta, A_\eta^*] > 2\operatorname{ord}\eta_1$ holds even when $\operatorname{ord}\theta_1 - \operatorname{ord}\eta_1 < 2\operatorname{ord}\eta_2$, see Case 3 after Example 17 in Section 4.

(iii) Under the same assumption as in (c), we obtain that

$$P_{\eta_1\mathscr{H}_{\theta_1}}[A_\eta, A_\eta^*]\mathscr{H}_{\theta_1} = \{0\}.$$

Indeed, since $A_{\eta_2}^{\theta_1} = \alpha I$ on $\mathscr{H}_{\theta_1}$, we have

$$\begin{aligned} P_{\eta_1 \mathscr{H}_{\theta_1}}[A_\eta, A_\eta^*]f &= P_{\eta_1 \mathscr{H}_{\theta_1}} A_\eta A_\eta^* f - P_{\eta_1 \mathscr{H}_{\theta_1}} A_\eta^* A_\eta f \\ &= \eta_1 A_{\eta_2}^{\theta_1} A_{\eta_2}^{\theta_1 *}(\overline{\eta_1} P_{\eta_1 \mathscr{H}_{\theta_1}} f) - P_{\eta_1 \mathscr{H}_{\theta_1}} A_{\eta_2}^{\theta_1 *} A_{\eta_2}^{\theta_1} f \\ &= |\alpha|^2 \left(P_{\eta_1 \mathscr{H}_{\theta_1}} f - P_{\eta_1 \mathscr{H}_{\theta_1}} f \right) = 0 \end{aligned}$$

for all $f \in \mathscr{H}_{\theta_1}$.

As an immediate consequence of Theorem 12, we have the following.

COROLLARY 14. *Let θ_1, η_1 be finite Blaschke products which are relatively prime. Let η_2 be an inner function which is relatively prime with θ_1. Put $\theta = \theta_1 \eta_1$ and $\eta = \eta_1 \eta_2$. Let A_η be the TTO on $\mathscr{H}_\theta$. If* $\operatorname{ord} \theta_1 = \operatorname{ord} \eta_1 + 1$*, then* $\operatorname{rank}[A_\eta, A_\eta^*] = 2 \operatorname{ord} \eta_1$.

Proof. Since $\operatorname{rank}[A_\eta, A_\eta^*] \leqslant \operatorname{ord} \theta = \operatorname{ord} \theta_1 + \operatorname{ord} \eta_1 = 2 \operatorname{ord} \eta_1 + 1$, Theorem 1 implies $\operatorname{rank}[A_\eta, A_\eta^*] \leqslant 2 \operatorname{ord} \eta_1$. It follows from Theorem 12(a) that

$$2 \operatorname{ord} \eta_1 \leqslant \operatorname{rank}[A_\eta, A_\eta^*] \leqslant 2 \operatorname{ord} \eta_1,$$

which gives the desired result. The proof is complete. □

Finally, we study the case when finite Blaschke product η has no nontrivial common inner divisor with θ satisfying $\operatorname{ord} \eta < \operatorname{ord} \theta < 2 \operatorname{ord} \eta$ and obtain a rank inequality. In this case, the rank of $[A_\eta, A_\eta^*]$ may take any even number between $2(\operatorname{ord} \theta - \operatorname{ord} \eta)$ and $2 \operatorname{ord} \theta$, see Example 18 in Section 4.

In the proof, we will use $||f|| = (\int_{\mathbb{T}} |f|^2 d\sigma)^{\frac{1}{2}}$ for $f \in L^2$.

THEOREM 15. *Let θ, η be finite Blaschke products which are relatively prime. Let A_η be the TTO defined on $\mathscr{H}_\theta$. If* $\operatorname{ord} \eta < \operatorname{ord} \theta < 2 \operatorname{ord} \eta$*, then*

$$\operatorname{rank}[A_\eta, A_\eta^*] \geqslant 2(\operatorname{ord} \theta - \operatorname{ord} \eta).$$

Proof. Let $N = \operatorname{ord} \theta$, $L = \operatorname{ord} \eta$. Write $\theta = \prod_{n=1}^N b_{\alpha_n}$, $\eta = \prod_{n=1}^L b_{\beta_n}$. Put $M = \{f \in \mathscr{H}_\theta : \eta f \in \mathscr{H}_\theta\}$. Then Lemma 5 shows $\dim M = N - L$ and

$$\{f \in M : \eta f \in M\} = \{0\}. \tag{15}$$

Letting $X = \mathscr{H}_\theta \ominus M$ and $Y = \mathscr{H}_\theta \ominus \eta M$, we see

$$\mathscr{H}_\theta = M \oplus X = Y \oplus \eta M \tag{16}$$

and

$$\dim X = \dim Y = N - \dim M = N - N + L = L. \tag{17}$$

Since $\eta M \perp \eta X$, we have $\eta M \perp A_\eta X$ and then $A_\eta X \subset Y$. Also, since $Y \perp \eta M$, we have $A_\eta^* Y \perp M$ and so $A_\eta^* Y \subset X$. Since θ and η are relatively prime, Lemma 11 (a) shows

$$A_\eta : X \to Y \text{ and } A_\eta^* : Y \to X \text{ are one-to-one and onto.} \tag{18}$$

Using (16) and (18), we see

$$\begin{aligned}[A_\eta, A_\eta^*]\eta f &= \eta f - A_\eta^* A_\eta (P_M \eta f \oplus P_X \eta f) \\ &= \eta f - P_M \eta f - A_\eta^* A_\eta P_X \eta f \\ &= P_X \eta f - A_\eta^* A_\eta P_X \eta f \in X\end{aligned}$$

for every $f \in M$. Thus

$$[A_\eta, A_\eta^*]\eta M \subset X. \tag{19}$$

If $P_X \eta g = 0$ for some $g \in M$, then $\eta g \in M$ and $g = 0$ by (16). It follows that $P_X : \eta M \to X$ is one-to-one and hence $\dim P_X \eta M = N - L$. It is easy to see $\|A_\eta h\| < \|h\|$ for every $h \in X$ with $h \neq 0$. It follows that

$$\begin{aligned}\|[A_\eta, A_\eta^*]\eta f\| &\geqslant \|P_X \eta f\| - \|A_\eta^* A_\eta P_X \eta f\| \\ &\geqslant \|P_X \eta f\| - \|A_\eta P_X \eta f\| \\ &> 0\end{aligned}$$

for every $f \in M$, which means $[A_\eta, A_\eta^*]|_{\eta M}$ is one-to-one. Hence

$$\dim [A_\eta, A_\eta^*]\eta M = \dim \eta M = N - L. \tag{20}$$

On the other hand, we see by (18)

$$\begin{aligned}[A_\eta, A_\eta^*]g &= A_\eta A_\eta^* g - A_\eta^* A_\eta (P_M g + P_X g) \\ &= A_\eta A_\eta^* g - P_M g - A_\eta^* A_\eta P_X g \\ &= P_M (A_\eta A_\eta^* g - g) \oplus (P_X A_\eta A_\eta^* g - A_\eta^* A_\eta P_X g) \\ &\in M \oplus X\end{aligned}$$

for every $g \in Y$. Put

$$Q_1 = \{g \in Y : A_\eta A_\eta^* g - g \in X\}$$

and $Q_2 = Y \ominus Q_1$. Then we have $Y = Q_1 \oplus Q_2$ and

$$[A_\eta, A_\eta^*]Q_1 \subset X. \tag{21}$$

If $P_M [A_\eta, A_\eta^*] g = 0$ for some $g \in Q_2$, then $P_M (A_\eta A_\eta^* g - g) = 0$ and then $A_\eta A_\eta^* g - g \in X$. Hence $g \in Q_1$ and $g = 0$, which means $P_M [A_\eta, A_\eta^*]|_{Q_2}$ is one-to-one. So, (21) shows

$$\dim P_M [A_\eta, A_\eta^*] Y = \dim P_M [A_\eta, A_\eta^*] Q_2 = \dim Q_2.$$

Thus $X \cap [A_\eta, A_\eta^*] Q_2 = \{0\}$ and $\dim [A_\eta, A_\eta^*] Q_2 = \dim Q_2$. This, together with (19) and (21), implies

$$\begin{aligned}\operatorname{rank} [A_\eta, A_\eta^*] &= \dim \big([A_\eta, A_\eta^*]\eta M + [A_\eta, A_\eta^*] Q_1 + [A_\eta, A_\eta^*] Q_2\big) \\ &= \dim \big([A_\eta, A_\eta^*]\eta M + [A_\eta, A_\eta^*] Q_1\big) + \dim [A_\eta, A_\eta^*] Q_2 \\ &= \dim \big([A_\eta, A_\eta^*]\eta M + [A_\eta, A_\eta^*] Q_1\big) + \dim Q_2.\end{aligned}$$

Put

$$R_1 = \big\{g \in Q_1 : [A_\eta, A_\eta^*] g \in [A_\eta, A_\eta^*]\eta M\big\}$$

and $R_2 = Q_1 \ominus R_1$. Then $Q_1 = R_1 \oplus R_2$. By the similar arguments as we have done above and (20), we get

$$\begin{aligned}
&\dim\left([A_\eta, A_\eta^*]\eta M + [A_\eta, A_\eta^*]Q_1\right) \\
&\quad = \dim[A_\eta, A_\eta^*]\eta M + \dim[A_\eta, A_\eta^*]R_2 \\
&\quad = N - L + \dim R_2 \\
&\quad = N - L + \dim Q_1 - \dim R_1.
\end{aligned}$$

Therefore, we have by (17)

$$\begin{aligned}
\operatorname{rank}[A_\eta, A_\eta^*] &= N - L + \dim Q_1 + \dim Q_2 - \dim R_1 \\
&= N - L + \dim Y - \dim R_1 \\
&= N - \dim R_1.
\end{aligned}$$

Now, in order to complete the proof, it suffices to show that

$$\dim R_1 \leqslant 2L - N. \tag{22}$$

First note that $Q_1 = \{g \in Y : [A_\eta, A_\eta^*]g \in X\}$. By (19) we have

$$R_1 = \{g \in Y : [A_\eta, A_\eta^*]g \in [A_\eta, A_\eta^*]\eta M\}.$$

and $M \perp [A_\eta, A_\eta^*]\eta M$. Thus $[A_\eta, A_\eta^*]M \perp \eta M$ and then $[A_\eta, A_\eta^*]M \subset Y$. It follows that $M \perp [A_\eta, A_\eta^*](Y \ominus [A_\eta, A_\eta^*]M)$ and hence

$$[A_\eta, A_\eta^*](Y \ominus [A_\eta, A_\eta^*]M) \subset X.$$

On the other hand, for a nonzero $f \in [A_\eta, A_\eta^*]M$, we have $[A_\eta, A_\eta^*]f \notin X$. Indeed, if $[A_\eta, A_\eta^*]f \in X$, then $[A_\eta, A_\eta^*]f \perp M$ and $f \perp [A_\eta, A_\eta^*]M$, thus $f = 0$. Thus we have

$$R_1 = \{g \in Y \ominus [A_\eta, A_\eta^*]M : [A_\eta, A_\eta^*]g \in [A_\eta, A_\eta^*]\eta M\}. \tag{23}$$

By (15), we have that

$$\begin{aligned}
\|[A_\eta, A_\eta^*]f\| &= \|A_\eta A_\eta^* f - f\| \\
&\geqslant \|f\| - \|A_\eta A_\eta^* f\|
\end{aligned}$$

for every nonzero $f \in M$. Hence dim $[A_\eta, A_\eta^*]M =$ dim $M = N - L$ and

$$\begin{aligned}
\dim(Y \ominus [A_\eta, A_\eta^*]M) &= \dim Y - \dim[A_\eta, A_\eta^*]M \\
&= 2L - N.
\end{aligned}$$

This, together with (23), gives (22) as desired. The proof is complete. □

Combining Theorem 15 with Theorem 1, we obtain the following simple application as before.

COROLLARY 16. *Let θ, η be finite Blaschke products being relatively prime. Let A_η be the TTO defined on $\mathscr{H}_\theta$. If $\operatorname{ord}\theta = 2\operatorname{ord}\eta - 1$, then*

$$\operatorname{rank}[A_\eta, A_\eta^*] = 2(\operatorname{ord}\eta - 1).$$

4. Two examples

In this section, we give two examples which will be related to results obtained in the previous section. For a point $a \in \mathbb{D}$, let

$$K_a(z) = \frac{1}{1-\bar{a}z}, \qquad z \in \mathbb{D}.$$

EXAMPLE 17. Let $\alpha_1, \cdots, \alpha_4$ be nonzero distinct points in $\mathbb{D}$ and $\theta = \prod_{n=1}^4 b_{\alpha_n}$. Let η be an inner function and A_η be the TTO defined on $\mathscr{H}_\theta$. Then the following statements hold.

(a) If $\eta(\alpha_1) = \eta(\alpha_2) = \eta(\alpha_3) \neq \eta(\alpha_4)$, then $\operatorname{rank}[A_\eta, A_\eta^*] = 2$.

(b) If $\eta(\alpha_1) = \eta(\alpha_2) \neq \eta(\alpha_3)$ and $\eta(\alpha_4)$, then $\operatorname{rank}[A_\eta, A_\eta^*] = 4$.

Proof. First we note that $\mathscr{H}_\theta = \sum_{n=1}^4 \mathbb{C}K_{\alpha_n}$. For each n, we write $A_\eta K_{\alpha_n} = \sum_{j=1}^4 c_{nj} K_{\alpha_j}$ for some $c_{nj} \in \mathbb{C}$.

We first prove (a). Let $g \in \mathscr{H}_\theta$ and write $g = \sum_{n=1}^4 d_n K_{\alpha_n} \in \mathscr{H}_\theta$ for some $d_n \in \mathbb{C}$. Since $A_\eta^* K_a = \overline{\eta(a)} K_a$ for all $a \in \mathbb{D}$, one can see that $g \in \ker[A_\eta, A_\eta^*]$ if and only if the following two conditions hold;

$$d_4 c_{41} = d_4 c_{42} = d_4 c_{43} = 0, \tag{24}$$

$$d_1 c_{14} + d_2 c_{24} + d_3 c_{34} = 0. \tag{25}$$

Notice that one of c_{41}, c_{42}, c_{43} is not zero, otherwise $A_\eta K_{\alpha_4} = c_{44} K_{\alpha_4}$. Since

$$c_{44} K_{\alpha_4}(\alpha_1) = A_\eta K_{\alpha_4}(\alpha_1) = \langle A_\eta K_{\alpha_4}, K_{\alpha_1} \rangle = \eta(\alpha_1) K_{\alpha_4}(\alpha_1)$$

and

$$c_{44} K_{\alpha_4}(\alpha_4) = A_\eta K_{\alpha_4}(\alpha_4) = \langle A_\eta K_{\alpha_4}, K_{\alpha_4} \rangle = \eta(\alpha_4) K_{\alpha_4}(\alpha_4),$$

we have $\eta(\alpha_1) = c_{44} = \eta(\alpha_4)$, which is a contradiction. Hence one of c_{41}, c_{42}, c_{43} is not zero and then $d_4 = 0$ by (24). Then, considering d_1, d_2, d_3 satisfying (25) and taking $d_4 = 0$, we have nonzero g in $\ker[A_\eta, A_\eta^*]$. Also, we have

$$[A_\eta, A_\eta^*] K_{\alpha_4} = \sum_{j=1}^4 \left(\overline{\eta(\alpha_4)} c_{4j} - c_{4j} \overline{\eta(\alpha_j)} \right) K_{\alpha_j} \neq 0.$$

Since the rank of $[A_\eta, A_\eta^*]$ is one of 0, 2, 4 by Theorem 1, the observation above shows that the rank of $[A_\eta, A_\eta^*]$ must be 2, as desired.

Now, in order to prove (b), we let $g = \sum_{n=1}^4 d_n K_{\alpha_n} \in \mathscr{H}_\theta$ as before. By direct computations using $A_\eta^* K_a = \overline{\eta(a)} K_a$ again, one can see that $g \in \ker[A_\eta, A_\eta^*]$ if and only if the following four conditions hold;

$$d_3 c_{31}[\overline{\eta(\alpha_3)} - \overline{\eta(\alpha_1)}] + d_4 c_{41}[\overline{\eta(\alpha_4)} - \overline{\eta(\alpha_1)}] = 0,$$
$$d_3 c_{32}[\overline{\eta(\alpha_3)} - \overline{\eta(\alpha_1)}] + d_4 c_{42}[\overline{\eta(\alpha_4)} - \overline{\eta(\alpha_1)}] = 0,$$
$$d_1 c_{13}[\overline{\eta(\alpha_1)} - \overline{\eta(\alpha_3)}] + d_2 c_{23}[\overline{\eta(\alpha_1)} - \overline{\eta(\alpha_3)}] + d_4 c_{43}[\overline{\eta(\alpha_4)} - \overline{\eta(\alpha_3)}] = 0,$$
$$d_1 c_{14}[\overline{\eta(\alpha_1)} - \overline{\eta(\alpha_4)}] + d_2 c_{24}[\overline{\eta(\alpha_1)} - \overline{\eta(\alpha_4)}] + d_3 c_{34}[\overline{\eta(\alpha_3)} - \overline{\eta(\alpha_4)}] = 0.$$

Now we claim that $c_{13}c_{24} \neq c_{14}c_{23}$ and $c_{31}c_{42} \neq c_{32}c_{41}$. Then, the above shows that all d_j equals 0, so $\ker[A_\eta, A_\eta^*] = \{0\}$ and $[A_\eta, A_\eta^*]\mathscr{H}_\theta = \mathscr{H}_\theta$, which will give the desired result.

Since

$$A_\eta K_{\alpha_n}(\alpha_m) = \sum_{j=1}^{4} c_{nj} K_{\alpha_j}(\alpha_m)$$

and

$$A_\eta K_{\alpha_n}(\alpha_m) = \langle A_\eta K_{\alpha_n}, K_{\alpha_m}\rangle = \eta(\alpha_m) K_{\alpha_n}(\alpha_m)$$

for $1 \leqslant n, m \leqslant 4$, we have the following linear equations

$$\sum_{j=1}^{4} c_{nj} K_{\alpha_j}(\alpha_m) = \eta(\alpha_m) K_{\alpha_n}(\alpha_m) \tag{26}$$

for $m = 1, \cdots, 4$. Set

$$|G| := \begin{vmatrix} K_{\alpha_1}(\alpha_1) & K_{\alpha_2}(\alpha_1) & K_{\alpha_3}(\alpha_1) & K_{\alpha_4}(\alpha_1) \\ K_{\alpha_1}(\alpha_2) & K_{\alpha_2}(\alpha_2) & K_{\alpha_3}(\alpha_2) & K_{\alpha_4}(\alpha_2) \\ K_{\alpha_1}(\alpha_3) & K_{\alpha_2}(\alpha_3) & K_{\alpha_3}(\alpha_3) & K_{\alpha_4}(\alpha_3) \\ K_{\alpha_1}(\alpha_4) & K_{\alpha_2}(\alpha_4) & K_{\alpha_3}(\alpha_4) & K_{\alpha_4}(\alpha_4) \end{vmatrix}.$$

By the proof of Proposition 7 of [4], we have following identity;

$$\begin{aligned} &\det \begin{pmatrix} \frac{1}{1-\overline{a_1}b_1} & \frac{1}{1-\overline{a_2}b_1} & \cdots & \frac{1}{1-\overline{a_n}b_1} \\ \frac{1}{1-\overline{a_1}b_2} & \frac{1}{1-\overline{a_2}b_2} & \cdots & \frac{1}{1-\overline{a_n}b_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1-\overline{a_1}b_n} & \frac{1}{1-\overline{a_2}b_n} & \cdots & \frac{1}{1-\overline{a_n}b_n} \end{pmatrix} \\ &= \Big(\prod_{j=1}^{n} \frac{1}{(1-\overline{a_j}b_j)}\Big) \prod_{k=1}^{n-1}\prod_{i>k} \frac{(\overline{a_i}-\overline{a_k})(b_i-b_k)}{(1-\overline{a_i}b_k)(1-\overline{a_k}b_i)} \end{aligned} \tag{27}$$

for every $b_j \in \mathbb{D}$ and distinct points $a_j \in \mathbb{D}$. So $|G| \neq 0$. By solving equation (26) and using simple calculations, we can see

$$\begin{aligned} c_{24} &= \frac{1}{|G|} \begin{vmatrix} K_{\alpha_1}(\alpha_1) & K_{\alpha_2}(\alpha_1) & K_{\alpha_3}(\alpha_1) & \eta(\alpha_1)K_{\alpha_2}(\alpha_1) \\ K_{\alpha_1}(\alpha_2) & K_{\alpha_2}(\alpha_2) & K_{\alpha_3}(\alpha_2) & \eta(\alpha_1)K_{\alpha_2}(\alpha_2) \\ K_{\alpha_1}(\alpha_3) & K_{\alpha_2}(\alpha_3) & K_{\alpha_3}(\alpha_3) & \eta(\alpha_3)K_{\alpha_2}(\alpha_3) \\ K_{\alpha_1}(\alpha_4) & K_{\alpha_2}(\alpha_4) & K_{\alpha_3}(\alpha_4) & \eta(\alpha_4)K_{\alpha_2}(\alpha_4) \end{vmatrix} \\ &= \frac{1}{|G|}\Big[(\eta(\alpha_4)-\eta(\alpha_1))K_{\alpha_2}(\alpha_4)|A| - (\eta(\alpha_3)-\eta(\alpha_1))K_{\alpha_2}(\alpha_3)|B|\Big] \end{aligned}$$

where

$$|A| = \begin{vmatrix} K_{\alpha_1}(\alpha_1) & K_{\alpha_2}(\alpha_1) & K_{\alpha_3}(\alpha_1) \\ K_{\alpha_1}(\alpha_2) & K_{\alpha_2}(\alpha_2) & K_{\alpha_3}(\alpha_2) \\ K_{\alpha_1}(\alpha_3) & K_{\alpha_2}(\alpha_3) & K_{\alpha_3}(\alpha_3) \end{vmatrix},$$

$$|B| = \begin{vmatrix} K_{\alpha_1}(\alpha_1) & K_{\alpha_2}(\alpha_1) & K_{\alpha_3}(\alpha_1) \\ K_{\alpha_1}(\alpha_2) & K_{\alpha_2}(\alpha_2) & K_{\alpha_3}(\alpha_2) \\ K_{\alpha_1}(\alpha_4) & K_{\alpha_2}(\alpha_4) & K_{\alpha_3}(\alpha_4) \end{vmatrix}.$$

Also, by the similar argument above, we can see

$$c_{13} = \frac{1}{|G|}\begin{vmatrix} K_{\alpha_1}(\alpha_1) & K_{\alpha_2}(\alpha_1) & \eta(\alpha_1)K_{\alpha_1}(\alpha_1) & K_{\alpha_4}(\alpha_1) \\ K_{\alpha_1}(\alpha_2) & K_{\alpha_2}(\alpha_2) & \eta(\alpha_1)K_{\alpha_1}(\alpha_2) & K_{\alpha_4}(\alpha_2) \\ K_{\alpha_1}(\alpha_3) & K_{\alpha_2}(\alpha_3) & \eta(\alpha_3)K_{\alpha_1}(\alpha_3) & K_{\alpha_4}(\alpha_3) \\ K_{\alpha_1}(\alpha_4) & K_{\alpha_2}(\alpha_4) & \eta(\alpha_4)K_{\alpha_1}(\alpha_4) & K_{\alpha_4}(\alpha_4) \end{vmatrix}$$

$$= \frac{1}{|G|}\Big[(\eta(\alpha_1)-\eta(\alpha_4))K_{\alpha_1}(\alpha_4)|C| - (\eta(\alpha_1)-\eta(\alpha_3))K_{\alpha_1}(\alpha_3)|D|\Big]$$

where

$$|C| = \begin{vmatrix} K_{\alpha_1}(\alpha_1) & K_{\alpha_2}(\alpha_1) & K_{\alpha_4}(\alpha_1) \\ K_{\alpha_1}(\alpha_2) & K_{\alpha_2}(\alpha_2) & K_{\alpha_4}(\alpha_2) \\ K_{\alpha_1}(\alpha_3) & K_{\alpha_2}(\alpha_3) & K_{\alpha_4}(\alpha_3) \end{vmatrix},$$

$$|D| = \begin{vmatrix} K_{\alpha_1}(\alpha_1) & K_{\alpha_2}(\alpha_1) & K_{\alpha_4}(\alpha_1) \\ K_{\alpha_1}(\alpha_2) & K_{\alpha_2}(\alpha_2) & K_{\alpha_4}(\alpha_2) \\ K_{\alpha_1}(\alpha_4) & K_{\alpha_2}(\alpha_4) & K_{\alpha_4}(\alpha_4) \end{vmatrix}.$$

Similarly, we also check

$$c_{14} = \frac{1}{|G|}\Big[(\eta(\alpha_4)-\eta(\alpha_1))K_{\alpha_1}(\alpha_4)|A| - (\eta(\alpha_3)-\eta(\alpha_1))K_{\alpha_1}(\alpha_3)|B|\Big],$$

$$c_{23} = \frac{1}{|G|}\Big[(\eta(\alpha_1)-\eta(\alpha_4))K_{\alpha_2}(\alpha_4)|C| - (\eta(\alpha_1)-\eta(\alpha_3))K_{\alpha_2}(\alpha_3)|D|\Big].$$

Now, by comparing quantities above, one can see that $c_{13}c_{24} \neq c_{14}c_{23}$ if and only if $|A||D| \neq |B||C|$. On the other hand, we see from (27)

$$|A||D| = \frac{|b_{\alpha_2}(\alpha_1)|^4|b_{\alpha_3}(\alpha_2)|^2|b_{\alpha_4}(\alpha_2)|^2}{(1-|\alpha_1|^2)^2(1-|\alpha_2|^2)^2(1-|\alpha_3|^2)(1-|\alpha_4|^2)}$$

and

$$|B||C| = \frac{|b_{\alpha_2}(\alpha_1)|^4|b_{\alpha_3}(\alpha_2)|^2|b_{\alpha_4}(\alpha_2)|^2}{(1-|\alpha_1|^2)^2(1-|\alpha_2|^2)^2|1-\overline{\alpha_3}\alpha_4|^2}.$$

Since $\alpha_3 \neq \alpha_4$ if and only if $(1-|\alpha_3|^2)(1-|\alpha_4|^2) \neq |1-\overline{\alpha_3}\alpha_4|^2$, the above shows $|A||D| \neq |B||C|$ and hence $c_{13}c_{24} \neq c_{14}c_{23}$, as desired.

Using the same arguments above together with assumption $\alpha_1 \neq \alpha_2$, we can see that $c_{31}c_{42} \neq c_{32}c_{41}$ either. The proof is complete. $\square$

In more special cases of Example 17, we reprove several results what we have obtained in this paper. We will consider six cases in which η is a finite Blaschke product with $\operatorname{ord}\eta = 3$.

Case 1. If $\eta(\alpha_1) = \eta(\alpha_2) = \eta(\alpha_3) = 0 \neq \eta(\alpha_4)$, then $\theta = \eta b_{\alpha_4}$. By Example 17 (a), we have

$$\operatorname{rank}[A_\eta, A_\eta^*] = 2 = 2\operatorname{ord} b_{\alpha_4},$$

which is a special case of Theorem 10 (c) or Theorem 12 (b).

Case 2. If $\eta(\alpha_1) = \eta(\alpha_2) = \eta(\alpha_3) \neq \eta(\alpha_4) = 0$, then $\theta = \theta_1 b_{\alpha_4}$ and $\eta = b_{\alpha_4}\eta_2$ where $\theta_1 = \prod_{n=1}^{3} b_{\alpha_n}$ and η are relatively prime. Noting $\operatorname{ord}\theta_1 > \operatorname{ord} b_{\alpha_4}$ and Example 17 (a) gives

$$\operatorname{rank}[A_\eta, A_\eta^*] = 2 = 2\operatorname{ord} b_{\alpha_4},$$

we see that Theorem 12 (a) is sharp.

Case 3. If $\eta(\alpha_1) = \eta(\alpha_2) \neq \eta(\alpha_3)$, $\eta(\alpha_4) = 0$ and $\prod_{n=1}^{3}\eta(\alpha_n) \neq 0$, then $\theta = \theta_1 b_{\alpha_4}, \eta = b_{\alpha_4}\eta_2$ where θ_1 and η are relatively prime. Because $\operatorname{ord}\theta_1 > \operatorname{ord} b_{\alpha_4}$ and Example 17 (b) induces

$$\operatorname{rank}[A_\eta, A_\eta^*] = 4 > 2\operatorname{ord} b_{\alpha_4},$$

we see that inequality can occur in Theorem 12 (a).

Case 4. If $\eta(\alpha_1) = \eta(\alpha_2) \neq \eta(\alpha_3) = \eta(\alpha_4) = 0$, then $\theta = \theta_1\eta_1$ and $\eta = \eta_1\eta_2$ where $\eta_1 = b_{\alpha_3} b_{\alpha_4}$ and θ_1 and η are relatively prime. Noting $\operatorname{ord}\theta_1 = \operatorname{ord}\eta_1$ and Example 17 (b) tells

$$\operatorname{rank}[A_\eta, A_\eta^*] = 2\operatorname{ord}\theta_1,$$

this is a special case of Theorem 12 (b).

Case 5. If $\eta(\alpha_1) = \eta(\alpha_2) = \eta(\alpha_3) \neq \eta(\alpha_4)$ and $\prod_{n=1}^{4}\eta(\alpha_n) \neq 0$, then θ and η are relatively prime. Since $\operatorname{ord}\eta < \operatorname{ord}\theta < 2\operatorname{ord}\eta$ and Example 17 (a) yields

$$\operatorname{rank}[A_\eta, A_\eta^*] = 2 = 2(\operatorname{ord}\theta - \operatorname{ord}\eta),$$

this case gives the sharpness in Theorem 15.

Case 6. If $\eta(\alpha_1) = \eta(\alpha_2) \neq \eta(\alpha_3)$ and $\eta(\alpha_4)$, $\prod_{n=1}^{4}\eta(\alpha_n) \neq 0$, then θ and η are relatively prime. Noting $\operatorname{ord}\eta < \operatorname{ord}\theta < 2\operatorname{ord}\eta$ and Example 17 (b) gives

$$\operatorname{rank}[A_\eta, A_\eta^*] = 4 > 2(\operatorname{ord}\theta - \operatorname{ord}\eta),$$

we have the inequality in Theorem 15.

Also, in conjunction with Theorem 15, we have the following example.

EXAMPLE 18. Choose $L < N < 2L$ such that there is a nonnegative integer N_1 satisfying $2(N + N_1 - L) \leqslant N$. Let θ, θ_1 be finite Blaschke products with $\operatorname{ord}\theta = N$ and $\operatorname{ord}\theta_1 = N_1$. Fix $\alpha \in \mathbb{D}\setminus\{0\}$ and let $b_\alpha \circ (\theta\theta_1) = \eta\zeta$ such that $\operatorname{ord}\eta = L$. Then $b_\alpha \circ (\theta\theta_1)$ and θ are relatively prime and $\operatorname{ord} b_\alpha \circ (\theta\theta_1) = N + N_1$. If A_η and A_ζ are TTOs on $\mathscr{H}_\theta$, we can see by the similar argument as in the proof of Corollary 9

$$\operatorname{rank}[A_\eta, A_\eta^*] = \operatorname{rank}[A_\zeta, A_\zeta^*].$$

Since $2(N + N_1 - L) \leqslant N$, Theorem 7 says

$$\operatorname{rank}[A_\zeta, A_\zeta^*] = 2\operatorname{ord}\zeta = 2(N + N_1 - L)$$

and hence

$$\operatorname{rank}[A_\eta, A_\eta^*] = 2(N + N_1 - L) \geqslant 2(N - L).$$

For example, if $N = 10$ and $L = 9$, then we may take N_1 as $0, 1, 2, 3$ or 4.

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Yong Chen
Department of Mathematics
Hangzhou Normal University
Hangzhou, 311121, P. R. China
e-mail: ychen227@gmail.com, ychen@hznu.edu.cn

Kei Ji Izuchi
Department of Mathematics
Niigata University
Niigata, 950-2181, Japan
e-mail: izuchi@m.sc.niigata-u.ac.jp

Young Joo Lee
Department of Mathematics
Chonnam National University
Gwangju, 61186, Korea
e-mail: leeyj@chonnam.ac.kr