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NEW DETERMINANTAL INEQUALITIES CONCERNING HERMITIAN AND POSITIVE SEMI–DEFINITE MATRICES

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Abstract. Let A,B be $n\times n$ matrices such that A is positive semi-definite and B is Hermitian. In this note, it is shown, among other inequalities, the following determinantal inequality

$$\det(A^k+(AB)^2)\geqslant \det(A^k+A^2B^2)$$

for all $k\in[1,\infty[$.

1. Introduction

Audenaert [2] proved the following determinantal inequality, for $n\times n$ positive semi-definite matrices A and B,

$$\det(A^2+|BA|)\leqslant \det(A^2+AB). \tag{1}$$

M. Lin [6] generalized Audenaert's result by proving

$$\det(A^2+|BA|^p)\leqslant \det(A^2+A^pB^p) \qquad 0\leqslant p\leqslant 2,$$

and also complemented (1) by proving that

$$\det(A^2+|AB|)\geqslant \det(A^2+AB).$$

In [1], the authors gave a further generalization of (1) by proving:

$$\det(A^{kp}+|BA|^p)\leqslant \det(A^{kp}+A^pB^p) \qquad k\geqslant 1,\ \ 0\leqslant p\leqslant 2,$$

and in addition, they formulated the following conjecture.

CONJECTURE 1. *Let A,B be two $n\times n$ positive semi-definite matrices. Then for any $k\in[1,\infty[$,*

$$\det(A^k+(AB)^2)\geqslant \det(A^k+A^2B^2). \tag{2}$$

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The main purpose of this paper is to confirm this conjecture in a slightly more general setting; namely in the case where A is positive semi-definite and B is Hermitian. It is worthy to note that inequality (2) turned into equality when $k=1$.

To proceed, we shall first fix some notation. Let M_n be the space of all $n\times n$ complex matrices where its identity matrix is denoted by I_n. The modulus of a complex matrix X is defined as $|X|=(X^*X)^{1/2}$. As usual, we shall write $X>0$ (resp. $X\geqslant 0$) to indicate that X is positive definite (resp. positive semi-definite). For Hermitian matrices $X,Y\in M_n$, by $X\geqslant Y$ we mean that $X-Y$ is positive semi-definite matrix. If the eigenvalues $\lambda_1(X),\lambda_2(X),\dots,\lambda_n(X)$ of a matrix X are real, then without loss of generality we shall always assume that they are arranged in decreasing order, that is

$$\lambda_1(X)\geqslant\lambda_2(X)\geqslant\ldots\geqslant\lambda_n(X).$$

For a Hermitian matrix $X\in M_n$, we shall denote by $\lambda(X)$ to be the real vector of order n defined by

$$\lambda(X)=(\lambda_1(X),\lambda_2(X),\dots,\lambda_n(X))^t.$$

If, in addition, $X\geqslant 0$, then we define $\lambda^{\frac{1}{2}}(X):=\left(\lambda_1^{\frac{1}{2}}(X),\lambda_2^{\frac{1}{2}}(X),\dots,\lambda_n^{\frac{1}{2}}(X)\right)^t$.

Majorization relations are great tools for deriving determinantal inequalities, see, for example, [10, Chapter 10] for details on this subject. If $\lambda(A)$, $\lambda(B)$ $\in\mathbb{R}^n$, then

1. by $\lambda(A)\prec_w\lambda(B)$ we mean that A is *weakly majorized* by B, that is

$$\sum_{i=1}^k\lambda_i(A)\leqslant\sum_{i=1}^k\lambda_i(B)\qquad\text{for all } k=1,2,\dots,n. \tag{3}$$

   Moreover, we shall say that A is *majorized* by B and we will write $\lambda(A)\prec\lambda(B)$ if (3) is true and equality holds for $k=n$.

2. By $\lambda(A)\prec_{wlog}\lambda(B)$, we mean that A is *weakly log-majorized* by B, that is

$$\prod_{i=1}^k\lambda_i(A)\leqslant\prod_{i=1}^k\lambda_i(B)\qquad\text{for all } k=1,2,\dots,n. \tag{4}$$

   In addition, we shall write $\lambda(A)\prec_{log}\lambda(B)$ and we will say that A is *log-majorized* by B if (4) is true and equality holds for $k=n$.

The antisymmetric tensor product technique is very powerful in deriving log-majorization inequalities. The k^{th} antisymmetric tensor product of an $n\times n$ matrix A is denoted by $\wedge^kA$ for all $k=1,2,\dots,n$ (see for example [3] for details on this topic). The following are some essential properties of this product where the first one is known as the Binet-Cauchy formula.

1. $\wedge^k(AB)=\wedge^kA\wedge^kB$.
2. $(\wedge^kA)^{-1}=\wedge^kA^{-1}$ in case A is invertible.

3. $(\wedge^k A)^p = \wedge^k A^p$ for every $A \geqslant 0$ and $p > 0$.

4. If A is positive semi-definite or Hermitian matrix then so is $\wedge^k A$.

5. If $A \geqslant 0$, then $\lambda_1(\wedge^k A) = \prod_{i=1}^{k} \lambda_i(A)$.

It is worthy to mention here that a common practice for proving a log-majorization inequality such as $\lambda(X) \prec_{log} \lambda(Y)$ where $X, Y \geqslant 0$, is to prove that

$$\lambda_1(X) \leqslant \lambda_1(Y) \text{ and } \det(X) = \det(Y).$$

The rest of the paper is organized as follows. In Section 2, we shall present our main result which is the proof of Conjecture 1. In addition, we will show that (2) is reversed when $k \in [0,1]$. In the third section, we study some inequalities for larger classes of matrices A and B. In last section, we conclude with some remarks and an open problem.

2. Main results

As mentioned earlier, there is a close connection between majorization and determinantal inequalities. We shall start with the following lemma which shows the existence of such a link and can be found in [6].

LEMMA 1. *Let A and B be two complex n-square matrices. Then the following holds.*

1. *If $\lambda(A), \lambda(B) \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\lambda(A) \prec \lambda(B)$ then $\det(A) \geqslant \det(B)$.*
2. *If $\lambda(A), \lambda(B) \in \mathbb{R}^n_+$ such that $\lambda(A) \prec_{wlog} \lambda(B)$ then*

$$\det(I_n + A) \leqslant \det(I_n + B).$$

The following majorization inequality given in the next lemma, stands for a larger set of matrices than the one originally proved by L. Plevnik in [8]. The steps in the proof are essentially the same, but we include them here for the sake of completeness and also to assert our claim that it is valid for a wider class of matrices.

LEMMA 2. *Let X, Y be in $\mathbb{M}_n$. If either one of the following two conditions*

1. *X is Hermitian, $Y \geqslant 0$ and p, $q \in [0, \infty[$, or*
2. *X, Y are Hermitian and p, q are even positive integers,*

is satisfied, then it holds that

$$\lambda(XY^pXY^q) \prec_{wlog} \lambda(X^2Y^{p+q}).$$

Proof. By appealing to a standard argument (anti-symmetric tensor product), then it suffices to prove that

$$\lambda_1(XY^pXY^q) \leqslant \lambda_1(X^2Y^{p+q}).$$

Without loss of generality, we shall assume that X is invertible as the general case can be done by continuity argument. In addition, we shall assume that $p > q$ and $\lambda_1(X^2Y^{p+q}) = 1$. Now, obviously proving our claim is equivalent to showing that

$$\lambda_1(XY^pXY^q) \leqslant 1.$$

The fact that the largest eigenvalue of the matrix X^2Y^{p+q} is equal to 1, clearly implies that

$$\lambda_j(X^2Y^{p+q}) \leqslant 1 \text{ for all } 1 \leqslant j \leqslant n.$$

But this is equivalent to $XY^{p+q}X \leqslant I_n$ which in turn gives

$$Y^{p+q} \leqslant (X^{-1})^2. \tag{5}$$

Next, applying the well-known Löwner-Heinz inequality (see, for example [10, p. 211]) on (5) for a power $\frac{p}{p+q} < 1$, yields

$$(Y^{p+q})^{\frac{p}{p+q}} \leqslant ((X^{-1})^2)^{\frac{p}{p+q}}.$$

Obviously, for both cases in the lemma we see that $(Y^{p+q})^{\frac{p}{p+q}} = Y^p$, and here it is worthy to draw the attention to the fact that $((X^{-1})^2)^{\frac{p}{p+q}}$ is defined since $(X^{-1})^2 > 0$.

Thus, we obtain

$$Y^p \leqslant ((X^{-1})^2)^{\frac{p}{p+q}}. \tag{6}$$

Again, taking a power $\frac{q}{p} < 1$ in both sides of (6), we get

$$(Y^p)^{\frac{q}{p}} \leqslant \left(((X^{-1})^2)^{\frac{p}{p+q}}\right)^{\frac{q}{p}}.$$

As before, in both cases of the lemma we have

$$Y^q \leqslant \left((X^{-1})^2\right)^{\frac{q}{p+q}} = \left((X^2)^{\frac{q}{p+q}}\right)^{-1}.$$

Hence,

$$(X^2)^{\frac{q}{p+q}} \leqslant Y^{-q}. \tag{7}$$

Therefore,

$$\begin{aligned}
\lambda_1(XY^pXY^q) &= \lambda_1\left(Y^{q/2}XY^pXY^{q/2}\right) \\
&\leqslant \lambda_1\left(Y^{q/2}X((X^{-1})^2)^{\frac{p}{p+q}}XY^{q/2}\right) && \text{Using (6)} \\
&= \lambda_1\left((Y^{q/2}(X^2)^{\frac{q}{p+q}}Y^{q/2}\right) \\
&\leqslant \lambda_1\left(Y^{q/2}Y^{-q}Y^{q/2}\right) && \text{Using (7)} \\
&= \lambda_1(I_n) \\
&= 1.
\end{aligned}$$

Thus, the proof is complete. □

Next, we need the following lemma whose proof can be found in [10, p. 352].

LEMMA 3. *Let* $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *where* A, B *and* C *are* n*-square complex matrices. If* $M \geqslant 0$ *then*

$$|\lambda(B)| \prec_{wlog} \lambda^{\frac{1}{2}}(A) \circ \lambda^{\frac{1}{2}}(C),$$

where $\circ$ *is the componentwise product. In particular, we have*

$$|\lambda_1(B)| \leqslant \lambda_1^{\frac{1}{2}}(A) \cdot \lambda_1^{\frac{1}{2}}(C).$$

Now, we prove the following lemma which is also essential for our main results.

LEMMA 4. *Let* K, L *be in* $\mathbb{M}_n$. *If either one of the following two conditions*

1. K *is Hermitian,* $L > 0$, $p', q' \in [0, \infty[$ *and* $p' \geqslant 2q'$ *or*
2. K, L *are Hermitian,* q' *and* $p' \geqslant 2q'$ *are even positive integers,*

is satisfied, then we have

$$\lambda(K^2 L^{p'-q'}) \prec_{wlog} \lambda(KL^{p'} KL^{-q'}).$$

Proof. By Schur's complement, we know that

$$M = \begin{bmatrix} L^{\frac{p'}{2}} KL^{-q'} KL^{\frac{p'}{2}} & L^{\frac{p'}{2}} K^2 L^{\frac{p'}{2}-q'} \\ L^{\frac{p'}{2}-q'} K^2 L^{\frac{p'}{2}} & L^{\frac{p'}{2}-q'} KL^{q'} KL^{\frac{p'}{2}-q'} \end{bmatrix} \geqslant 0.$$

Now in view of Lemma 3, we can write

$$\lambda_1(L^{p'} KL^{-q'} K) \cdot \lambda_1(L^{p'-2q'} KL^{q'} K) \geqslant \lambda_1(K^2 L^{p'-q'})^2. \tag{8}$$

Taking $X = K$, $Y = L$, $q = q'$ and $p = p' - 2q' \geqslant 0$ in Lemma 2, and keeping in mind that either Condition 1 or 2 is satisfied, we obtain

$$\lambda(KL^{p'-2q'} KL^{q'}) \prec_{wlog} \lambda(K^2 L^{p'-q'}),$$

which is certainly implies that

$$\lambda_1(L^{p'-2q'} KL^{q'} K) \leqslant \lambda_1(K^2 L^{p'-q'}).$$

Therefore, using (8) we conclude that

$$\lambda_1(L^{p'} KL^{-q'} K)\lambda_1(L^{p'-2q'} KL^{q'} K) \geqslant \lambda_1(K^2 L^{p'-q'})^2 \geqslant \lambda_1(K^2 L^{p'-q'})\lambda_1(L^{p'-2q'} KL^{q'} K).$$

Thus,

$$\lambda_1\left(L^{p'} KL^{-q'} K\right) \geqslant \lambda_1\left(K^2 L^{p'-q'}\right).$$

Finally, by a standard antisymmetric tensor product argument, the proof is achieved. □

As a result, we have the following theorem.

THEOREM 1. *Let A be a positive semi-definite matrix and B be a Hermitian matrix. Then*

1. $\det(A^k+|AB|^2) \geqslant \det(A^k+A^2B^2), \qquad k\in[0,1].$
2. $\det(A^k+|AB|^2) \geqslant \det(A^k+A^2B^2), \qquad k\in[4,\infty[\,.$
3. *If A invertible, then* $\det(A^k+|AB|^2) \leqslant \det(A^k+A^2B^2), \qquad k\in\,]-\infty,0].$

Proof. For the first two inequalities, we shall assume first that A is invertible and then the general case follows easily by a continuity argument.

1. $k\in[0,1]$:

   Replacing K with B, L with A, q' with k, and p' with $2\geqslant 2k$ in Lemma 4 gives

$$\lambda(B^2A^{2-k}) \prec_{wlog} \lambda(BA^2BA^{-k}).$$

   Now using Part 2 of Lemma 1 yields

$$\det(I_n+B^2A^{2-k}) \leqslant \det(I_n+BA^2BA^{-k}).$$

   By multiplying both sides by $\det(A^k)>0$, we obtain the result.

2. $k\in[4,\infty[$:

   Taking $K=B$, $L=A^{-1}$, $q'=2$, and $p'=k\geqslant 2\cdot 2=4$ in Lemma 4 gives

$$\lambda\left(B^2(A^{-1})^{k-2}\right) \prec_{wlog} \lambda\left(B(A^{-1})^{-2}B(A^{-1})^k\right).$$

   Again using first Part 2 of Lemma 1 which yields

$$\det(I_n+B^2A^{2-k}) \leqslant \det(I_n+BA^2BA^{-k}),$$

   and then multiplying both sides by $\det(A^k)>0$, will prove the result.

3. $k\in]-\infty,0]$:

   Replacing X with B, Y with A, p with $-k\geqslant 0$ and $q=2>0$ in Lemma 2 gives

$$\lambda(A^{-k}BA^2B) \prec_{wlog} \lambda(A^{2-k}B^2). \tag{9}$$

   Similarly, using Part 2 of Lemma 1 for (9) yields this time

$$\det(I_n+A^{-k}BA^2B) \leqslant \det(I_n+A^{2-k}B^2).$$

   Multiplying both sides with $\det(A^k)>0$, leads to the result. □

For our purposes, we need the next two lemmas.

LEMMA 5. *Let A and B be in M_n such that A is positive semi-definite and B is Hermitian, and let α_1 and β_1 be any two real numbers. If, for all $k \in [\alpha_1, \beta_1]$,*

$$\det(A^k + |AB|^2) \geqslant \det(A^k + A^2B^2), \tag{10}$$

then,

$$\det(A^{k'} + (AB)^2) \geqslant \det(A^{k'} + A^2B^2),$$

for all $k' \in [\alpha_2, \beta_2]$ with $\alpha_2 = \frac{\alpha_1}{2} + 1$ and $\beta_2 = \frac{\beta_1}{2} + 1$.

Proof. Let $k' \in [\frac{\alpha_1}{2} + 1, \frac{\beta_1}{2} + 1]$, then $2(k'-1) \in [\alpha_1, \beta_1]$. Replacing A with $A^{1/2}$ and k with $2(k'-1)$ in (10) gives

$$\det\left((A^{1/2})^{2(k'-1)} + |A^{1/2}B|^2\right) \geqslant \det\left((A^{1/2})^{2(k'-1)} + (A^{1/2})^2B^2\right). \tag{11}$$

On the other hand, we can write

$$\begin{aligned}
\det(A^{k'} + (AB)^2) &= \det(A^{k'} + ABAB)\\
&= \det(A) \cdot \det(A^{k'-1} + BAB)\\
&= \det(A) \cdot \det\left((A^{1/2})^{2(k'-1)} + |A^{1/2}B|^2\right)\\
&\geqslant \det(A) \cdot \det\left((A^{1/2})^{2(k'-1)} + (A^{1/2})^2B^2\right) \qquad \text{Using (11)}\\
&= \det(A) \cdot \det(A^{k'-1} + AB^2)\\
&= \det(A^{k'} + A^2B^2). \quad \square
\end{aligned}$$

LEMMA 6. *Let A and B be in M_n such that $A \geqslant 0$ and B is Hermitian, and let α and β be any two positive real numbers. If*

$$\det(A^k + (AB)^2) \geqslant \det(A^k + A^2B^2)$$

is true for $k \in [\alpha, \beta]$, then

$$\det(A^k + |AB|^2) \geqslant \det(A^k + (AB)^2) \geqslant \det(A^k + A^2B^2)$$

is also true for $k \in [\alpha, \beta]$.

Proof. Using Schur's complement, we know that (see also [9, Theorem 5.12])

$$\begin{bmatrix} |BA|^2 & (AB)^2 \\ (BA)^2 & |AB|^2 \end{bmatrix} \geqslant 0 \quad \text{and} \quad \begin{bmatrix} A^k & A^k \\ A^k & A^k \end{bmatrix} \geqslant 0.$$

So,

$$\begin{bmatrix} A^k + |BA|^2 & A^k + (AB)^2 \\ A^k + (BA)^2 & A^k + |AB|^2 \end{bmatrix} \geqslant 0,$$

and then

$$\begin{bmatrix} \det(A^k+|BA|^2) & \det(A^k+(AB)^2) \\ \det(A^k+(BA)^2) & \det(A^k+|AB|^2) \end{bmatrix} \geqslant 0.$$

Using the fact that $\det(A^k+(AB)^2)=\det(A^k+(BA)^2)$, we conclude that

$$\det(A^k+|BA|^2)\cdot\det(A^k+|AB|^2) \geqslant \det(A^k+(AB)^2)^2.$$

As $\det(A^k+|BA|^2)=\det(A^k+A^2B^2) \leqslant \det(A^k+(AB)^2)$ for all $k\in[\alpha,\beta]$, then we obtain

$$\det(A^k+|AB|^2) \geqslant \det(A^k+(AB)^2).$$

Thus, for all real numbers $k\in[\alpha,\beta]$, we have that

$$\det(A^k+|AB|^2) \geqslant \det(A^k+(AB)^2) \geqslant \det(A^k+A^2B^2). \quad \square$$

Now we are ready to prove one of the main results of this paper.

THEOREM 2. *Let A and B be in M_n such that A is positive semi-definite and B is Hermitian. Then, for $k\in[0,\infty[$, we have that*

$$\det(A^k+|AB|^2) \geqslant \det(A^k+A^2B^2). \tag{12}$$

Proof. For the proof, we distinguish between the two cases: $k\in[2,\infty[$ and $k\in[0,2]$.

Case 1: $k\in[2,\infty[$:

We construct a recursive sequence $(\alpha_n)_{n\geqslant 1}$ with $\alpha_1=4$, and $\alpha_{i+1}=\frac{\alpha_i}{2}+1$. Then, in view of Part 2 in Theorem 1, we know that (12) is true for all $k\in[\alpha_1,\infty[$. Next, we shall show in two steps the validity of inequality (12) for $k\in[\alpha_2,\infty[\ \supset\ [\alpha_1,\infty[$.

Step 1: Combining Part 2 of Theorem 1 and Lemma 5, implies that for all $k\in[\alpha_2,\infty[$ where $\alpha_2=\frac{\alpha_1}{2}+1=3$, we have

$$\det(A^k+(AB)^2) \geqslant \det(A^k+A^2B^2).$$

Step 2: Making use of Lemma 6, we first conclude that (12) is true for all $k\in[\alpha_2,\infty[$.

Repeating the same process as before, we see that inequality (12) is true for all

$$\begin{aligned}
&k\in[\alpha_3,\infty[\,, \quad \left(\alpha_3=\frac{\alpha_1}{2^2}+\frac{1}{2}+1=\frac{5}{2}\right)\\
&k\in[\alpha_4,\infty[\,, \quad \left(\alpha_4=\frac{\alpha_1}{2^3}+\frac{1}{2^2}+\frac{1}{2}+1=\frac{9}{4}\right)\\
&\vdots\\
&k\in[\alpha_{n+2},\infty[\,, \quad \left(\alpha_{n+2}=\frac{\alpha_1}{2^{n+1}}+\frac{1}{2^n}+\frac{1}{2^{n-1}}+\ldots+\frac{1}{2}+1=2+\frac{1}{2^n}\right).
\end{aligned}$$

As n approaches $+\infty$, the sequence $(\alpha_n)_n$ tends to 2. Therefore, (12) is true for $k \in [2,\infty[$.

Case 2: $k \in [0,2]$:

Here we construct another recursive sequence $(\alpha_n)_{n\geqslant 1}$ with $\alpha_1 = 0$, and $\alpha_{i+1} = \frac{\alpha_i}{2} + 1$. Again, Part 2 of Theorem 1 says that (12) is true for all $k \in [\alpha_1, \alpha_2]$.

Using a similar argument as earlier, we conclude that inequality (12) is valid for all

$$\begin{aligned}
&k \in [\alpha_2, \alpha_3], \quad \left(\alpha_3 = 1 + \frac{1}{2}\right)\\
&k \in [\alpha_3, \alpha_4], \quad \left(\alpha_4 = 1 + \frac{1}{2} + \frac{1}{2^2}\right)\\
&k \in [\alpha_4, \alpha_5], \quad \left(\alpha_5 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right)\\
&\vdots\\
&k \in [\alpha_{n-1}, \alpha_n], \quad \left(\alpha_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} = 2 - \frac{1}{2^n}\right).
\end{aligned}$$

As n approaches $+\infty$, the sequence α_n also tends to 2. Thus, for all $A \geqslant 0$, B Hermitian matrix and for $k \in [0,2]$, we have that

$$\det(A^k + |AB|^2) \geqslant \det(A^k + A^2B^2). \quad \square$$

Consequently, we have the following corollary.

COROLLARY 1. *Let A be a positive semi-definite matrix and B be a Hermitian matrix. Then*

1. $\det(A^k + |AB|^2) \geqslant \det(A^k + (AB)^2) \geqslant \det(A^k + A^2B^2), \qquad k \in [1,\infty[$.
2. $\det(A^k + |AB|^2) \geqslant \det(A^k + A^2B^2) \geqslant \det(A^k + (AB)^2), \qquad k \in [0,1]$.
3. *For $k \in]-\infty, 0]$ and in the event of A being invertible, we have that*
   - $\det(A^k + A^2B^2) \geqslant \det(A^k + |AB|^2)$, *and*
   - $\det(A^k + A^2B^2) \geqslant \det(A^k + (AB)^2)$.

3. Further remarks

In this section, we are tempted to study inequality (2) for a larger class of matrices, and to see when it might fail. In fact, we have the following theorem.

THEOREM 3. *Let A and B be two Hermitian matrices. Then for all positive even integers k,*

$$\det(A^k + |AB|^2) \geqslant \det(A^k + A^2B^2).$$

Proof. The proof for even integers $k \geqslant 4$ is similar to that of Part 1 in Theorem 1; we just need to apply the second case of Lemma 3 when A and B are Hermitian matrices and p, q are even integers. To close the proof, we still have to show the inequality for $k = 0$ and $k = 2$. However, for $k = 0$, it is easy to see that

$$\det(I_n + |AB|^2) = \det(I_n + A^2B^2).$$

Next, for $k = 2$, assume first that A and B are invertible matrices. Observe that in this case $A^2 + BA^2B > 0$ and $A^2 + AB^2A > 0$ for all A and B invertible Hermitian matrices. So that

$$\lambda(A^2 + BA^2B),\ \lambda(A^2 + AB^2A) \in \mathbb{R}^n_+ \setminus \{0\}.$$

From [7], we know that the following majorization inequality

$$\lambda(XX^* + YY^*) \prec \lambda(X^*X + Y^*Y)$$

is valid for X, $Y \in \mathbb{M}_n$ with X^*Y Hermitian. Replacing X and Y with A and BA respectively gives

$$\lambda(A^2 + BA^2B) \prec \lambda(A^2 + AB^2A). \tag{13}$$

Now applying Part 1 of Lemma 1 on (13) yields

$$\det(A^2 + |AB|^2) \geqslant \det(A^2 + AB^2A) = \det(A^2 + A^2B^2).$$

Finally, by a continuity argument, the proof is complete. $\square$

REMARK 1. It is worthy to note the following.

1. For Hermitian matrices A and B and for positive odd integers k, the inequality

$$\det(A^k + |AB|^2) \geqslant \det(A^k + A^2B^2)$$

as well as its reverse, are not true in general. This can be easily seen by taking $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix}$, and then a simple check shows that

- for $k = 1$, $\det(A^1 + A^2B^2) = 4872 < \det(A^1 + |AB|^2) = 5766$, and
- for $k = 3$, $\det(A^3 + A^2B^2) = -1872 > \det(A^3 + |AB|^2) = -10650$.

2. On the other hand, for Hermitian matrices A and B, the following inequality

$$\det(A^k + (AB)^2) \geqslant \det(A^k + A^2B^2)$$

may also fail for even integers $k \geqslant 0$. This can be easily seen by taking first

$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & -i \\ 3 & i & 2 \end{bmatrix}$. A simple inspection shows that $k = 2$, we have

$$\det(A^2 + (AB)^2) = 14082 > \det(A^2 + A^2B^2) = 12708.$$

Secondly, if $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -i \\ 3 & i & 2 \end{bmatrix}$, then again for $k = 2$, a simple check shows that

$$\det(A^2 + (AB)^2) = 82.75 < \det(A^2 + A^2B^2) = 125.$$

4. An open problem

Recall that in [1], it has been proven that for all $k \in [1, \infty[$ and $p \in [0, 2]$,

$$\det(A^{kp} + |BA|^p) \leqslant \det(A^{kp} + A^p B^p). \tag{14}$$

So, Theorem 2 can be thought of as a complement of (14) for $p = 2$. We can also find another complement of (14) for $p = 1$ as the following theorem shows.

THEOREM 4. *Let A and B be two positive semi-definite matrices. Then for all $k \in [1, \infty[$, we have that*

$$\det(A^k + |AB|) \geqslant \det(A^k + AB).$$

Proof. As mentioned earlier, we know that

$$\begin{bmatrix} |BA| & AB \\ BA & |AB| \end{bmatrix} \geqslant 0 \quad \text{and} \quad \begin{bmatrix} A^k & A^k \\ A^k & A^k \end{bmatrix} \geqslant 0.$$

So that

$$\begin{bmatrix} A^k + |BA| & A^k + AB \\ A^k + BA & A^k + |AB| \end{bmatrix} \geqslant 0,$$

and then

$$\begin{bmatrix} \det(A^k + |BA|) & \det(A^k + AB) \\ \det(A^k + BA) & \det(A^k + |AB|) \end{bmatrix} \geqslant 0.$$

As a result, we obtain

$$\det(A^k + |BA|) \cdot \det(A^k + |AB|) \geqslant \det(A^k + AB)^2.$$

Now the required determinantal inequality follows easily by noting that for all $k \in [1, \infty[$,

$$\det(A^k + |BA|) \leqslant \det(A^k + AB). \quad \square$$

Finally, we conclude the paper with the following open question.

CONJECTURE 2. *Let A, B be two positive semi-definite matrices. Then for all $k \in [1, \infty[$, we have that*

$$\det(A^{kp} + |AB|^p) \geqslant \det(A^{kp} + A^p B^p), \qquad 0 \leqslant p \leqslant 2. \quad \square$$

At the end, it is worthy to note that it appears that the methods used here do not seem to work for this conjecture even for the case $k = \frac{2}{p}$.

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