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WEAK SUBNORMALITY OF INFINITE 4–BANDED MATRICES

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Abstract. In this paper, we consider a class of operators whose matrix representations comprise 4-banded matrices, i.e., sparse matrices whose non-zero entries are confined to four diagonals. In particular, we focus on the hyponormality and weak subnormality when each diagonal forms a hyponormal weighted shift.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces and $\mathscr{B}(\mathscr{H},\mathscr{K})$ be the algebra of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$, and write $\mathscr{B}(\mathscr{H}) \equiv \mathscr{B}(\mathscr{H},\mathscr{H})$. For $T \in \mathscr{B}(\mathscr{H})$, the self-commutator of T is defined by

$$[T^*,T] := T^*T - TT^*.$$

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *normal* if $[T^*,T]=0$, *hyponormal* if $[T^*,T] \geqslant 0$, and *subnormal* if T has a normal extension, i.e., $T = N|_{\mathscr{H}}$, where N is a normal operator on some Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ such that $\mathscr{H}$ is invariant for N. Thus the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$\begin{cases} [T^*,T] = AA^* \\ A^*T = BA^* \\ [B^*,B] + A^*A = 0. \end{cases} \tag{1}$$

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *weakly subnormal* if there exist operators $A \in \mathscr{B}(\mathscr{H}',\mathscr{H})$ and $B \in \mathscr{B}(\mathscr{H}')$ such that the first two conditions in (1) hold, or equivalently, there is an extension $\widehat{T}$ of T such that

$$\widehat{T}^*\widehat{T}h = \widehat{T}\widehat{T}^*h \quad \text{for all } h \in \mathscr{H}.$$

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The operator $\widehat{T}$ is said to be a *partially normal extension* of T. Clearly,

$$\text{subnormal} \Longrightarrow \text{weakly subnormal} \Longrightarrow \text{hyponormal}.$$

The class of weakly subnormal operators has been studied in an attempt to bridge the gap between subnormality and hyponormality ([2], [3], [4]).

Let $\{e_n : n = 0, 1, 2, \cdots\}$ be an orthonormal basis for $\mathscr{H}$ and let S be a weighted shift with positive weight sequence $\{w_n\}$, that is,

$$Se_n = w_n e_{n+1} \quad \text{for } n \geqslant 0.$$

Then for $k \geqslant 1$, $S^{*k}S^k$ and S^kS^{*k} are both diagonal operators such that

$$\begin{aligned} &S^{*k}S^k e_n = w_n^2 \cdots w_{n+k-1}^2 e_n && \text{for } n \geqslant 0, \\ &S^kS^{*k} e_n = 0 && \text{for } 0 \leqslant n < k, \\ &S^kS^{*k} e_n = w_{n-k}^2 \cdots w_{n-1}^2 e_n && \text{for } n \geqslant k. \end{aligned} \tag{2}$$

Observe that S is hyponormal if and only if $\{w_n\}$ is increasing. Let S be a weighted shift and let M and N be positive integers. Write

$$T \equiv aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N},$$

where a, b, c, d are nonzero complex numbers such that $a\overline{b} = c\overline{d}$. In this case, the matrix of T forms a 4-banded matrix. The hyponormality of this type of operators has been studied in [5], [6], [7]. Since $a\overline{b} = c\overline{d}$, a direct calculation shows that

$$[T^*, T] = \left(|a|^2 - |c|^2\right)[S^{*M}, S^M] - \left(|d|^2 - |b|^2\right)[S^{*N}, S^N]. \tag{3}$$

Thus T is normal if and only if $|a| = |c|$. From this viewpoint, we will assume that $|a| \neq |c|$, to avoid the triviality of our argument. In this note, we consider a class of operators whose matrix representations comprise 4-banded matrices. In particular, we focus on the hyponormality and weak subnormality when each diagonal forms a hyponormal weighted shift.

2. The main results

We first observe:

THEOREM 2.1. (Propagation phenomenon) *Suppose S is a hyponormal weighted shift and $N > M$. Let*

$$T := aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N} \quad (a\overline{b} = c\overline{d} \text{ with } |a| > |c|).$$

If T is hyponormal, then S has no $2M$-consecutive equal weights. In particular, if $M = 1$, then the weight sequence of S is strictly increasing.

Proof. Let S be a hyponormal weighted shift with positive weight sequence $\{w_n\}$. Then it follows from (3) that

$$[T^*,T] = \big(|a|^2-|c|^2\big)[S^{*M},S^M] - \big(|d|^2-|b|^2\big)[S^{*N},S^N].$$

Put

$$\alpha := |a|^2-|c|^2 \text{ and } \beta := |d|^2-|b|^2.$$

Then it follows from (2) that $[T^*,T]$ is a diagonal operator whose diagonal entries μ_n are given by

$$\mu_n = \begin{cases} \alpha\prod_{k=0}^{M-1}\omega_{n+k}^2 - \beta\prod_{k=0}^{N-1}\omega_{n+k}^2 & \text{if } 0\leqslant n\leqslant M-1 \\ \alpha\big(\prod_{k=0}^{M-1}\omega_{n+k}^2 - \prod_{k=0}^{M-1}\omega_{n-M+k}^2\big) - \beta\prod_{k=0}^{N-1}\omega_{n+k}^2 & \text{if } M\leqslant n\leqslant N-1 \\ \alpha\big(\prod_{k=0}^{M-1}\omega_{n+k}^2 - \prod_{k=0}^{M-1}\omega_{n-M+k}^2\big) - \beta\big(\prod_{k=0}^{N-1}\omega_{n+k}^2 - \prod_{k=0}^{N-1}\omega_{n-N+k}^2\big) & \text{if } N\leqslant n. \end{cases} \tag{4}$$

Hence $[T^*,T]\geqslant 0$ if and only if $\mu_n\geqslant 0$ for all $n=0,1,2,\cdots$. Suppose that n_0 is the smallest integer such that $\omega_{n_0}=\omega_{n_0+1}=\cdots=\omega_{n_0+2M-1}$. There are two cases to consider.

Case 1: If $0\leqslant n_0\leqslant N-M-1$, then it follows from (4) that

$$-\beta\prod_{k=0}^{N-1} w_{n_0+M+k}^2 \geqslant 0.$$

Thus we have $\beta=0$, so that $|a|=|c|$, a contradiction.

Case 2: If $N-M\leqslant n_0$, then it follows from (4) that

$$-\beta\Big(\prod_{k=0}^{N-1}\omega_{n_0+M+k}^2 - \prod_{k=0}^{N-1}\omega_{n_0+M-N+k}^2\Big)\geqslant 0.$$

But since $\{w_n\}$ is increasing and $\beta\neq 0$, it follows that $\omega_{n_0+M-N}=\omega_{n_0+M-N+1}=\cdots=\omega_{n_0+M+N-1}$, a contradiction. The second assertion follows at once from the first assertion. This completes the proof. □

REMARK 2.2. The condition "$|a|>|c|$" is essential in Theorem 2.1. For example, if $|a|<|c|$, then we may have a hyponormal operator T for a flat-subnormal shift S. Indeed, let S be a unilateral shift and

$$T := S^M + 2S^N + 2S^{*M} + S^{*N} \quad (M<N).$$

Then it follows from (3) that

$$[T^*,T] = 3\big([S^{*N},S^N]-[S^{*M},S^M]\big).$$

Thus T is hyponormal. Observe that

$$\ker[T^*,T] = \bigvee\{e_0,e_1,\cdots,e_{M-1},e_N,e_{N+1},\cdots\},$$

where $\bigvee$ denotes the closed linear span. Thus $T(\ker[T^*,T])$ is not contained in $\ker[T^*,T]$. But since $\ker[T^*,T]$ is always invariant for every weakly subnormal operator T (cf. [4]), we see that T is not weakly subnormal.

THEOREM 2.3. *Let S be a weighted shift and*

$$T := aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N} \quad (a\overline{b} = c\overline{d} \text{ and } M < N).$$

If $\ker[T^*,T]$ is invariant for T, then $\ker[T^*,T]$ reduces T.

Proof. Suppose S is a weighted shift with weight sequence $\{\omega_n\}$ and $\ker[T^*,T]$ is an invariant subspace for T. If $\ker[T^*,T] = \{0\}$, this is trivial. Let $\ker[T^*,T] \neq \{0\}$. Note that $[T^*,T]$ is a diagonal operator with respect to the standard bases $\{e_n\}$. Write

$$[T^*,T] \equiv \operatorname{diag}(\mu_0, \mu_1, \mu_2, \cdots).$$

Then it suffices to show that

$$\mu_{n_0} \neq 0 \Rightarrow Te_{n_0} \in \operatorname{ran}[T^*,T].$$

Let $\mu_{n_0} \neq 0$. If $n_0 \geqslant N$, then

$$Te_{n_0} = ae_{n_0+M} + be_{n_0+N} + \overline{c}e_{n_0-M} + \overline{d}e_{n_0-N}.$$

Suppose $Te_{n_0} \notin \operatorname{ran}[T^*,T]$. Then at least one of the following is zero:

$$\mu_{n_0+M},\ \mu_{n_0+N},\ \mu_{n_0-M},\ \mu_{n_0-N}.$$

If $\mu_{n_0+M} = 0$, then $e_{n_0+M} \in \ker[T^*,T]$, so that $Te_{n_0+M} \in \ker[T^*,T]$. Thus $e_{n_0} \in \ker[T^*,T]$, and hence $\mu_{n_0} = 0$, a contradiction. Similarly, we can prove the rest of the cases. This completes the proof. $\square$

THEOREM 2.4. *Let S be a weighted shift with strictly increasing weight sequence. Put*

$$T := aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N} \quad (a\overline{b} = c\overline{d} \text{ and } M < N).$$

If $\ker[T^*,T]$ is invariant for T, then $\ker[T^*,T] = \{0\}$.

Proof. Suppose S is a weighted shift with strictly increasing weight sequence $\{\omega_n\}$. Put

$$\alpha := |a|^2 - |c|^2 \text{ and } \beta := |d|^2 - |b|^2.$$

Then by the proof of Theorem 2.1, we have that

$$[T^*,T] = \operatorname{diag}(\mu_0, \mu_1, \mu_2, \cdots),$$

where the μ_n are given by the equation (4). Suppose that $\ker[T^*,T]$ is invariant for T and $\ker[T^*,T] \neq \{0\}$. Then there exists $0 \leqslant n_0 \leqslant M-1$ such that $\mu_{n_0} = 0$. Since $\ker[T^*,T]$ is invariant for T, we have $\mu_{n_0+N} = 0$. It thus follows from (4) that

$$\frac{\alpha}{\beta} = \prod_{k=M}^{N-1} \omega_{n_0+k}^2 = \frac{\prod_{k=0}^{N-1} \omega_{n_0+N+k}^2 - \prod_{k=0}^{N-1} \omega_{n_0+k}^2}{\prod_{k=0}^{M-1} \omega_{n_0+N+k}^2 - \prod_{k=0}^{M-1} \omega_{n_0+N-M+k}^2},$$

or equivalently,

$$\prod_{k=M}^{N-1} \omega_{n_0+k}^2 \left(\prod_{k=0}^{M-1} \omega_{n_0+N+k}^2 - \prod_{k=0}^{M-1} \omega_{n_0+N-M+k}^2 + \prod_{k=0}^{M-1} \omega_{n_0+k}^2 \right) = \prod_{k=0}^{N-1} \omega_{n_0+N+k}^2.$$

But since $M < N$ and $\{\omega_n\}$ is strictly increasing, it follows that

$$\prod_{k=0}^{M-1} \omega_{n_0+k}^2 < \prod_{k=0}^{M-1} \omega_{n_0+N-M+k}^2.$$

We thus have that

$$\prod_{k=M}^{N-1} \omega_{n_0+k}^2 \left(\prod_{k=0}^{M-1} \omega_{n_0+N+k}^2 \right) > \prod_{k=0}^{N-1} \omega_{n_0+N+k}^2,$$

a contradiction. This completes the proof. $\square$

We would like to ask the following questions.

QUESTION 2.5. For which hyponormal weighted shift S, is

$$T := aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N} \quad (a\overline{b} = c\overline{d} \neq 0 \text{ and } M < N)$$

weakly subnormal?

For Question 2.5, a good candidate for S is the Cowen and Long's shift [1], i.e., the weight sequence $\{w_k\}$ of S is given by

$$w_k = \left(\sum_{j=0}^{k} \gamma^{2j} \right)^{\frac{1}{2}} \quad (k = 0,1,2,\cdots; 0 < \gamma < 1).$$

LEMMA 2.6. *([3, Lemma 2.1]) If $T \in \mathscr{B}(\mathscr{H})$ is weakly subnormal then T has a partially normal extension $\widehat{T}$ on $\mathscr{K}$ of the form*

$$\widehat{T} = \begin{pmatrix} T & [T^*,T]^{\frac{1}{2}} \\ 0 & B \end{pmatrix} \quad on \quad \mathscr{K} := \mathscr{H} \oplus \mathscr{H}.$$

We are ready for:

THEOREM 2.7. *Let S be the weighted shift with weight sequence*

$$w_k = \left(\sum_{j=0}^{k} \gamma^{2j}\right)^{\frac{1}{2}} \quad (k = 0,1,2,\cdots; 0 < \gamma < 1).$$

Let

$$T := aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N} \quad (a\overline{b} = c\overline{d} \neq 0 \text{ and } M < N).$$

Then T is weakly subnormal if and only if T is hyponormal and $\ker[T^*,T] = \{0\}$.

Proof. Put

$$\alpha := |a|^2 - |c|^2 \text{ and } \beta := |d|^2 - |b|^2.$$

Then by the proof of Theorem 2.1, we have that

$$[T^*,T] = \mathrm{dia}(\mu_0, \mu_1, \mu_2, \cdots),$$

where the μ_n are given by the equation (4). Suppose that T is a hyponormal with $\ker[T^*,T] = \{0\}$. Then it follows from Lemma 2.6 that T is weakly subnormal if and only if T has a partially normal extension $\widehat{T}$ on $\mathscr{K}$ of the form

$$\widehat{T} = \begin{pmatrix} T & [T^*,T]^{\frac{1}{2}} \\ 0 & B \end{pmatrix} \quad \text{on} \quad \mathscr{K} := \mathscr{H} \oplus \mathscr{H}.$$

It thus follows that T is weakly subnormal if and only if there exist $B \in \mathscr{B}(\mathscr{H})$ such that

$$[T^*,T]^{\frac{1}{2}} T = B[T^*,T]^{\frac{1}{2}}.$$

Since $\ker[T^*,T]^{\frac{1}{2}} = \ker[T^*,T] = \{0\}$, it follows that

$$[T^*,T]^{\frac{1}{2}} T = B[T^*,T]^{\frac{1}{2}} \Longleftrightarrow B = [T^*,T]^{\frac{1}{2}} T [T^*,T]^{-\frac{1}{2}}.$$

Thus T is weakly subnormal if and only if

$$[T^*,T]^{\frac{1}{2}} T [T^*,T]^{-\frac{1}{2}} \text{ is bouned.} \tag{5}$$

Now we will show that

$$\lim_{n\to\infty} \frac{\sqrt{\mu_{n+1}}}{\sqrt{\mu_n}} = \gamma. \tag{6}$$

It follows from (4) that for $n \geqslant N$,

$$\begin{aligned} \mu_n = & \frac{\alpha}{(1-\gamma^2)^M}\left(\prod_{k=1}^{M}\left(1-\gamma^{2(n+k)}\right) - \prod_{k=1}^{M}\left(1-\gamma^{2(n-M+k)}\right)\right) \\ & - \frac{\beta}{(1-\gamma^2)^N}\left(\prod_{k=1}^{N}\left(1-\gamma^{2(n+k)}\right) - \prod_{k=1}^{N}\left(1-\gamma^{2(n-N+k)}\right)\right). \end{aligned} \tag{7}$$

Let

$$f(x) := \prod_{k=1}^{M}\left(1-\gamma^{2(x+k)}\right) - \prod_{k=1}^{M}\left(1-\gamma^{2(x-M+k)}\right).$$

Then we have that

$$f'(x) = 2\gamma^{2x}\log\gamma\left(\sum_{k=1}^{M}\gamma^{2(k-M)}\cdot\prod_{i=1,i\neq k}^{M}(1-\gamma^{2(x-M+i)}) - \sum_{k=1}^{M}\gamma^{2k}\cdot\prod_{i=1,i\neq k}^{M}(1-\gamma^{2(x+i)})\right).$$

Thus

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \lim_{x\to\infty}\frac{f'(x+1)}{f'(x)} = \gamma^2,$$

and hence

$$\lim_{n\to\infty}\frac{\prod_{k=1}^{M}\left(1-\gamma^{2(n+k+1)}\right)-\prod_{k=1}^{M}\left(1-\gamma^{2(n-M+k+1)}\right)}{\prod_{k=1}^{M}\left(1-\gamma^{2(n+k)}\right)-\prod_{k=1}^{M}\left(1-\gamma^{2(n-M+k)}\right)} = \gamma^2. \tag{8}$$

Similarly, we also have that

$$\lim_{n\to\infty}\frac{\prod_{k=1}^{N}\left(1-\gamma^{2(n+k+1)}\right)-\prod_{k=1}^{N}\left(1-\gamma^{2(n-N+k+1)}\right)}{\prod_{k=1}^{N}\left(1-\gamma^{2(n+k)}\right)-\prod_{k=1}^{N}\left(1-\gamma^{2(n-N+k)}\right)} = \gamma^2. \tag{9}$$

It thus follows from (7), (8) and (9) that

$$\lim_{n\to\infty}\frac{\mu_{n+1}}{\mu_n} = \gamma^2,$$

which proves (6). Thus, by (6), we have that

$$\begin{aligned}\left|\left|[T^*,T]^{\frac{1}{2}}T[T^*,T]^{-\frac{1}{2}}\right|\right| &\leqslant |a|\left|\left|[T^*,T]^{\frac{1}{2}}S^M[T^*,T]^{-\frac{1}{2}}\right|\right| + |b|\left|\left|[T^*,T]^{\frac{1}{2}}S^N[T^*,T]^{-\frac{1}{2}}\right|\right| \\ &\quad + |c|\left|\left|[T^*,T]^{\frac{1}{2}}S^{*M}[T^*,T]^{-\frac{1}{2}}\right|\right| + |d|\left|\left|[T^*,T]^{\frac{1}{2}}S^{*N}[T^*,T]^{-\frac{1}{2}}\right|\right| \\ &< \infty,\end{aligned}$$

which gives (5). Thus T is weakly subnormal. For the converse, suppose that T is weakly subnormal. Then T is hyponormal and $\ker[T^*,T]$ is invariant for T. But since $\{\omega_n\}$ is strictly increasing, it follows from Theorem 2.4 that $\ker[T^*,T] = \{0\}$. This completes the proof. □

We now have:

COROLLARY 2.8. *Let S be the weighted shift with weight sequence*

$$w_k = \left(\sum_{j=0}^{k}\gamma^{2j}\right)^{\frac{1}{2}} \quad (k = 0,1,2,\cdots;0<\gamma<1).$$

Let

$$T := \lambda S^M + S^N + S^{*M} + \overline{\lambda} S^{*N} \quad (\lambda \in \mathbb{C} \text{ and } M < N).$$

Then the following are equivalent:

(a) T *is hyponormal;*

(b) T *is weakly subnormal;*

(c) $|\lambda| \leqslant 1$.

Proof. (a) $\Leftrightarrow$ (c): Observe that

$$[T^*, T] = \big(|\lambda|^2 - 1\big)\mathrm{diag}(\delta_0, \delta_1, \delta_2, \cdots), \tag{10}$$

where the δ_n are given by

$$\delta_n = \begin{cases} \prod_{k=0}^{M-1} \omega_{n+k}^2 - \prod_{k=0}^{N-1} \omega_{n+k}^2 & \text{if } 0 \leqslant n \leqslant M-1 \\ \prod_{k=0}^{M-1} \omega_{n+k}^2 - \prod_{k=0}^{M-1} \omega_{n-M+k}^2 - \prod_{k=0}^{N-1} \omega_{n+k}^2 & \text{if } M \leqslant n \leqslant N-1 \\ \prod_{k=0}^{M-1} \omega_{n+k}^2 - \prod_{k=0}^{M-1} \omega_{n-M+k}^2 - \prod_{k=0}^{N-1} \omega_{n+k}^2 + \prod_{k=0}^{N-1} \omega_{n-N+k}^2 & \text{if } N \leqslant n. \end{cases}$$

Thus for $0 \leqslant n \leqslant N-1$,

$$\delta_n \leqslant \prod_{k=0}^{M-1} \omega_{n+k}^2 - \prod_{k=0}^{N-1} \omega_{n+k}^2 < 0,$$

and for $n \geqslant N$,

$$\begin{aligned} \delta_n &= \prod_{k=0}^{N-1} \omega_{n-N+k}^2 - \prod_{k=0}^{M-1} \omega_{n-M+k}^2 - \prod_{k=0}^{N-1} \omega_{n+k}^2 + \prod_{k=0}^{M-1} \omega_{n+k}^2 \\ &< \prod_{k=0}^{M-1} \omega_{n-M+k}^2 \left(\prod_{k=M}^{N-1} \omega_{n-M+k}^2 - 1 \right) - \prod_{k=0}^{M-1} \omega_{n+k}^2 \left(\prod_{k=M}^{N-1} \omega_{n+k}^2 - 1 \right) \\ &< \prod_{k=0}^{M-1} \omega_{n+k}^2 \left(\prod_{k=M}^{N-1} \omega_{n-M+k}^2 - \prod_{k=M}^{N-1} \omega_{n+k}^2 \right) \\ &< 0. \end{aligned}$$

Thus it follows from (10) that T is hyponormal if and only if $|\lambda| \leqslant 1$.

(c) $\Rightarrow$ (b): Suppose that $|\lambda| \leqslant 1$. If $|\lambda| = 1$ then by (10), $[T^*, T] = 0$, so that T is normal, and hence weakly subnormal. Let $|\lambda| < 1$. Then it follows from (10) that T is hyponormal and $\ker[T^*, T] = \{0\}$. Thus by Theorem 2.7, we have that T is weakly subnormal.

(b) $\Rightarrow$ (a): Clear. $\square$

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