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A CLASS OF NORMAL DILATION MATRICES AFFIRMING THE MARCUS–DE OLIVEIRA CONJECTURE

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Abstract. In this article, we provide a class of normal dilation matrices affirming the Marcus-de Oliveira conjecture.

Throughout, n will denote a positive integer. The determinant conjecture of Marcus and de Oliveira states that the determinant of the sum of two n by n normal matrices A and B belongs to the convex hull of the $n!$ σ-points, $z_\sigma := \prod_{i=1}^{n}(a_i + b_{\sigma(i)})$, indexed by $\sigma \in S_n$, where a_i's and b_j's are eigenvalues of A and B, respectively (see [9], [3], [11]). We briefly write as $(A,B) \in MOC$ if the pair of normal matrices A,B affirms the Marcus and de Oliveira conjecture, i.e.,

$$\det(A+B) \in co(\{z_\sigma | \sigma \in S_n\}).$$

In [8], Fiedler showed that, for two hermitian matrices A,B

$$\Delta(A,B) := \{\det(A+UBU^*) | U \in U_n(\mathbb{C})\}$$

is a line segment with σ-points as endpoints, where $U_n(\mathbb{C})$ denotes the set of all unitary matrices of dimension $n \times n$. This result, in fact, motivates the conjecture. As a consequence of Fiedler's result, $(A,B) \in MOC$ for any pair of skew-hermitian matrices A,B.

In [1], N. Bebiano, A. Kovacec, and J. da Providencia provided that if A is positive definite and B a non-real scalar multiple of a hermitian matrix, then $(A,B) \in MOC$. They also obtained that if eigenvalues of A are pairwise distinct complex numbers lying on a line l and all eigenvalues of B lie on a parallel to l, then $(A,B) \in MOC$. S.W. Drury showed that $(A,B) \in MOC$ for the case that A is hermitian and B is non-real scalar multiple of a hermitian matrix (essentially hermitian matrix) in [5] and the case that $A = sU$ and $B = tV$ for $s,t \in \mathbb{C}$ and $U,V \in U_n(\mathbb{C})$ in [6].

It is also known that, for normal matrices $A,B \in M_n(\mathbb{C})$ (the set of all $n \times n$ matrices over $\mathbb{C}$), $(A,B) \in MOC$: if $\det(A+B) = 0$ ([7]); if the point z_σ lie all on a straight line ([10]); if $n = 2,3$ ([3, 2]); if A or B has only two distinct eigenvalues, one of them simple, ([3]). However, it seems that there is no new affirmative class of normal matrices to this conjecture after the year 2007.

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Let X be a square $n\times n$ complex matrix and s be a complex number. It is a direct calculation to see that

$$N(X,s) := \begin{pmatrix} X & (X-sI)^* \\ (X-sI)^* & X \end{pmatrix}$$

is a normal matrix of size $2n\times 2n$ and thus it is a normal dilation of X. We will see (in the proof of the main result) that the eigenvalues of $N(X,s)$ lie on both real and imaginary axis and thus this matrix need not be essentially hermitian or a scalar multiple of a unitary matrix. In this short note, we show that:

THEOREM 1. *Let $X,Y\in M_n(\mathbb{C})$ and $s,t\in\mathbb{C}$. Then $(N(X,s),N(Y,t))\in MOC$.*

Note that if $A\in M_n(\mathbb{C})$ is normal then UAU^* is also normal for any $U\in U_n(\mathbb{C})$. Then $VN(X,s)V^*$ is also a normal dilation of X for any $V\in U_{2n}(\mathbb{C})$. Moreover, since the conjecture is invariant under simultaneous unitary similarity, we also deduce from Theorem 1 that $(VN(X,s)V^*,VN(Y,t)V^*)\in MOC$ for any $V\in U_{2n}(\mathbb{C})$.

To prove the main result, we will use the following lemmas.

LEMMA 2. *Let $A,B\in M_n(\mathbb{C})$ and $C,D\in M_m(\mathbb{C})$ be normal. If $(A,B)\in MOC$ and $(C,D)\in MOC$, then $(A\oplus C,B\oplus D)\in MOC$.*

Proof. Suppose that $\{a_i\,|\,1\leqslant i\leqslant n\}$, $\{b_i\,|\,1\leqslant i\leqslant n\}$, $\{c_i\,|\,1\leqslant i\leqslant m\}$ and $\{d_i\,|\,1\leqslant i\leqslant m\}$ are ordered set of the eigenvalues of A,B,C and D, respectively. Denote $e_i := a_i$, $f_i := b_i$ for $i=1,\dots,n$ and $e_{n+j}=c_j$, $f_{n+j}=d_j$ for $j=1,\dots,m$. Then, $\{e_i\,|\,1\leqslant i\leqslant n+m\}$ and $\{f_i\,|\,1\leqslant i\leqslant n+m\}$ are ordered set of the eigenvalues of $A\oplus C$ and $B\oplus D$, respectively. For each $\sigma\in S_n,\pi\in S_m$ and $\theta\in S_{n+m}$, denote z_σ,v_π and w_θ the product $\prod_{i=1}^n(a_i+b_{\sigma(i)})$, $\prod_{i=1}^m(c_i+d_{\pi(i)})$ and $\prod_{i=1}^{n+m}(e_i+f_{\theta(i)})$, respectively. Suppose that $(A,B)\in MOC$ and $(C,D)\in MOC$, then

$$\det(A+B)=\sum_{\sigma\in S_n} t_\sigma z_\sigma \text{ and } \det(C+D)=\sum_{\pi\in S_m} s_\pi v_\pi,$$

where $t_\sigma,s_\pi\in[0,1]$ such that $\sum_{\sigma\in S_n}t_\sigma=1$ and $\sum_{\sigma\in S_m}s_\pi=1$. Note that

$$\begin{aligned}\det(A\oplus C+B\oplus D) &= \det((A+B)\oplus(C+D))\\ &= \det(A+B)\cdot\det(C+D)\\ &= (\sum_{\sigma\in S_n} t_\sigma z_\sigma)(\sum_{\pi\in S_m} s_\pi v_\pi)\\ &= \sum_{\sigma\in S_n,\pi\in S_m}(t_\sigma s_\pi)(z_\sigma v_\pi).\end{aligned}$$

For each $\sigma\in S_n$ and $\pi\in S_m$, define a permutation $\theta(\sigma,\pi)\in S_{n+m}$ by

$$\theta(\sigma,\pi) := \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n+m \\ \sigma(1) & \cdots & \sigma(n) & n+\pi(1) & \cdots & n+\pi(m) \end{pmatrix}$$

Then $w_{\theta(\sigma,\pi)} = z_\sigma v_\pi$. Since, for each $\sigma \in S_n$ and $\pi \in S_m$, $t_\sigma s_\pi \in [0,1]$ and

$$\sum_{\sigma\in S_n,\pi\in S_m} (t_\sigma s_\pi) = (\sum_{\sigma\in S_n} t_\sigma)(\sum_{\sigma\in S_m} s_\pi) = (1)(1) = 1,$$

we conclude that

$$\det(A\oplus C + B\oplus D) \in co\{w_{\theta(\sigma,\pi)} \,|\, \sigma \in S_n, \pi \in S_m\} \subseteq co\{w_\theta \,|\, \theta \in S_{n+m}\}.$$

Hence $(A\oplus C, B\oplus D) \in MOC$. □

To be a self contained material, we record a result of S. W. Drury.

THEOREM 3. [4] *Let A and B be hermitian matrices with the given eigenvalues $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ respectively. Let $(t_1,\ldots,t_n)$ be the eigenvalues of $A+B$. Then*

$$\prod_{j=1}^{n}(\lambda + t_j) \in co\{\prod_{j=1}^{n}(\lambda + a_j + b_{\sigma(j)})|\sigma \in S_n\},$$

where co denotes the convex hull in the space of polynomials and λ is an indeterminate.

As a corollary of the above theorem, we have that:

LEMMA 4. *Let $X,Y \in M_n(\mathbb{C})$ and $\alpha,\beta \in \mathbb{C}$. Then $(X-X^*+\alpha I_n, Y-Y^*+\beta I_n) \in MOC$ and $(X+X^*+\alpha I_n, Y+Y^*+\beta I_n) \in MOC$.*

Proof. Since $X+X^*$ and $Y+Y^*$ are hermitian, by Theorem 3, we deduce directly that $(X+X^*+\alpha I_n, Y+Y^*+\beta I_n) \in MOC$. Since $X-X^*$ and $Y-Y^*$ are skew-hermitian, $i(X-X^*)$ and $i(Y-Y^*)$ are hermitian. Again, by Theorem 3, $(X-X^*+\alpha I_n, Y-Y^*+\beta I_n) \in MOC$. □

Proof of Theorem 1. Let U be the block matrix in $M_{2n}(\mathbb{C})$ defined by

$$U := \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix}.$$

It is a direct computation to see that U is a unitary matrix and

$$U^*\begin{pmatrix} M & N \\ N & M \end{pmatrix}U = (M-N)\oplus(M+N),$$

for any $M,N \in M_n(\mathbb{C})$. Let $A := X-X^*+(\bar{s})I_n$, $B := Y-Y^*+\bar{t}I_n$, $C := X+X^*-(\bar{s})I_n$, and $D := Y+Y^*-\bar{t}I_n$. By Lemma 4, the pair of normal matrices (A,B) and (C,D) satisfy the conjecture. Hence, by Lemma 2, $(A\oplus C, B\oplus D) \in MOC$. Therefore,

$$(N(X,s), N(Y,t)) = (U(A\oplus C)U^*, U(B\oplus D)U^*) \in MOC,$$

which completes the proof. □

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