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EXPLICIT CONSTRUCTION OF TIGHT NONUNIFORM FRAMELET PACKETS ON LOCAL FIELDS

O. AHMAD AND N. A. SHEIKH

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Abstract. The main objective of this paper is to provide the explicit construction of nonuniform tight framelet packets on local fields via wavelet spaces and NUMRA spaces.

1. Introduction

The theory and applications of frames have been developed nicely during the last sixty years. This development turns out to be an active research area dealing with a generalization of the concept of an orthonormal basis. The idea comes by having an additional lower bound of the Bessel sequences. Frames were introduced in 1952 by Duffin and Schaeffer in the context of nonharmonic Fourier series [12]. They used frames as a tool in the study of nonharmonic Fourier series. Almost 30 years later, Young wrote his book [23] in 1980, which contains a beautiful development of abstract frames and their applications to nonharmonic Fourier series. In 1986, Daubechies, Grossmann and Meyer constructed frames for $L^2(\mathbb{R})$ based on time frequency or time-scale translates of functions [11]. These developments and others spurred a dramatic advancement of wavelet and frame theory in the following years.

Recently there has been an interest in the applications of redundant dyadic wavelet systems. Although many applications of wavelets use wavelet bases, other types of applications work better with redundant wavelet families, of which tight wavelet frames are the easiest to use. Tight wavelet frames are different from orthonormal wavelet bases in one important respect; they are (in general) redundant systems but with the same fundamental structure as wavelet systems.The most common method to construct tight wavelet frames relies on the so-called extension principles. The resulting tight wavelet frames are based on a multiresolution analysis, and the generators are often called mother framelets. Wavelet frames provide poor frequency localization in applications.For instance, in context of signal processing, the pyramid-structured framelet transform decomposes the signal into a set of frequency channels that have narrower bandwidths in the lower frequency region. The transform is suitable for a signal whose main information is concentrated in the low frequency regions. But it may not be suitable for information whose domain frequency channels are focused on the middle frequency region. To overcome this disadvantage, the concept of wavelet frames must be

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generalized to include a library of wavelet frames, called framelet packets. Coifman, Meyer and Wickerhauser [8, 9] introduced the concept of framelet packets where orthonormal wavelet packets were considered and then lots of results on wavelet packets emerged [5, 6, 7, 10, 15, 21].

On the other hand, the past decade has also witnessed a tremendous interest in the problem of constructing wavelet bases and frames on various spaces other than $\mathbb{R}$. For example, R. L. Benedetto and J. J. Benedetto [3] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Jiang et al. [13] pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. Subsequently, tight wavelet frames on local fields of positive characteristic were constructed by Shah and Debnath [18] using extension principles. Also the concept of nonuniform multiresolution analysis (NUMRA) on local fields of positive characteristic given by Shah and Abdullah [16]. For more about frames and frames on local fields, we refer to [1, 2, 4, 14, 17, 19, 20]. In this paper we introduce the notion of tight nonuniform framelet packets on local fields. We provide the method of constructing nonuniform tight framelet packets via wavelet spaces and NUMRA spaces.

The remainder of the paper is structured as follows. In Section 2, we discuss the notations and basic facts on local fields. In Section 3, we provide the construction of tight nonuniform framelet packets on local fields by splitting wavelet spaces. We provide the construction of tight nonuniform framelet packets on local fields by decomposing NUMRA spaces in section section 4.

2. Notations and basic facts on local fields

A local field K is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. If K is of positive characteristic, then K is a field of formal Laurent series over a finite field $GF(p^c)$. If $c=1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let K be a fixed local field with the ring of integers $\mathfrak{D} = \{x \in K : |x| \leqslant 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimedean norm $|\cdot| : K \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;
(b) $|xy| = |x||y|$ for all $x, y \in K$;
(c) $|x+y| \leqslant \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in K : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in K. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element

$\mathfrak{p}$ of K such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathscr{U} = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathscr{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \{x \in K : |x| < q^{-k}\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [22]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that χ_u is any character on K^+, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [22], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leqslant n < q, \;\; n = a_0 + a_1 p + \ldots + a_{c-1} p^{c-1}, \;\; 0 \leqslant a_k < p, \;\; \text{and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1 \zeta_1 + \ldots + a_{c-1} \zeta_{c-1}) \mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \ldots + b_s q^s$, $n \in \mathbb{N}_0$, $0 \leqslant b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \ldots + u(b_s)\mathfrak{p}^{-s}. \tag{2.2}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leqslant s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}, n \geqslant 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.3}$$

The Fourier transform of $f \in L^1(\mathbb{K})$ is denoted by $\hat{f}(\xi)$ and defined by

$$\mathscr{F}\{f(x)\} = \hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\,dx. \tag{2.4}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)\,dx.$$

The properties of Fourier transforms on local field K are much similar to those of on the classical field $\mathbb{R}$. In fact, the Fourier transform on local fields of positive characteristic have the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(\mathbb{K})$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \leqslant \|f\|_1$.
- If $f \in L^1(\mathbb{K})$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\leqslant q^k} f(x)\overline{\chi_\xi(x)}\,dx, \tag{2.5}$$

where $f_k = f\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\,dx. \tag{2.6}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} |f(x)|^2 dx = \sum_{n\in\mathbb{N}_0} \left|\hat{f}\big(u(n)\big)\right|^2. \tag{2.7}$$

For an integer $N \geqslant 1$ and an odd integer r with $1 \leqslant r \leqslant qN-1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{u(r)}{N}\right\} + \mathscr{Z}.$$

where $\mathscr{Z} = \{u(n) : n \in \mathbb{N}_0\}$. It is easy to verify that Λ is not a group on local field K, but is the union of $\mathscr{Z}$ and a translate of $\mathscr{Z}$. Following is the definition of nonuniform multiresolution analysis (NUMRA) on local fields of positive characteristic given by Shah and Abdullah [16].

DEFINITION 2.1. For an integer $N \geqslant 1$ and an odd integer r with $1 \leqslant r \leqslant qN-1$ such that r and N are relatively prime, an associated NUMRA on local field K of positive characteristic is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$ such that the following properties hold:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{K})$;
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-1}N\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) There exists a function ϕ in V_0 such that $\{\phi(\cdot-\lambda):\lambda\in\Lambda\}$, is an orthonormal basis for V_0.

It is worth noticing that, when $N=1$, one recovers from the definition above the definition of an MRA on local fields of positive characteristic $p>0$ [13]. When, $N>1$, the dilation is induced by $\mathfrak{p}^{-1}N$ and $|\mathfrak{p}^{-1}|=q$ ensures that $qN\Lambda\subset\mathscr{Z}\subset\Lambda$.

As in the standard scheme, one expects the existence of $qN-1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $\mathfrak{p}^{-1}N$ form an orthonormal basis for $L^2(\mathbb{K})$.

Let us define the spaces

$$l^2(\Lambda)=\left\{z:\Lambda\to\mathbb{C}:\sum_{\lambda\in\Lambda}|z(\lambda)|^2<\infty\right\}$$

and

$$L^2(\Omega)=\left\{f:\Omega\to\mathbb{C}:\int_{\Omega}|f(\xi)|^2d\xi<\infty\right\},$$

where Ω is a Lebesgue measurable subset of K with finite positive measure. These spaces are Hilbert spaces with the inner products defined by

$$\langle z,w\rangle=\sum_{\lambda\in\Lambda}z(\lambda)\overline{w(\lambda)}\quad\text{for } z,w\in l^2(\Lambda)$$

and

$$\langle f,g\rangle=\int_{\Omega}f(\xi)\overline{g(\xi)}\,d\xi\quad\text{for } f,g\in L^2(\Omega),$$

respectively.

DEFINITION 2.2. The Fourier transform on $l^2(\Lambda)$ is a map $\wedge: l^2(\Lambda)\to L^2(\Omega)$ defined by

$$\hat{z}(\xi)=\sum_{\lambda\in\Lambda}z(\lambda)\overline{\chi_\lambda(\xi)},\quad z\in l^2(\lambda)$$

and its inverse is given by

$$f^{\vee}(\lambda)=\left\langle f,\overline{\chi_\lambda(\xi)}\right\rangle=\int_{\Omega}f(\xi)\chi_\lambda(\xi)\,d\xi,\quad f\in L^2(\Omega).$$

For all $z,w\in l^2(\Lambda)$, the Parseval and Plancherel formulae are respectively, given by

$$\langle z,w\rangle=\sum_{\lambda\in\Lambda}z(\lambda)\overline{w(\lambda)}=\int_{\Omega}f(\xi)\overline{g(\xi)}d\xi=\langle\hat{z},\hat{w}\rangle,$$

$$||z||^2=\sum_{\lambda\in\Lambda}|z(\lambda)|^2=\int_{\Omega}|\hat{z}(\xi)|^2d\xi=||\hat{z}||^2.$$

For $\lambda \in \Lambda$, we define the translation operator $T_{qN\lambda} : l^2(\Lambda) \to l^2(\Lambda)$ by

$$T_{qN\lambda} z(\sigma) = z(\sigma - qN\lambda), \quad \forall \sigma \in \Lambda.$$

Then, it can be easily verified that for $z, w \in l^2(\Lambda)$, we have

$$\left(T_{qN\lambda} z\right)^{\wedge}(\xi) = \overline{\chi_{qN\lambda}(\xi)}\hat{z}(\xi) \quad \text{and} \quad \left\langle T_{qN\lambda} z, T_{qN\sigma} w\right\rangle = \left\langle T_{qN(\lambda-\sigma)} z, w\right\rangle.$$

DEFINITION 2.3. For given $\Psi := \left\{\psi_1, \ldots, \psi_{qN-1}\right\} \subset L^2(\mathbb{K})$. A system of the form

$$\mathscr{F}(\Psi, \lambda) = \left\{\psi_{\ell,j,\lambda} := (qN)^{j/2} \psi_\ell\left((\mathfrak{p}^{-1}N)^j x - \lambda\right), \, j \in \mathbb{Z}, \lambda \in \Lambda, 1 \leqslant \ell \leqslant qN-1\right\}. \tag{2.8}$$

is called a *nonuniform wavelet system* on local field $\mathbb{K}$, where $\mathfrak{p}$ is prime and ψ is called the *generator of the system*.

DEFINITION 2.4. The nonuniform wavelet system $\mathscr{F}(\Psi, \lambda)$ defined by (2.8) is called a *non uniform wavelet frame*, if there exist positive numbers $0 < A \leqslant B < \infty$ such that for all $f \in L^2(\mathbb{K})$

$$A\|f\|_2^2 \leqslant \sum_{\ell=1}^{qN-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \left|\left\langle f, \psi_{\ell,j,\lambda}\right\rangle\right|^2 \leqslant B\|f\|_2^2. \tag{2.9}$$

The largest A and the smallest B for which (2.9) holds are called *nonuniform wavelet frame bounds*. A wavelet frame is a *tight nonuniform wavelet frame* if A and B are chosen such that $A = B$ and then the generators $\left\{\psi_1, \psi_2, \ldots, \psi_{qN-1}\right\}$ are often referred as *tight nonuniform framelets*. If only the right-hand inequality in (2.8) holds, then $\mathscr{F}(\Psi, \lambda)$ is called a *Bessel sequence*.

Next, we give a brief account of the NUMRA-based wavelet frames generated by the wavelet masks on local fields. Following the unitary extension principle, one often starts with a refinable function or even with a refinement mask to construct desired wavelet frames. A function $\varphi \in L^2(\mathbb{K})$ is called a *nonuniform refinable function*, if it satisfies an equation of the type

$$\varphi(x) = \sqrt{qN} \sum_{\lambda \in \Lambda} a_\lambda \, \varphi\left((\mathfrak{p}^{-1}N)x - \lambda\right), \tag{2.10}$$

where $\left\{a_\lambda : \lambda \in \Lambda\right\} \in l^2(\Lambda)$. In the frequency domain, equation (2.9) can be written as

$$\hat{\varphi}(\xi) = m_0\left(\frac{\mathfrak{p}\xi}{N}\right) \hat{\varphi}\left(\frac{\mathfrak{p}\xi}{N}\right), \tag{2.11}$$

where

$$m_0(\xi) = \frac{1}{\sqrt{qN}} \sum_{\lambda \in \Lambda} a_\lambda \, \overline{\chi_\lambda(\xi)}, \tag{2.12}$$

is an integral periodic function in $L^2(\mathfrak{D})$ and is often called the *refinement mask* of φ. Observe that $\chi_k(0) = \hat{\phi}(0) = 1$. By letting $\xi = 0$ in equations (2.10) and (2.11), we obtain $\sum_{\lambda\in\Lambda} a_\lambda = 1$. Further, it is proved in [18] that a function $\varphi \in L^2(\mathbb{K})$ generates an NUMRA in $L^2(\mathbb{K})$ if and only if

$$\sum_{\lambda\in\Lambda} \left|\hat{\varphi}(\xi+\lambda)\right|^2 = 1, \text{ for } a.e.\ \xi \in \mathfrak{D}, \text{ and } \hat{\varphi}(0) = \lim_{\xi\to 0} \hat{\varphi}(\xi) = 1, \quad \xi \in \mathbb{K}. \tag{2.13}$$

Suppose $\Psi = \left\{\psi_1, \dots, \psi_{qN-1}\right\}$ is a set of NUMRA functions derived from

$$\hat{\psi}_\ell(\xi) = m_\ell\left(\frac{\mathfrak{p}\xi}{N}\right)\hat{\varphi}\left(\frac{\mathfrak{p}\xi}{N}\right), \tag{2.14}$$

where

$$m_\ell(\xi) = \frac{1}{\sqrt{qN}} \sum_{\lambda\in\Lambda} a_\lambda^\ell\, \overline{\chi_\lambda(\xi)}, \quad 1 \leqslant \ell \leqslant qN-1 \tag{2.15}$$

are the integral periodic functions in $L^2(\mathfrak{D})$ and are called the *nonuniform framelet symbols* or *nonuniform wavelet masks*. With $m_\ell(\xi), 0 \leqslant \ell \leqslant qN-1$ as the wavelet masks, we formulate the matrix $\mathscr{M}(\xi)$ as

$$\mathscr{M}(\xi) = \begin{pmatrix} m_0(\xi) & m_0\big(\xi+\mathfrak{p}u(1)\big) & \dots & m_0\big(\xi+\mathfrak{p}u(s-1)\big) \\ m_1(\xi) & m_1\big(\xi+\mathfrak{p}u(1)\big) & \dots & m_1\big(\xi+\mathfrak{p}u(s-1)\big) \\ \vdots & \vdots & \ddots & \vdots \\ m_{qN-1}(\xi) & m_{qN-1}\big(\xi+\mathfrak{p}u(1)\big) & \dots & m_{qN-1}\big(\xi+\mathfrak{p}u(s-1)\big) \end{pmatrix}. \tag{2.16}$$

The so-called unitary extension principle (UEP) provides a sufficient condition on $\Psi = \left\{\psi_1, \psi_2 \dots, \psi_{qN-1}\right\}$ such that the nonuniform wavelet system $\mathscr{F}(\Psi,\lambda)$ given by (2.8) constitutes a tight frame for $L^2(\mathbb{K})$.

The construction of nonuniform framelet systems often starts with the construction of NUMRA, which is built on nonuniform refinable functions. A function $\varphi \in L^2(\mathbb{K})$ is called *refinable* if it satisfies a refinement equation given by (2.10) The so-called *unitary extension principle* (UEP) provides a sufficient condition on $\Psi = \{\psi_1, \dots, \psi_{qN-1}\}$ such that the resulting nonuniform wavelet system $\mathscr{F}(\Psi,\lambda)$ forms a tight frame of $L^2(\mathbb{K})$. In this connection, Shah and Debnath [18] gave an explicit construction scheme for the construction of tight framelets on local fields using unitary extension principles in the following way.

THEOREM 2.1. *Suppose that the refinable function ϕ and the framelet symbols m_ℓ, $0 \leqslant \ell \leqslant qN-1$ satisfy equations* (2.11)–(2.13). *Define $\psi_1, \dots, \psi_{qN-1}$ by equation* (2.14). *Let $\mathscr{M}(\xi)$ be the modulation matrix such that*

$$\mathscr{M}(\xi)\mathscr{M}^*(\xi) = I_q, \quad \textit{for} \quad a.e.\ \xi \in \sigma(V_0) \tag{2.17}$$

where

$$\sigma(V_0) := \{\xi \in \mathfrak{D} : \sum_{\lambda\in\Lambda} |\hat{\phi}(\xi+\lambda)|^2 \neq 0\},$$

then the wavelet system $\mathscr{F}(\Psi,\lambda)$ *given by equation* (2.7) *constitutes a normalized tight wavelet frame for* $L^2(\mathbb{K})$.

Moreover, if the nonuniform framelet symbols $m_\ell, 0 \leqslant \ell \leqslant qN-1$, satisfy the UEP condition (2.17). Then, for any $\xi \in K$, we have

$$\sum_{k=0}^{qN-1} \left| m_\ell \left(\frac{\mathfrak{p}}{N} (\xi + u(k)) \right) \right|^2 \leqslant 1, \tag{2.18}$$

and

$$\sum_{\ell=0}^{qN-1} m_\ell \left(\frac{\mathfrak{p}}{N} (\xi + u(r)) \right) \overline{m_\ell \left(\frac{\mathfrak{p}}{N} (\xi + u(s)) \right)} = \delta_{r,s}, \quad 0 \leqslant r,s \leqslant qN-1. \tag{2.19}$$

For each $j \in \mathbb{Z}$, we define the spaces

$$V_j = \overline{\text{span}}\left\{ \phi_{j,\lambda} : \lambda \in \Lambda \right\},$$

and

$$W_{j,\ell} = \overline{\text{span}}\left\{ \psi_{\ell,j,\lambda} : \lambda \in \Lambda \right\}, \quad 0 \leqslant \ell \leqslant qN-1.$$

Therefore, in view of tight frame decomposition algorithm, we can write

$$V_j = V_{j-1} + \sum_{\ell=1}^{qN-1} W_{\ell,j-1}. \tag{2.20}$$

It immediately follows from the above decomposition that these qN spaces are in general not orthogonal. Therefore, by the repeated applications of (2.20), we can further split the V_j spaces as:

$$\begin{aligned} V_j &= V_{j-1} + \sum_{\ell=1}^{qN-1} W_{\ell,j-1} = V_{j-2} + \sum_{r=j-2}^{j-1} \sum_{\ell=1}^{qN-1} W_{\ell,r} \\ &= \ldots = V_{j_0} + \sum_{r=j_0}^{j-1} \sum_{\ell=1}^{qN-1} W_{\ell,r} = \sum_{r=-\infty}^{j-1} \sum_{\ell=1}^{qN-1} W_{\ell,r}. \end{aligned}$$

3. Construction of tight nonuniform framelet packets via wavelet space

This section is devoted to the construction of tight nonuniform framelet packets on local field spaces by splitting the wavelet spaces $W_{\ell,j}$. We start this section with the following lemma which will be useful for obtaining main result.

LEMMA 3.1. *Let* $g \in L^2(\mathbb{K})$ *and* $\left\{ g_{j,\lambda} : \lambda \in \Lambda \right\}$ *be a Bessel's sequence in* $L^2(\mathbb{K})$ *i.e.,*

$$\sum_{\lambda \in \Lambda} \left| \hat{g}(\xi + \lambda) \right|^2 \leqslant B, \quad \xi \in K \tag{3.1}$$

for any fixed $j \in \mathbb{Z}$. *Let* $m_\ell, 0 \leqslant \ell \leqslant qN-1$ *be the nonuniform framelet masks associated with the nonuniform refinable function* φ *and the tight nonuniform framelets* $\psi_\ell, 1 \leqslant \ell \leqslant qN-1$ *satisfying the UEP condition* (2.17). *Suppose*

$$g_\ell(x) = qN \sum_{\lambda \in \Lambda} m_\ell(\lambda)\, g(\mathfrak{p}^{-1}Nx - \lambda), \tag{3.2}$$

$$G_\ell = \overline{span}\{g_{\ell,j-1,\lambda} : \lambda \in \Lambda\}, \tag{3.3}$$

and $G = \overline{span}\{g_{j,\lambda} : \lambda \in \Lambda\}$, *for* $0 \leqslant \ell \leqslant qN-1$. *Then*

1. *For* $0 \leqslant \ell \leqslant qN-1$, *each set* $\{g_{\ell,j-1,\lambda} : \lambda \in \Lambda\}$ *forms a Bessel's sequence with* $\|g_\ell\|_2^2 \leqslant B$ *and* $\|g\|_2^2 \leqslant B$.

2. *For any sequence* $z \in l^2(\mathscr{Z})$, *there exists* qN *sequences* $\{z_\ell\}_{\ell=0}^{qN-1}$ *defined by*

$$z_\ell(u(k)) = \sqrt{q}N \sum_{k \in \mathbb{N}_0} \overline{m_\ell}(\lambda - \mathfrak{p}^{-1}Nu(k))\, z(\lambda), \quad \lambda \in \Lambda \tag{3.4}$$

such that

$$\|z\|_{l^2(\mathscr{Z})}^2 = \sum_{\ell=0}^{qN-1} \|z_\ell\|^2, \tag{3.5}$$

and

$$\sum_{\lambda \in \Lambda} z(\lambda)\, g_{j,\lambda} = \sum_{\ell=0}^{qN-1} \sum_{\lambda \in \Lambda} z_\ell(\lambda) g_{\ell,j-1,\lambda}. \tag{3.6}$$

3. *In particular for any* $f \in L^2(\mathbb{K})$, *let* $z(\lambda) = \langle f, g_{j,\lambda}\rangle, \lambda \in \Lambda$, *then* $z \in l^2(\mathscr{Z})$ *and equations* (3.4)–(3.6) *gives*

$$z_\ell(\lambda) = \langle f, g_{\ell,j-1,\lambda}\rangle, \quad \lambda \in \Lambda, 0 \leqslant \ell \leqslant qN-1, \tag{3.7}$$

$$\sum_{\lambda \in \Lambda} |\langle f, g_{j,\lambda}\rangle|^2 = \sum_{\ell=0}^{qN-1} \sum_{\lambda \in \Lambda} |\langle f, g_{\ell,j-1,\lambda}\rangle|^2, \tag{3.8}$$

and

$$\sum_{\lambda \in \Lambda} \langle f, g_{j,\lambda}\rangle g_{j,\lambda} = \sum_{\ell=0}^{qN-1} \sum_{\lambda \in \Lambda} \langle f, g_{\ell,j-1,\lambda}\rangle\, g_{\ell,j-1,\lambda}, \tag{3.9}$$

respectively.

4. G *has the decomposition*

$$G = \sum_{i=0}^{qN-1} G_i.$$

Proof. (a) By invoking Plancherel's formula, we have

$$\begin{aligned}\|g\|_2^2 &= \|\hat{g}\|_2^2\\ &= \int_K \left|\hat{g}(\xi)\chi_\lambda(\xi)\right|^2 d\xi\\ &= \int_{\mathfrak{D}} \sum_{\lambda\in\Lambda} \left|\hat{g}(\xi+\lambda)\right|^2 \left|\chi_\lambda(\xi)\right|^2 d\xi.\end{aligned}$$

Using equation (3.1) and the fact that the set $\{\chi_\lambda : \lambda \in \Lambda\}$ is an orthonormal basis on $\mathfrak{D}$, we obtain $\|g\|_2^2 \leqslant B$.

On taking Fourier transform of equation (3.2), we obtain

$$\hat{g}_\ell(\xi) = m_\ell\left(\frac{\mathfrak{p}\xi}{N}\right)\hat{g}\left(\frac{\mathfrak{p}\xi}{N}\right). \tag{3.10}$$

Using equations (2.18) and (3.1), we have

$$\begin{aligned}\sum_{\lambda\in\Lambda}\left|\hat{g}_\ell(\xi+\lambda)\right|^2 &= \sum_{\lambda\in\Lambda}\left|m_\ell\left(\frac{\mathfrak{p}}{N}(\xi+u(\lambda))\right)\right|^2\left|\hat{g}\left(\frac{\mathfrak{p}}{N}(\xi+u(\lambda))\right)\right|^2\\ &= \sum_{\sigma=0}^{qN-1}\sum_{\lambda\in\Lambda}\left|m_\ell\left(\frac{\mathfrak{p}}{N}(\xi+u(qN\lambda+\sigma))\right)\right|^2\left|\hat{g}\left(\frac{\mathfrak{p}}{N}(\xi+u(kqN\lambda+\sigma))\right)\right|^2\\ &= \sum_{\sigma=0}^{qN-1}\left|m_\ell\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma)\right)\right|^2\sum_{\lambda\in\Lambda}\left|\hat{g}\left(\frac{\mathfrak{p}}{N}(\xi+u(qN\lambda+\sigma))\right)\right|^2\\ &= \sum_{\sigma=0}^{q-1}\left|m_\ell\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right)\right|^2\sum_{\lambda\in\Lambda}\left|\hat{g}\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))+u(\lambda)\right)\right|^2\\ &\leqslant B\sum_{\sigma=0}^{qN-1}\left|\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right)\right|^2\\ &\leqslant B, \quad \text{for } \ell=0,1,\ldots,L.\end{aligned}$$

(b) For each $0 \leqslant \ell \leqslant qN-1$, the Fourier transform of equation (3.4) gives

$$\hat{z}_\ell(\xi) = (qN)^{-1/2}\sum_{\sigma=0}^{qN-1} m_\ell\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right)\hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right). \tag{3.11}$$

By summing equation (3.11) over $\ell = 0$ to $qN-1$ and using (2.19), we obtain

$$\begin{aligned}\sum_{\ell=0}^{qN-1}\left|\hat{z}_\ell(\xi)\right|^2 = (qN)^{-1}&\sum_{\ell=0}^{qN-1}\sum_{r,\sigma=0}^{qN-1}\overline{m_\ell}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\\ &\times m_\ell\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right)\overline{\hat{z}}\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right)\end{aligned}$$

$$
\begin{aligned}
&= (qN)^{-1}\sum_{r,\sigma=0}^{qN-1}\hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\overline{\hat{z}}\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right)\\
&\quad\times\sum_{\ell=0}^{qN-1}m_\ell\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right)\overline{m_\ell}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\\
&= (qN)^{-1}\sum_{r,\sigma=0}^{qN-1}\hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(\sigma))\right)\overline{\hat{z}}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\delta_{r,\sigma}\\
&= (qN)^{-1}\sum_{r=0}^{qN-1}\left|\hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\right|^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{\ell=0}^{qN-1}\left\|\hat{z}_\ell\right\|^2_{\ell^2(\mathscr{Z})} &= \sum_{\ell=0}^{qN-1}\sum_{\lambda\in\Lambda}\left|\hat{z}_\ell(\lambda)\right|^2\\
&= \sum_{\ell=0}^{qN-1}\int_{\mathfrak{D}}\left|\hat{z}_\ell(\lambda)\right|^2 d\xi\\
&= \int_{\mathfrak{D}}\sum_{\ell=0}^{qN-1}\left|\hat{z}_\ell(\lambda)\right|^2 d\xi\\
&= (qN)^{-1}\int_{\mathfrak{D}}\sum_{r=0}^{qN-1}\left|\hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\right|^2 d\xi\\
&= \int_{\mathfrak{D}}\left|\hat{z}(\xi)\right|^2 d\xi\\
&= \int_{\mathfrak{D}}\left|\sum_{\sigma\in\Lambda}z(\sigma)\chi_\sigma(\xi)\right|^2 d\xi\\
&= \sum_{\sigma\in\Lambda}\left|z(\sigma)\right|^2\\
&= \left\|\hat{z}\right\|^2_{\ell^2(\mathscr{Z})}.
\end{aligned}
$$

In frequency domain, (3.6) can be written as:

$$
\begin{aligned}
&(qN)^{-j/2}\hat{z}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\hat{g}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\\
&= (qN)^{\frac{1-j}{2}}\sum_{\ell=0}^{qN-1}\hat{z}_\ell\left((\mathfrak{p}^{-1}N)^{1-j}\xi\right)\hat{g}_\ell\left((\mathfrak{p}^{-1}N)^{1-j}\xi\right).
\end{aligned}
\tag{3.12}
$$

Thus, in order to show that (3.6) holds, it suffices to verify only the equality (3.12).

$$\begin{aligned}
R.H.S. &= (qN)^{\frac{1-j}{2}} \sum_{\ell=0}^{qN-1} \hat{z}_\ell\left((\mathfrak{p}^{-1}N)^{1-j}\xi\right) \hat{g}_\ell\left((\mathfrak{p}^{-1}N)^{1-j}\xi\right)\\
&= (qN)^{\frac{1-j}{2}} \sum_{\ell=0}^{qN-1} \hat{z}_\ell\left((\mathfrak{p}^{-1}N)^{1-j}\xi\right) m_\ell\left((\mathfrak{p}^{-1}N)^{-j}\xi\right) \hat{g}_\ell\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\\
&= (qN)^{\frac{1-j}{2}} \hat{g}_\ell\left((\mathfrak{p}^{-1}N)^{-j}\xi\right) \sum_{\ell=0}^{qN-1} \left[(qN)^{-1} \sum_{r=0}^{qN-1} \hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\right.\\
&\qquad \left.\times \overline{m_\ell}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\right] m_\ell\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\\
&= (qN)^{-j/2} \hat{g}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right) \sum_{r=0}^{qN-1} \hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right)\\
&\qquad \times \sum_{\ell=0}^{qN-1} \left[\overline{m_\ell}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right) m_\ell\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\right]\\
&= (qN)^{-j/2} \hat{g}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right) \sum_{r=0}^{qN-1} \hat{z}\left(\frac{\mathfrak{p}}{N}(\xi+u(r))\right) \delta_{r,0}\\
&= (qN)^{-j/2} \hat{g}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right) \hat{z}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right) = L.H.S.
\end{aligned}$$

(c). For the proof of the part (c) of the lemma, it is sufficient to verify equation (3.7) only. The equations (3.8) and (3.9) are direct consequences of equations (3.5) and (3.6) which have been proved. Moreover, from equation (3.4), we have

$$\begin{aligned}
z_\ell(\lambda) &= (qN)^{1/2} \sum_{\sigma\in\Lambda} \overline{m_\ell}(\sigma - \mathfrak{p}^{-1}N\lambda) z(\lambda)\\
&= (qN)^{1/2} \sum_{\sigma\in\Lambda} \overline{m_\ell}(\sigma - \mathfrak{p}^{-1}N\lambda) \langle f, g_{j,\sigma}\rangle\\
&= \left\langle f, (qN)^{1/2} \sum_{\sigma\in\Lambda} \overline{m_\ell}(\sigma - \mathfrak{p}^{-1}N\lambda) g_{j,\sigma}\right\rangle\\
&= \langle f, g_{\ell,j-1,\lambda}\rangle, \quad 0 \leqslant \ell \leqslant qN-1.
\end{aligned}$$

(d). This is immediate from equations (3.2) and (3.3).

Now we proceed to construct tight nonuniform framelet packets for $L^2(\mathbb{K})$ via NUMRA generated by the nonuniform framelet symbols . To do this, let $\{\psi_\ell, m_\ell\}_{\ell=0}^{qN-1}$ satisfy the conditions of the unitary extension principle and $\omega_0 = \phi$. Define the functions $\Gamma_n(x)$, $n = 0,1,2,\ldots,$ associated with the refinable function φ recursively by

$$\hat{\Gamma}_n(\xi) = \hat{\omega}_{qNr+\ell}(\xi) = m_\ell\left((\mathfrak{p}^{-1}N)^{-1}\xi\right) \Gamma_r\left((\mathfrak{p}^{-1}N)^{-1}\xi\right), \quad 0 \leqslant \ell \leqslant qN-1,\ r \in \mathbb{N}_0. \tag{3.13}$$

Note that for $r = 0$ and $0 \leqslant \ell \leqslant qN-1$, we have

$$\hat{\omega}_\ell(\xi) = m_\ell\left((\mathfrak{p}^{-1}N)^{-1}\xi\right) \omega_0\left((\mathfrak{p}^{-1}N)^{-1}\xi\right) = m_\ell\left((\mathfrak{p}^{-1}N)^{-1}\xi\right) \varphi\left((\mathfrak{p}^{-1}N)^{-1}\xi\right) \tag{3.14}$$

which shows that $\Gamma_\ell(x) = \psi_\ell(x),\ 0 \leqslant \ell \leqslant qN-1$.

For $n \in \mathbb{N}_0$, define a family of subspaces of $L^2(\mathbb{K})$ by

$$U_n = \overline{\text{span}}\big\{\Gamma_{n,0,\lambda} : \lambda \in \Lambda\big\}. \tag{3.15}$$

Clearly $U_0 = V_0$ and $U_\ell = W_{\ell,0}$, for $1 \leqslant \ell \leqslant qN-1$. Since $\mathscr{F}(\Psi,\lambda)$ is a tight nonuniform wavelet frame constructed via UEP in an NUMRA generated by φ. Therefore, we have

$$\sum_{\lambda\in\Lambda} \big|\hat{\Gamma}_0\big(\xi+\lambda\big)\big|^2 \leqslant 1, \quad \xi \in K.$$

By invoking Lemma 3.1, for $n = 1,2,\ldots$, we obtain

$$\sum_{\lambda\in\Lambda} \big|\hat{\Gamma}_n\big(\xi+\lambda\big)\big|^2 \leqslant 1, \quad U_n^1 = \sum_{t=qNn}^{(qN)(n+1)-1} U_t,$$

and for any $f \in L^2(\mathbb{K})$,

$$\sum_{\lambda\in\Lambda} \big|\langle f, \Gamma_{n,1,\lambda}\rangle\big|^2 = \sum_{t=qNn}^{(qN)(n+1)-1} \sum_{\lambda\in\Lambda} \big|\langle f, \Gamma_{t,0,\lambda}\rangle\big|^2.$$

A repeated application of the Splitting Lemma 3.1 for $j = 1,2,\ldots,$ yields

$$U_n^j = \sum_{t=(qN)^j n}^{(qN)^j(n+1)-1} U_t \tag{3.16}$$

and for any $f \in L^2(\mathbb{K})$

$$\sum_{\lambda\in\Lambda} \big|\langle f, \Gamma_{n,j,\lambda}\rangle\big|^2 = \sum_{t=(qN)^j n}^{(qN)^j(n+1)-1} \sum_{\lambda\in\Lambda} \big|\langle f, \Gamma_{t,0,\lambda}\rangle\big|^2. \tag{3.17}$$

Substituting $n = 0$ in equations (3.16) and (3.17), we get

$$V_j = \sum_{t=0}^{(qN)^j-1} U_t \tag{3.18}$$

and

$$\sum_{\lambda\in\Lambda} \big|\langle f, \varphi_{j,\lambda}\rangle\big|^2 = \sum_{t=0}^{(qN)^j-1} \sum_{\lambda\in\Lambda} \big|\langle f, \Gamma_{t,0,\lambda}\rangle\big|^2 \tag{3.19}$$

for any $f \in L^2(\mathbb{K})$, respectively. Moreover, for $n = \ell$, where $\ell = 1,\ldots,qN-1$, equations (3.16) and (3.17) yield

$$W_{\ell,j} = W_{\ell,0}^j = U_\ell^j = \sum_{t=(qN)^j\ell}^{(qN)^j(\ell+1)-1} U_t, \tag{3.20}$$

and for any $f \in L^2(\mathbb{K})$

$$\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{\ell,j,\lambda}\rangle\right|^2=\sum_{\lambda\in\Lambda}\left|\langle f,\Gamma_{\ell,j,\lambda}\rangle\right|^2=\sum_{t=(qN)^j\ell}^{(qN)^j(\ell+1)-1}\sum_{\lambda\in\Lambda}\left|\langle f,\Gamma_{t,0,\lambda}\rangle\right|^2. \tag{3.21}$$

From equation (3.21), it follows that each wavelet space $W_{\ell,j},\ j\in\mathbb{N}_0, 1\leqslant \ell\leqslant qN-1$ can be further splitted into $(qN)^j$ subspaces $U_t, t\in[(qN)^j\ell,(qN)^j(\ell+1)-1]$. If we keep the parameter j fixed, say $J>0$, we will obtain

$$L^2(\mathbb{K})=\sum_{t=0}^{(qN)^J-1}U_t+\sum_{\ell=1}^{qN-1}\sum_{j\geqslant J}W_{\ell,j}. \quad \square \tag{3.22}$$

THEOREM 3.1. *Let $\mathscr{F}(\Psi,\lambda)$ be a tight wavelet frame constructed via UEP in an NUMRA and $m_1,m_2,\ldots,m_{qN-1}$ are the nonuniform framelet symbols satisfying the UEP condition* (2.17). *Let $\{\Gamma_n : n\in\mathbb{N}_0\}$ be defined as in* (3.13). *Then for any fixed $J>0$, the family of functions*

$$\mathscr{G}=\left\{\Gamma_{n,0,\lambda}:0\leqslant n\leqslant (qN)^J-1,\,\lambda\in\Lambda\right\}\bigcup\left\{\psi_{\ell,j,\lambda}:1\leqslant\ell\leqslant qN-1,\,j\geqslant J,\,\lambda\in\Lambda\right\}$$

forms a tight frame for $L^2(\mathbb{K})$.

Proof. By Theorem 2.1, the nonuniform wavelet system $\mathscr{F}(\Psi,\lambda)$ constitutes a tight nonuniform wavelet frame for $L^2(\mathbb{K})$. Therefore by equation (3.18), we have for any $f\in L^2(\mathbb{K})$

$$\begin{aligned}\|f\|_2^2&=\sum_{\lambda\in\Lambda}\left|\langle f,\varphi_{0,\lambda}\rangle\right|^2+\sum_{\ell=1}^{qN-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{\ell,j,\lambda}\rangle\right|^2\\&=\sum_{\lambda\in\Lambda}\left|\langle f,\varphi_{J,\lambda}\rangle\right|^2+\sum_{\ell=1}^{qN-1}\sum_{j\geqslant J}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{\ell,j,\lambda}\rangle\right|^2\\&=\sum_{n=0}^{(qN)^J-1}\sum_{n\in\mathbb{N}_0}\left|\langle f,\Gamma_{n,0,\lambda}\rangle\right|^2+\sum_{\ell=1}^{qN-1}\sum_{j\geqslant J}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{\ell,j,\lambda}\rangle\right|^2.\end{aligned}$$

This completes the proof of the theorem. $\square$

DEFINITION 3.1. The functions $\{\Gamma_n : n\in\mathbb{N}_0\}$ are called as the *basic nonuniform framelet packets* on the local field associated with the refinable function φ..

Now we proceed to construct a class of tight frames for $L^2(\mathbb{K})$ by choosing other $L^2(\mathbb{K})$ space decompositions with the help of basiv nonuniform framelet packets. For simplicity, let us consider a disjoint partition P_J of a finite set of non-negative integers

$$\Omega_J=\left\{r\in\mathbb{N}_0:0\leqslant r\leqslant (qN)^J-1\right\} \tag{3.23}$$

into disjoint of the form

$$\mathscr{H}_{j,n} = \Big\{(qN)^j n, \ldots, (qN)^j(n+1)-1\Big\}, \quad j,n \in \mathbb{N}_0,$$

i.e.,

$$P_J = \Big\{\mathscr{H}_{j,n} : \bigcup \mathscr{H}_{j,n} = \Omega_J\Big\}, \tag{3.24}$$

Then, it follows from equations (3.16) and (3.21) that

$$\begin{aligned} L^2(\mathbb{K}) &= \sum_{t=0}^{(qN)^J-1} U_t + \sum_{\ell=1}^{qN-1}\sum_{j\geqslant J} W_{\ell,j} \\ &= \sum_{\mathscr{H}_{J,n}\in P_J}\ \sum_{t=(qN)^j n}^{(qN)^J(n+1)-1} U_t + \sum_{\ell=1}^{qN-1}\sum_{j\geqslant J} W_{\ell,j} \\ &= \sum_{\mathscr{H}_{J,n}\in P_J} U_t^j + \sum_{\ell=1}^{qN-1}\sum_{j\geqslant J} W_{\ell,j}. \end{aligned}$$

THEOREM 3.2. *Suppose $\mathscr{F}(\Psi,\lambda)$ is a tight nonuniform wavelet frame constructed via UEP in an NUMRA and $m_1, m_2, \ldots, m_{qN-1}$ are the nonuniform framelet symbols satisfying the UEP condition (2.17). Let $\{\Gamma_n : n \in \mathbb{N}_0\}$ be defined as in equation (3.13). For any fixed $J > 0$, P_J is a partition of Ω_J, where Ω_J and P_J are defined in equations (3.23) and (3.24), respectively. Then the family of functions*

$$\mathscr{F}_{P_J} = \Big\{\Gamma_{n,0,\lambda} : \mathscr{H}_{J,n} \in P_J,\, \lambda \in \Lambda\Big\} \bigcup \Big\{\psi_{\ell,j,\lambda} : 1 \leqslant \ell \leqslant qN-1,\, j \geqslant J,\, \lambda \in \Lambda\Big\}$$

constitutes a tight frame for $L^2(\mathbb{K})$.

Proof. For any arbitrary $f \in L^2(\mathbb{K})$, we have

$$\begin{aligned} \sum_{\mathscr{H}_{J,n}\in P_J}\sum_{\lambda\in\Lambda} \big|\langle f, \Gamma_{n,j,\lambda}\rangle\big|^2 &= \sum_{\mathscr{H}_{J,n}\in P_J}\ \sum_{n=(qN)^j n}^{(qN)^j(n+1)-1}\ \sum_{\lambda\in\Lambda} \big|\langle f, \Gamma_{n,0,\lambda}\rangle\big|^2 \\ &= \sum_{n=0}^{(qN)^J-1} \big|\langle f, \Gamma_{n,0,\lambda}\rangle\big|^2. \end{aligned}$$

By invoking Theorem 3.1, we get the desired result. □

4. Construction of tight nonuniform framelet packets via NUMRA space

In this section, we construct tight nonuniform framelet packets by decomposing the NUMRA space V_J directly for a fixed level $J > 0$ to the level 0.

At the first level of decomposition, by Lemma 3.1, V_J is decomposed into the qN spaces $W_{J-1,\mathbf{s}}, \mathbf{s} \in \Delta_1$ where

$$\Delta_1 = \Big\{\mathbf{s} = (s_J, s_{J-1}, \ldots s_1) : 0 \leqslant s_J \leqslant qN-1,\, s_{J-1} = \ldots = s_1 = 0\Big\}.$$

For this choice of $\mathbf{s} = (s_J, s_{J-1}, \dots, s_1)$, we define

$$\mathbf{s}(n) = s_n, \quad n = 1,2,\dots,J,$$
$$\Gamma_{\mathbf{s}}(x) = (qN)^{1/2} \sum_{\sigma\in\Lambda} h_\sigma^{\mathbf{s}(1)}\, \varphi\big(\mathfrak{p}^{-1}Nx - \sigma\big),$$

and

$$W_{J-1,\mathbf{s}} := \overline{\operatorname{span}}\big\{\Gamma_{\mathbf{s},J-1,\lambda} : \lambda \in \Lambda\big\}.$$

Therefore, for any $f \in L^2(\mathbb{K})$, we have

$$\sum_{\lambda\in\Lambda} \big|\langle f, \varphi_{J,\lambda}\rangle\big|^2 = \sum_{\mathbf{s}\in\Delta_1}\sum_{\lambda\in\Lambda} \big|\langle f, \Gamma_{\mathbf{s},J-1,\lambda}\rangle\big|^2.$$

At the second level of decomposition, by Lemma 3.1, each space $W_{J-1,\mathbf{s}}, \mathbf{s} \in \Delta_1$ is decomposed with the constructed mask $\mathbf{M}$ into spaces $W_{J-2,\mathbf{s}'}, \mathbf{s}' \in \Delta_2^{\mathbf{s}}$, where $\Delta_2^{\mathbf{s}}$ is a subset of Δ_2 defined by

$$\Delta_2^{\mathbf{s}} = \big\{\mathbf{s}' \in \Delta_2 : \mathbf{s}'(1) = \mathbf{s}(1)\big\}$$

and Δ_2 is a J-tuple index set defined by

$$\Delta_2 = \Big\{\mathbf{s} = (s_J, s_{J-1}, \dots, s_1) : 0 \leqslant s_{J-1}, s_J \leqslant qN-1,\, s_{J-2} = \dots = s_1 = 0\Big\},$$

$$\Gamma_{\mathbf{s}'}(x) = (qN)^{1/2} \sum_{\sigma\in\Lambda} h_\sigma^{\mathbf{s}'(2)}\, \varphi\big(\mathfrak{p}^{-1}Nx - \sigma\big),$$
$$W_{J-2,\mathbf{s}'} := \overline{\operatorname{span}}\big\{\Gamma_{\mathbf{s}',J-2,\lambda} : \lambda \in \Lambda\big\}.$$

Thus, for any $f \in L^2(\mathbb{K})$, we have

$$\sum_{\lambda\in\Lambda} \big|\langle f, \omega_{\mathbf{s},J-1,\lambda}\rangle\big|^2 = \sum_{\mathbf{s}'\in\Delta_2^{\mathbf{s}}}\sum_{\lambda\in\Lambda} \big|\langle f, \omega_{\mathbf{s}',J-2,\lambda}\rangle\big|^2.$$

Finally, at the m-th level $(2 \leqslant m \leqslant J)$ of decomposition, by Lemma 3.1, each space $W_{J-m+1,\mathbf{s}}$, $\mathbf{s} \in \Delta_{m-1}$ is decomposed with the constructed mask $\mathbf{M}$ into spaces $W_{J-m,\mathbf{s}'}, \mathbf{s}' \in \Delta_m^{\mathbf{s}}$, where $\Delta_m^{\mathbf{s}}$ is a subset of Δ_m defined by

$$\Delta_m^{\mathbf{s}} = \Big\{\mathbf{s}' \in \Lambda_m : \mathbf{s}'(n) = \mathbf{s}(n), \ \text{ for } \ 1 \leqslant n \leqslant m-1\Big\}$$

and Δ_m is a J-tuple index set defined by

$$\Delta_m = \Big\{\mathbf{s} = (s_J, s_{J-1}, \dots, s_1) : 0 \leqslant s_{J-m} \leqslant qN-1,\, s_{J-m} = \dots = s_1 = 0\Big\},$$

$$\Gamma_{\mathbf{s}'}(x) = (qN)^{1/2} \sum_{\sigma\in\Lambda} h_\sigma^{\mathbf{s}'(m)}\, \varphi\big(\mathfrak{p}^{-1}Nx - \sigma\big),$$
$$W_{J-m,\mathbf{s}'} := \overline{\operatorname{span}}\big\{\Gamma_{\mathbf{s}',J-m,\lambda} : \lambda \in \Lambda\big\}.$$

Therefore for any $f \in L^2(\mathbb{K})$, we have

$$\sum_{\lambda\in\Lambda} \left|\left\langle f, \Gamma_{\mathbf{s},J-m+1,\lambda}\right\rangle\right|^2 = \sum_{\mathbf{s}'\in\Delta_m^{\mathbf{s}}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \Gamma_{\mathbf{s}',J-m,\lambda}\right\rangle\right|^2.$$

In particular, at the J-th level of decomposition, by Lemma 3.1, each space $W_{1,\mathbf{s}}, \mathbf{s} \in \Delta_{J-1}$ is decomposed with $\mathbf{M}$ into spaces $W_{0,\mathbf{s}'}, \mathbf{s}' \in \Delta_J^{\mathbf{s}}$, where $\Delta_J^{\mathbf{s}}$ is a subset of Δ_J defined by

$$\Delta_J^{\mathbf{s}} = \left\{\mathbf{s}' \in \Delta_J : \mathbf{s}'(n) = \mathbf{s}(n), \ \text{ for } 1 \leqslant n \leqslant J-1\right\}$$

and Δ_J is a J-tuple index set defined by

$$\Delta_J = \left\{\mathbf{s} = (s_J, s_{J-1}, \ldots, s_1) : 0 \leqslant s_t \leqslant qN-1, \, 1 \leqslant t \leqslant J\right\}, \tag{4.1}$$

$$\Gamma_{\mathbf{s}'}(x) = (qN)^{1/2} \sum_{\sigma\in\Lambda} h_\sigma^{\mathbf{s}'(J)} \, \varphi\left(\mathfrak{p}^{-1}Nx - \sigma\right),$$

$$W_{0,\mathbf{s}'} := \overline{\operatorname{span}}\left\{\Gamma_{\mathbf{s}',0,\lambda} : \lambda \in \Lambda\right\}.$$

Thus, for any $f \in L^2(\mathbb{K})$, we have

$$\sum_{\lambda\in\Lambda} \left|\left\langle f, \Gamma_{\mathbf{s},1,\lambda}\right\rangle\right|^2 = \sum_{\mathbf{s}'\in\Lambda_J^{\mathbf{s}}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \Gamma_{\mathbf{s}',0,\lambda}\right\rangle\right|^2.$$

Combining all the inner product equations in the above construction, we get

$$\sum_{\lambda\in\Lambda} \left|\langle f, \varphi_{J,\lambda}\rangle\right|^2 = \sum_{\mathbf{s}\in\Lambda_J} \sum_{\lambda\in\Lambda} \left|\langle f, \Gamma_{\mathbf{s},0,\lambda}\rangle\right|^2, \quad \text{for any } f \in L^2(\mathbb{K}). \tag{4.2}$$

In other words, we obtain another representation of V_J as

$$V_J := \overline{\operatorname{span}}\left\{\Gamma_{\mathbf{s},0,\lambda} : \mathbf{s} \in \Delta_J, \lambda \in \Lambda\right\}.$$

THEOREM 4.1. *Suppose $\mathscr{F}(\Psi,\lambda)$ is a tight nonuniform wavelet frame constructed via UEP in an NUMRA and $\mathbf{M} = [m_0, m_1, \ldots, m_{qN-1}]$ is the combined mask satisfying the UEP condition* (2.17)*. Then for any fixed $J > 0$, the family of functions*

$$\mathscr{F} = \left\{\Gamma_{\mathbf{s},0,\lambda} : \mathbf{s} \in \Delta_J\right\} \bigcup \left\{\psi_{\ell,j,\lambda} : \ell = 1, \ldots, qN-1, j \geqslant J, \lambda \in \Lambda\right\}$$

forms a tight frame for $L^2(\mathbb{K})$, where Δ_J is a index set defined in (4.1).

Proof. Since $\mathscr{F}(\Psi,\lambda)$ is a tight nonuniform wavelet frame of $L^2(\mathbb{K})$, then by (4.2), we have

$$\begin{aligned}\|f\|_2^2 &= \sum_{\lambda\in\Lambda} \left|\langle f, \phi_{J,\lambda}\rangle\right|^2 + \sum_{\ell=1}^{qN-1} \sum_{j\geqslant J} \sum_{\lambda\in\Lambda} \left|\left\langle f, \psi_{\ell,j,\lambda}\right\rangle\right|^2 \\ &= \sum_{\mathbf{s}\in\Delta_J} \sum_{\lambda\in\Lambda} \left|\left\langle f, \Gamma_{\mathbf{s},0,\lambda}\right\rangle\right|^2 + \sum_{\ell=1}^{qN-1} \sum_{j\geqslant J} \sum_{\lambda\in\Lambda} \left|\left\langle f, \psi_{\ell,j,k}\right\rangle\right|^2\end{aligned}$$

for any $f \in L^2(\mathbb{K})$. This completes the proof. $\square$

Similar to the recursive construction of tight nonuniform framelet packets, we can obtain tight nonuniform framelet packets by performing various disjoint partitions $\mathscr{N}_J$ of Δ_J with each partition separating Δ_J into disjoint subsets of the form

$$S_{j,\mathbf{s}} = \Big\{(s_J,\ldots,s_{j+1},s'_j,\ldots,s'_1) \in \Delta_J : \mathbf{s} = (s_J,\ldots,s_{j+1},0,\ldots,0) \in \Delta_{J-j}\Big\},$$

i.e.,

$$\mathscr{N}_J = \Big\{S_{j,\mathbf{s}} : \bigcup S_{j,\mathbf{s}} = \Delta_J\Big\}. \tag{4.3}$$

COROLLARY 4.1. *Suppose $\mathscr{F}(\Psi,\lambda)$ is a tight nonuniform wavelet frame constructed via UEP in an NUMRA and $\mathbf{M} = [m_0, m_1, \ldots, m_{qN-1}]$ is the combined mask satisfying the UEP condition* (2.17)*. Let $\mathscr{N}_J$ be a disjoint partition of Δ_J, where Δ_J and $\mathscr{N}_J$ are defined in* (4.1) *and* (4.3)*, respectively. Then the collection*

$$\mathscr{F}_{\mathscr{N}_J} = \Big\{\Gamma_{\mathbf{s},j,\lambda} : S_{j,\mathbf{s}} \in \mathscr{N}_J, \lambda \in \Lambda\Big\} \bigcup \Big\{\psi_{\ell,j,\lambda} : \ell = 1,\ldots,qN-1, j \geqslant J \in \mathbb{Z}, \lambda \in \Lambda\Big\}$$

generates a tight nonuniform frame for $L^2(\mathbb{K})$.

Proof. Since $\mathscr{N}_J$ is a disjoint partition of Δ_J, for any $f \in L^2(\mathbb{K})$, we have

$$\begin{aligned}\sum_{S_{j,\mathbf{s}} \in \mathscr{N}_J} \sum_{\lambda \in \Lambda} \left|\langle f, \Gamma_{\mathbf{s},j,\lambda}\rangle\right|^2 &= \sum_{S_{j,\mathbf{s}} \in \mathscr{N}_J} \sum_{\mathbf{s}' \in S_{j,\mathbf{s}}} \sum_{\lambda \in \Lambda} \left|\langle f, \Gamma_{\mathbf{s}',0,\lambda}\rangle\right|^2 \\ &= \sum_{\mathbf{s} \in \Delta_J} \sum_{\lambda \in \Lambda} \left|\langle f, \Gamma_{\mathbf{s},0,\lambda}\rangle\right|^2.\end{aligned}$$

By applying Theorem 4.1, we obatin the desired result. $\square$

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O. Ahmad
Department of Mathematics
National Institute of Technology
Srinagar-190006, Jammu and Kashmir, India
e-mail: siawoahmad@gmail.com

N. A. Sheikh
Department of Mathematics
National Institute of Technology
Srinagar-190006, Jammu and Kashmir, India
e-mail: neyaznit@yahoo.co.in