Operators
and
Matrices
Volume 15, Number 1 (2021), 151–162 doi:10.7153/oam-2021-15-11

THE G–DRAZIN INVERSES OF SPECIAL OPERATOR MATRICES

HUANYIN CHEN AND MARJAN SHEIBANI*

(*Communicated by I. M. Spitkovsky*)

Abstract. An element a in a Banach algebra $\mathscr{A}$ has g-Drazin inverse provided that there exists $b\in\mathscr{A}$ such that $b=bab$, $ab=ba$, $a-a^2b\in\mathscr{A}^{qnil}$. In this paper we give a computational formula for the g-Drazin inverse of operator matrix $\begin{pmatrix} E & I\\ F & 0\end{pmatrix}$ which was posed by Campbell in the research on singular differential equations.

1. Introduction

Let $\mathscr{A}$ be a Banach algebra with an identity. An element a in a Banach algebra $\mathscr{A}$ has g-Drazin inverse provided that there exists some $b\in\mathscr{A}$ such that $b=bab$, $ab=ba$, $a-a^2b\in\mathscr{A}^{qnil}$. Such b is unique, if exists, and we denote it by a^d. Here, $\mathscr{A}^{qnil}$ is the set of all quasinilpotents in $\mathscr{A}$, i.e.,

$$a\in\mathscr{A}^{qnil}\Leftrightarrow \lim_{n\to\infty}\parallel a^n\parallel^{\frac{1}{n}}=0\Leftrightarrow 1+\lambda a\in\mathscr{A}^{-1} \text{ for any } \lambda\in\mathbb{C}.$$

We always use $\mathscr{A}^d$ to stand for the set of all g-Drazin invertible elements in $\mathscr{A}$. We say a has Drazin inverse a^D if the preceding quasinilpotent is replaced by the set of all nilpotent elements in $\mathscr{A}$.

Let E,F be bounded linear operators and I be the identity operator over a Banach space X. It is attractive to investigate the Drazin and g-Drazin invertibility of the operator matrix $M=\begin{pmatrix} E & I\\ F & 0\end{pmatrix}$. It was firstly posed by Campbell that the solutions to singular systems of differential equations is determined by the Drazin invertibility of the preceding special matrix M (see [2]). In 2005, Castro-González and Dopazo gave the representations of the Drazin inverse for a class of operator matrix $\begin{pmatrix} I & I\\ F & 0\end{pmatrix}$ (see [3]). In 2011, Bu et al. investigate the Drazin inverse of the preceding operator matrix M under the condition $EF=FE$ (see [1]). Afterwards, Patricio and Hartwig studied the g-Drazin invertibility of such special operator matrix M under the conditions $F^{\pi}EFF^d=0$, $F^{\pi}FE=EFF^{\pi}$ (see [8]). Here, $F^{\pi}=I-FF^d$ is the spectral idempotent of F. In 2016, Zhang investigated the g-Drazin invertibility of M under

Mathematics subject classification (2010): 15A09, 47A11, 47A53.
Keywords and phrases: g-Drazin inverse, operator matrix, Banach algebra.
* Corresponding author.

Paper OaM-15-11

the conditions $F^dEF^\pi=0$, $F^\pi FE=0$ and $F^\pi EF^d=0$, $EFF^\pi=0$ (see [9, Theorem 2.6, Theorem 2.8]).

The motivation of this paper is to further study the g-Drazin invertibility of this special operator matrix M. We shall present new conditions under which an operator matrix over a Banach algebra has g-Drazin inverse, and we thereby apply to determine the g-Drazin invertibility of M under new conditions $F^dEF^\pi=0$, $EFF^\pi=0$. The representations of M^d are given as well.

Throughout the paper, all Banach algebras of bounded linear operators are complex. Let $M_2(\mathscr{A})$ be the Banach algebra of all 2×2 matrices over the Banach algebra $\mathscr{A}$. We denote by $\mathbb{C}$ the field of all complex numbers. $\mathbb{N}$ stands for the set of all natural numbers.

2. 2×2 block matrices

In this section we consider the g-Drazin inverse of block matrix in a Banach algebra which will be used in the sequel. We begin with

LEMMA 2.1. (see [9, Lemma 2.2]) *Let*

$$x=\begin{pmatrix} a & 0\\ c & b\end{pmatrix},\qquad y=\begin{pmatrix} b & c\\ 0 & a\end{pmatrix}\in M_2(\mathscr{A})$$

Then

$$x^d=\begin{pmatrix} a^d & 0\\ z & b^d\end{pmatrix}\ \text{ and }\ y^d=\begin{pmatrix} b^d & z\\ 0 & a^d\end{pmatrix},$$

where $z=(b^d)^2\big(\sum\limits_{i=0}^{\infty}(b^d)^ica^i\big)a^\pi+b^\pi\big(\sum\limits_{i=0}^{\infty}b^ic(a^d)^i\big)(a^d)^2-b^dca^d.$

LEMMA 2.2. (see [9, Lemma 2.5]) *Let $a,d\in\mathscr{A}^d$ and $b,c\in\mathscr{A}$. If $abc=0$, $bd=0$ and $bc\in\mathscr{A}^{qnil}$, then $M:=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathscr{A})^d$. In this case,*

$$M^d=\begin{pmatrix} \phi_1 a & \phi_1 b\\ \omega a+\psi_1 & d^d+\omega b\end{pmatrix},$$

where

$$\begin{aligned}
\phi_n &= \sum_{j=0}^{\infty}(bc)^j(a^d)^{2j+2n};\\
\psi_n &= \sum_{j=0}^{\infty}(d^d)^{2j+2n}(cb)^jc;\\
\omega &= \sum_{i=0}^{\infty}(cb+d^2)^ic(a^d)^{2i+3}+\sum_{i=0}^{\infty}d^\pi d^{2i+1}c\phi_{i+2}\\
&\quad -\sum_{i=0}^{\infty}d^2(cb+d^2)^i\psi_1(a^d)^{2i+3}+\sum_{i=0}^{\infty}\psi_{i+2}a^{2i+1}a^\pi\\
&\quad -\sum_{i=0}^{\infty}(d^d)^{2i+3}c(a^2+bc)^ia^\pi-\sum_{i=0}^{\infty}(d^d)^{2i+1}c(bc)^i\phi_1-\psi_1a^d.
\end{aligned}$$

We are ready to prove:

THEOREM 2.3. *Let $a,d \in \mathscr{A}^d$ and $b,c \in \mathscr{A}$. If $a^d b = 0$, $dcb = 0$, $caa^{\pi} = 0$ and $cb \in \mathscr{A}^{qnil}$, then $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathscr{A})^d$. In this case, $M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where*

$$\begin{aligned}
\alpha = &\sum_{i=0}^{\infty}(bca^{\pi}+a^2a^{\pi})^i bz_{2i+2} + \sum_{i=0}^{\infty}(bca^{\pi}+a^2a^{\pi})^i b(d^d)^{2i+3}ca^{\pi}\\
&+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a^{2i+1}a^{\pi}b(cb)^j z_{2j+2i+3} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a^{2i+1}a^{\pi}b(cb)^j(d^d)^{2j+2i+4}ca^{\pi},\\
\beta = &\sum_{i=0}^{\infty}(bca^{\pi}+a^2a^{\pi})^i b(d^d)^{2i+2} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a^{2i+1}b(cb)^j(d^d)^{2j+2i+3},\\
\gamma = &\sum_{i=0}^{\infty}(cb)^i z_{2i+1} + \sum_{i=0}^{\infty}(cb)^i(d^d)^{2i+2}ca^{\pi},\\
\delta = &\sum_{i=0}^{\infty}(cb)^i(d^d)^{2i+1}
\end{aligned}$$

and

$$\begin{aligned}
z_1 &= d^{\pi}\sum_{i=0}^{\infty} d^i c(a^d)^{i+2} - d^d ca^d,\\
z_{m+1} &= z_1(a^d)^m + d^d z_m \quad \textit{for any} \quad m \in \mathbb{N}.
\end{aligned}$$

Proof. Let $p = \begin{pmatrix} aa^d & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathscr{A})$. Then $p^2 = p$. By hypothesis, we have the Pierce decomposition of M relatively to the idempotent p:

$$\sigma(M) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_p,$$

where

$$A = \begin{pmatrix} a^2a^d & 0 \\ caa^d & d \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ ca^{\pi} & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} aa^{\pi} & 0 \\ 0 & 0 \end{pmatrix}.$$

We easily check that

$$ABC = \begin{pmatrix} a^2a^d & 0 \\ caa^d & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & cb \end{pmatrix} = 0,$$

$$BD = \begin{pmatrix} 0 & 0 \\ ca^{\pi} & 0 \end{pmatrix}\begin{pmatrix} aa^{\pi} & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

$$BC = \begin{pmatrix} 0 & 0 \\ ca^{\pi} & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & cb \end{pmatrix} \in M_2(\mathscr{A})^{qnil}.$$

By Lemma 2.1, A has g-Drazin inverse and $D^d = 0$. In light of Lemma 2.2, we have

$$\sigma(M)^d = \begin{pmatrix} \Phi_1 A & \Phi_1 B \\ \Omega A + \Psi_1 & \Omega B \end{pmatrix}_p,$$

where

$$\Phi_n = \sum_{j=0}^{\infty} (BC)^j (A^d)^{2j+2n},$$

$$\Psi_n = \sum_{j=0}^{\infty} (D^d)^{2j+2n} (CB)^j C = 0$$

and

$$\Omega = \sum_{i=0}^{\infty} (CB+D^2)^i C (A^d)^{2i+3} + \sum_{i=0}^{\infty} D^{2i+1} C \Phi_{i+2}.$$

Obviously, we have

$$(BC)^j = \begin{pmatrix} 0 & 0 \\ 0 & (cb)^j \end{pmatrix}.$$

Choose

$$z_1 = d^\pi \sum_{i=0}^{\infty} d^i c (a^d)^{i+2} - d^d c a^d,$$
$$z_{m+1} = z_1 (a^d)^m + d^d z_m \quad \text{for any} \quad m \in \mathbb{N}.$$

Then we verify that

$$(A^d)^m = \begin{pmatrix} (a^d)^m & 0 \\ z_m & (d^d)^m \end{pmatrix}.$$

Also we have

$$(CB+D^2)^i = \begin{pmatrix} ((bc+a^2)a^\pi)^i & 0 \\ 0 & 0 \end{pmatrix},$$

$$D^{2i+1} C = \begin{pmatrix} 0 & a^{2i+1} b \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$(BC)^j (A^d)^{2j+1} = \begin{pmatrix} 0 & 0 \\ (cb)^j z_{2j+1} & (cb)^j (d^d)^{2j+1} \end{pmatrix},$$

$$(BC)^j (A^d)^{2j+2} B = \begin{pmatrix} 0 & 0 \\ (cb)^j (d^d)^{2j+2} c a^\pi & 0 \end{pmatrix},$$

$$(CB+D^2)^i C (A^d)^{2i+2} = \begin{pmatrix} ((bc+a^2)a^\pi)^i b z_{2i+2} & ((bc+a^2)a^\pi)^i b (d^d)^{2i+2} \\ 0 & 0 \end{pmatrix},$$

$$D^{2i+1} C (BC)^j (A^d)^{2j+2i+3} = \begin{pmatrix} a^{2i+1} a^\pi b (cb)^j z_{2j+2i+3} & a^{2i+1} a^\pi b (cb)^j (d^d)^{2j+2i+3} \\ 0 & 0 \end{pmatrix},$$

$$(CB+D^2)^i C (A^d)^{2i+3} B = \begin{pmatrix} ((bc+a^2)a^\pi)^i b (d^d)^{2i+3} c a^\pi & 0 \\ 0 & 0 \end{pmatrix},$$

$$D^{2i+1} C (BC)^j (A^d)^{2j+2i+4} B = \begin{pmatrix} a^{2i+1} a^\pi b (cb)^j (d^d)^{2j+2i+4} c a^\pi & 0 \\ 0 & 0 \end{pmatrix},$$

By virtue of [9, Lemma 2.1], we have

$$\begin{aligned} M^d &= \Phi_1 A + \Phi_1 B + \Omega A + \Omega B \\ &= \sum_{j=0}^{\infty} (BC)^j (A^d)^{2j+1} + \sum_{j=0}^{\infty} (BC)^j (A^d)^{2j+2} B \\ &\quad + \sum_{i=0}^{\infty} (CB + D^2)^i C (A^d)^{2i+2} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} D^{2i+1} C (BC)^j (A^d)^{2j+2i+3} \\ &\quad + \sum_{i=0}^{\infty} (CB + D^2)^i C (A^d)^{2i+3} B + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} D^{2i+1} C (BC)^j (A^d)^{2j+2i+4} B \end{aligned}$$

By direct computation, $M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\alpha, \beta, \gamma, \delta$ as preceding written. □

COROLLARY 2.4. *Let $a, d \in \mathscr{A}^d$ and $b, c \in \mathscr{A}$. If $d^d c = 0$, $abc = 0$, $bdd^\pi = 0$ and $bc \in \mathscr{A}^{qnil}$, then $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathscr{A})^d$. In this case, $M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where*

$$\begin{aligned} \alpha &= \sum_{i=0}^{\infty} (bc)^i (a^d)^{2i+1}, \\ \beta &= \sum_{i=0}^{\infty} (bc)^i y_{2i+1} + \sum_{i=0}^{\infty} (bc)^i (a^d)^{2i+2} b d^\pi, \\ \gamma &= \sum_{i=0}^{\infty} (cbd^\pi + d^2 d^\pi)^i c (a^d)^{2i+2} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d^{2i+1} c (bc)^j (a^d)^{2j+2i+3}, \\ \delta &= \sum_{i=0}^{\infty} (cbd^\pi + d^2 d^\pi)^i c y_{2i+2} + \sum_{i=0}^{\infty} (cbd^\pi + d^2 d^\pi)^i c (a^d)^{2i+3} b d^\pi \\ &\quad + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d^{2i+1} d^\pi c (bc)^j y_{2j+2i+3} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d^{2i+1} d^\pi c (bc)^j (a^d)^{2j+2i+4} b d^\pi \end{aligned}$$

and

$$\begin{aligned} y_1 &= a^\pi \sum_{i=0}^{\infty} a^i b (d^d)^{i+2} - a^d b d^d, \\ y_{m+1} &= y_1 (d^d)^m + a^d y_m \quad \textit{for any} \quad m \in \mathbb{N}. \end{aligned}$$

Proof. Applying Theorem 2.3 to the matrix $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$, we see that it has g-Drazin inverse. Clearly, we have

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$M^d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & c \\ b & a \end{pmatrix}^d \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By direct computation, we complete the proof. □

We demonstrate Theorem 2.3 by the following numerical example.

EXAMPLE 2.5. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix},$$
$$c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C}).$$

Then M has g-Drazin inverse. In this case,

$$M^d = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

Proof. By the computation, we have $a^d b = 0$, $dcb = 0$, $caa^\pi = 0$ and $cb = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})^{qnil}$.

In view of Theorem 2.3, $M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Since $a^d = 0$, we see that $z_1 = d^\pi \sum\limits_{i=0}^{\infty} d^i c(a^d)^{i+2} - d^d ca^d = 0$; hence, $z_{m+1} = z_1(a^d)^m + d^d z_m = 0$ for any $m \in \mathbb{N}$. As $bc = a^2 = 0$, we have $\alpha = bdc = 0$. Also $\beta = (1+a)bd = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\gamma = (1+cb)dc = 0$. Moreover, $\delta = (1+cb)d = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$. Then

$$M^d = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}. \quad \square$$

3. Special operator matrices

Let E, F be bounded linear operators and I be the identity operator over a Banach space X. In this section we come now to the demonstration of our main result for the g-Drazin inverse of the operator matrix $\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. For future use, we record the following elementary result.

LEMMA 3.1. *Let E, F and EFF^d have g-Drazin inverses. If $F^d EF^\pi = 0$ and $EFF^\pi = 0$, then $FF^d E$, EF^π have g-Drazin inverses and*

$$(FF^d E)^d = FF^d E^d FF^d, \qquad (EF^\pi)^d = F^\pi E^d F^\pi.$$

Proof. By hypothesis, we have $FF^dEFF^d = FF^dE$, $FF^dEF^\pi = 0$ and $F^\pi EF^\pi = EF^\pi$. Let $e = FF^d$. Then we have the Pierce composition of E relatively to the idempotent e:

$$\sigma(E) = \begin{pmatrix} FF^dE & 0 \\ F^\pi EFF^d & EF^\pi \end{pmatrix}_e .$$

By using Cline's formula, FF^dE has g-Drazin inverse. In light of [4, Theorem 2.3], EF^π has g-Drazin inverse. By using [4, Theorem 2.3] again, we have

$$\sigma(E^d) = \begin{pmatrix} (FF^dE)^d & 0 \\ * & (EF^\pi)^d \end{pmatrix}_e .$$

Therefore

$$(FF^dE)^d = FF^dE^dFF^d, (EF^\pi)^d = F^\pi EF^\pi,$$

as asserted. □

THEOREM 3.2. *Let E,F and EFF^d have g-Drazin inverses. If $F^dEF^\pi = 0$ and $EFF^\pi = 0$, then $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ has g-Drazin inverse. In this case, $M^d = \begin{pmatrix} \Lambda & \Sigma \\ \Gamma & \Delta \end{pmatrix}$, where*

$$\begin{aligned}
\Lambda &= \sum_{i=0}^{\infty} (FF^\pi)^i X_{2i+1} + \sum_{i=0}^{\infty} F^i F^\pi (E^dF^\pi)^{2i+1}, \\
\Sigma &= \sum_{i=0}^{\infty} (FF^\pi)^i Y_{2i+1} + \sum_{i=0}^{\infty} F^i F^\pi (E^dF^\pi)^{2i+2}, \\
\Gamma &= \sum_{i=0}^{\infty} F^{i+1} F^\pi X_{2i+2} + \sum_{i=0}^{\infty} F^{i+1} F^\pi (E^dF^\pi)^{2i+2}, \\
\Delta &= \sum_{i=0}^{\infty} F^{i+1} F^\pi Y_{2i+2} + \sum_{i=0}^{\infty} F^{i+1} F^\pi (E^dF^\pi)^{2i+3}
\end{aligned}$$

and

$$\begin{aligned}
Z_1 &= \sum_{i=0}^{\infty} \begin{pmatrix} F^\pi E^\pi F^\pi (EF^\pi)^i EFF^d & F^\pi E^\pi F^\pi (EF^\pi)^i \\ 0 & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & F^d \\ FF^d & -FF^dEF^d \end{pmatrix}^{i+2} - \begin{pmatrix} 0 & F^\pi E^d F^\pi EF^d \\ 0 & 0 \end{pmatrix}, \\
Z_{m+1} &= Z_1 \begin{pmatrix} 0 & F^d \\ FF^d & -FF^dEF^d \end{pmatrix}^m + \begin{pmatrix} F^\pi E^d F^\pi & 0 \\ 0 & 0 \end{pmatrix} Z_m;
\end{aligned}$$

$$X_m = (Z_m)_{11}, \quad Y_m = (Z_m)_{12} \quad \textit{for any} \quad m \in \mathbb{N}.$$

Proof. Let $e = \begin{pmatrix} FF^d & 0 \\ 0 & I \end{pmatrix}$. Then we have the Pierce decomposition of M relatively to the idempotent e: $\sigma(M) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_e$, where

$$a = eMe, \quad b = eM(I-e), \quad c = (I-e)Me, \quad d = (I-e)M(I-e).$$

Since $F^dEF^\pi = 0$, we have $FF^dEFF^d = FF^dE(I-E^\pi) = FF^dE \; -F(F^dEF^\pi)$ $= FF^dE$. Thus we easily check that

$$a = \begin{pmatrix} FF^dE & FF^d \\ F^2F^d & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 \\ FF^\pi & 0 \end{pmatrix},$$
$$c = \begin{pmatrix} F^\pi EFF^d & F^\pi \\ 0 & 0 \end{pmatrix}, \qquad d = \begin{pmatrix} EF^\pi & 0 \\ 0 & 0 \end{pmatrix}.$$

We see that a has group inverse and

$$a^d = a^\# = \begin{pmatrix} 0 & F^d \\ FF^d & -FF^dEF^d \end{pmatrix}.$$

We note that the identity of the the corner ring containing eMe is e, and so

$$\begin{aligned} a^\pi &= e - aa^d \\ &= \begin{pmatrix} FF^d & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} FF^dE & FF^d \\ F^2F^d & 0 \end{pmatrix}\begin{pmatrix} 0 & F^d \\ FF^d & -FF^dEF^d \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & F^\pi \end{pmatrix}. \end{aligned}$$

In light of Lemma 3.1, FF^dE, EF^π have g-Drazin inverses and $(FF^dE)^d = FF^dE^d$, $(EF^\pi)^d = E^dF^\pi$. We compute that

$$ab = 0, \quad dcb = 0, \quad caa^\pi = 0, \quad cb = \begin{pmatrix} FF^\pi & 0 \\ 0 & 0 \end{pmatrix} \text{ is quasinilpotent.}$$

According to Theorem 2.3, M has g-Drazin inverse, and we have $M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where α, β, γ and δ are given in Theorem 2.3.

Clearly, we have

$$aa^\pi = 0, \qquad bca^\pi = \begin{pmatrix} 0 & 0 \\ 0 & FF^\pi \end{pmatrix}.$$

Moreover, we have

$$d^d = \begin{pmatrix} F^\pi E^dF^\pi & 0 \\ 0 & 0 \end{pmatrix}, \qquad d^\pi = \begin{pmatrix} F^\pi E^\pi F^\pi & 0 \\ 0 & 0 \end{pmatrix}.$$

Choose

$$Z_1 = d^\pi \sum_{i=0}^{\infty} d^i c (a^d)^{i+2} - d^d c a^d, \quad Z_{m+1} = Z_1 (a^d)^m + d^d Z_m;$$
$$X_m = (Z_m)_{11}, \quad Y_m = (Z_m)_{12}.$$

Then $Z_m = \begin{pmatrix} X_m & Y_m \\ * & * \end{pmatrix}$ for all $m \in \mathbb{N}$.

Hence,

$$\alpha = \sum_{i=0}^{\infty}\begin{pmatrix} 0 & 0 \\ F^{i+1}F^{\pi} & 0 \end{pmatrix} Z_{2i+2} + \sum_{i=0}^{\infty}\begin{pmatrix} 0 & 0 \\ 0 & F^{i+1}F^{\pi}(E^dF^{\pi})^{2i+3} \end{pmatrix}.$$

Also we have

$$ab(cb)^j(d^d)^{2i+2j+3} = \begin{pmatrix} FF^dE & FF^d \\ F^2F^d & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ F^{j+1}F^{\pi}(E^dF^{\pi})^{2i+2j+3} & 0 \end{pmatrix} = 0,$$

and so

$$\beta = \sum_{i=0}^{\infty}\begin{pmatrix} 0 & 0 \\ F^{i+1}F^{\pi}(E^dF^{\pi})^{2i+2} & 0 \end{pmatrix}.$$

We easily see that

$$ca^{\pi} = \begin{pmatrix} 0 & F^{\pi} \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned}\gamma &= Z_1 + (d^d)^2ca^{\pi} + \sum_{i=1}^{\infty}(cb)^iZ_{2i+1} + \sum_{i=1}^{\infty}(cb)^i(d^d)^{2i+2}ca^{\pi} \\ &= Z_1 + \begin{pmatrix} 0 & F^{\pi}(E^dF^{\pi})^2 \\ 0 & 0 \end{pmatrix} + \sum_{i=1}^{\infty}\begin{pmatrix} F^iF^{\pi} & 0 \\ 0 & 0 \end{pmatrix} Z_{2i+1} \\ &\quad + \sum_{i=1}^{\infty}\begin{pmatrix} 0 & F^iF^{\pi}(E^dF^{\pi})^{2i+2} \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\delta &= \sum_{i=0}^{\infty}(cb)^i(d^d)^{2i+1} \\ &= \sum_{i=0}^{\infty}\begin{pmatrix} F^iF^{\pi}(E^dF^{\pi})^{2i+1} & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

By [10, Lemma 2.1], we have $M^d = \begin{pmatrix} \Lambda & \Sigma \\ \Gamma & \Delta \end{pmatrix}$, where Λ, Σ, Γ and Δ are above given, as desired. □

COROLLARY 3.3. *Let E,F and EFF^d have g-Drazin inverses. If $F^dEF^{\pi} = 0$ and $EFF^{\pi} = 0$, then $M = \begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ has g-Drazin inverse. In this case,*

$$M^d = \begin{pmatrix} \Delta + E\Sigma & \Gamma + E\Lambda - \Delta E - E\Sigma E \\ \Sigma & \Lambda - \Sigma E \end{pmatrix},$$

where Λ,Σ,Γ and Δ are given as in Theorem 3.2.

Proof. Obviously, we have

$$\begin{pmatrix} E & F \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & -E \end{pmatrix}^{-1}\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & -E \end{pmatrix},$$

and so

$$\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}^d = \begin{pmatrix} E & I \\ I & 0 \end{pmatrix} \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}^d \begin{pmatrix} 0 & I \\ I & -E \end{pmatrix}.$$

Applying Theorem 3.2 to the matrix $\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$, we complete the proof. □

Let E, F and G be bounded linear operators over a Banach space X. We now derive

COROLLARY 3.4. *Let E, GF and $EGF(GF)^d$ have g-Drazin inverses. If $(GF)^d E (GF)^\pi = 0$ and $EGF(GF)^\pi = 0$, then $M = \begin{pmatrix} E & G \\ F & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. In view of Theorem 3.2, the operator matrix $\begin{pmatrix} E & I \\ GF & 0 \end{pmatrix}$ has g-Drazin inverse. We easily see that

$$\begin{pmatrix} E & I \\ GF & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} E & I \\ F & 0 \end{pmatrix},$$

it follows by Cline's formula (see [7, Theorem 2.1]) that $\begin{pmatrix} E & I \\ F & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix}$ has g-Drazin inverse. That is, $\begin{pmatrix} E & G \\ F & 0 \end{pmatrix}$ has g-Drazin inverse, as asserted. □

For any complex matrix, the Drazin inverse and g-Drazin inverse coincide with each other. Thus the preceding results are also valid for computing Drazin inverses. The following numerical example illustrates Theorem 3.2.

EXAMPLE 3.5. Let $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$, where

$$E = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C}).$$

Then M has Drazin inverse. In this case,

$$M^D = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. By the computation, we have $F^D E F^\pi = 0$ and $EFF^\pi = 0$. Construct X_m, Y_m as in Theorem 3.2, we easily see that $X_m = Y_m = 0$. Since $E^D = E^2 = E$ and

$F^2 = 0$, we have

$$\Lambda = F^{\pi}E^{D}F^{\pi} + FF^{\pi}(E^{D}F^{\pi})^{3} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Sigma = F^{\pi}(E^{D}F^{\pi})^{2} + FF^{\pi}(E^{D}F^{\pi})^{4} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Gamma = FF^{\pi}(E^{D}F^{\pi})^{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\Delta = FF^{\pi}(E^{D}F^{\pi})^{3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Therefore

$$M^{D} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

as desired. $\square$

Acknowledgement. The authors would like to thank the referee for his/her careful reading of the paper. The detailed remarks and corrections help to improve the paper so much. Huanyin Chen was supported by the Natural Science Foundation of Zhejiang Province, China (No. LY21A010018).

REFERENCES

[1] C. BU, K. ZHANG AND J. ZHAO, *Representation of the Drazin inverse on solution of a class singular differential equations*, Linear & Multilinear Algebra **59** (2011), 863–877.

[2] S. L. CAMPBELL, *The Drazin inverse and systems of second order linear differential equations*, Linear & Multilinear Algebra **14** (1983), 195–198.

[3] N. CASTRO-GONZÁLEZ AND E. DOPAZO, *Representations of the Drazin inverse for a class of block matrices*, Linear Algebra Appl. **400** (2005), 253–269.

[4] N. C. GONZÁLEZ AND J. J. KOLIHA, *New additive results for the g-Drazin inverse*, Proc. Roy. Soc. Edinburg, Sect. A, **134** (2004), 1085–1097.

[5] D. S. CVETKOVIC-ILIC, *Some results on the* $(2,2,0)$ *Drazin inverse problem*, Linear Algebra Appl. **438** (2013), 4726–4741.

[6] D. S. CVETKOVIĆ-ILIĆ, D. S. DJORDJEVIĆ AND Y. WEI, *Additive results for the generalized Drazin inverse in a Banach algebra*, Linear Algebra Appl. **418** (2006), 53–61.

[7] Y. LIAO, J. CHEN AND J. CUI, *Cline's formula for generalized Drazin inverse*, B Malays Math Sci So. **37** (2014), 37–42.

[8] P. PATRICIO AND R. E. HARTWIG, *The* $(2,2,0)$ *Drazin inverse problem*, Linear Algebra Appl. **437** (2012), 2755–2772.

[9] D. ZHANG, *Representations for generalized Drazin inverse of operator matrices over a Banach space*, Turk. J. Math. **40** (2016), 428–437.

[10] D. ZHANG AND D. MOSIĆ, *Explicit formulae for the generalized Drazin inverse of block matrices over a Banach algebra*, Filomat **32** (2018), 5907–5917.

(Received November 28, 2019)

Huanyin Chen
Department of Mathematics
Hangzhou Normal University
Hangzhou, China
e-mail: huanyinchen@aliyun.com

Marjan Sheibani
Women's University of Semnan (Farzanegan)
Semnan, Iran
e-mail: sheibani@fgusem.ac.ir