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COMMUTING AND SEMI–COMMUTING TOEPLITZ OPERATORS ON THE WEIGHTED HARMONIC BERGMAN SPACE

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Abstract. In this article, we show that two Toeplitz operators on the weighted harmonic Bergman space can commute only in the trivial case under certain conditions. The triviality here means a nonzero linear combination of their symbols is constant. Moreover, we give a characterization of semi-commuting Toeplitz operators with harmonic or analytic symbols.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and let dA be the normalized area measure on $\mathbb{D}$. For fixed $\alpha > -1$, let $L^2_\alpha = L^2(\mathbb{D}, dA_\alpha)$ be the Hilbert space of square integrable functions with respect to the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ with inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\alpha(z), \quad f, g \in L^2_\alpha.$$

The Bergman space A^2_α is the closed subspace of L^2_α consisting of all analytic functions. For $z \in \mathbb{D}$, the reproducing kernel $K_z^{(\alpha)}$ of A^2_α is given explicitly by

$$K_z^{(\alpha)}(w) = \frac{1}{(1-\bar{z}w)^{2+\alpha}},$$

which has the following reproducing property:

$$f(z) = \langle f, K_z^{(\alpha)}\rangle_\alpha \tag{1}$$

for every $f \in A^2_\alpha$ and $z \in \mathbb{D}$. The harmonic Bergman space b^2_α is another closed subspace of L^2_α consisting of harmonic functions on $\mathbb{D}$. It is easy to verify that

$$b^2_\alpha = A^2_\alpha + \overline{A^2_\alpha},$$

and b^2_α is also a reproducing Hilbert space with the reproducing kernel

$$R_w^{(\alpha)}(z) = K_w^{(\alpha)}(z) + \overline{K_w^{(\alpha)}(z)} - 1, \quad z, w \in \mathbb{D}. \tag{2}$$

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Let P_α and Q_α be the orthogonal projections from L^2_α onto A^2_α and b^2_α respectively. Then

$$P_\alpha(\varphi)(z) = \langle \varphi, K_z^{(\alpha)} \rangle_\alpha,$$

$$Q_\alpha(\varphi)(z) = \langle \varphi, R_z^{(\alpha)} \rangle_\alpha,$$

for $\varphi \in L^2_\alpha$. By (2), we have

$$Q_\alpha(\varphi)(z) = P_\alpha(\varphi)(z) + \overline{P_\alpha(\bar{\varphi})(z)} - P_\alpha(\varphi)(0).$$

See [1] for more information and related facts.

For $u \in L^2_\alpha$, we define the Toeplitz operator T_u with symbol u on b^2_α by

$$T_u(f) = Q_\alpha(uf), \quad f \in b^2_\alpha.$$

The operator T_u is densely defined with its domain containing all bounded harmonic functions. Clearly, if $u \in L^\infty(\mathbb{D})$, then T_u is a bounded operator with norm $\|T_u\| \leqslant \|u\|_\infty$.

The commuting problem of Toeplitz operators is an open question in the theory of Toeplitz operators. It can be stated as follows:

For $f, g \in L^\infty(\mathbb{D})$, when does $T_f T_g = T_g T_f$ hold?

Unlike the case of the Hardy space, the commuting problem on the Bergman space is still far form being totally answered. Nevertheless, many efforts have been made to resolve it.

On the Bergman space of the unit disk, the first complete result was obtained by Axler and Čučković in [2], where they characterized commuting Toeplitz operators with bounded harmonic symbols. Stroethoff [14] extended their result to essentially commuting Toeplitz operators. Axler et al. [3] showed that if two Toeplitz operators commute and the symbol of either of them is nonconstant analytic, then the other one is also analytic. Mellin transform turns out to be a powerful tool to study commuting Toeplitz operators with quasihomogeneous symbols (see [8], [11]).

In the setting of several complex variables, Zheng [15] completely characterized commuting Toeplitz operators with pluriharmonic symbols on the Bergman space of the unit ball in $\mathbb{C}^n$. Choe and Lee [6] studied the corresponding essentially commuting problem. Lee [9] further generalized the result in [15] to Toeplitz type operators acting on weighted Bergman spaces. Lu [12] considered the problem on the Bergman space of the bidisk. Choe et al. [4] obtained characterizations of commuting and essentially commuting Toeplitz operators on the Bergman space of the polydisk.

The harmonic Bergman space is more complicated than the analytic Bergman space, since the product of two harmonic functions is no longer harmonic, and this leads to the lost of effectiveness of many methods which work for the operators on the analytic Bergman space. On the harmonic Bergman space of the unit disk, Ohno [13] first characterized commuting Toeplitz operators with analytic symbols, either of which is a monomial. Then Choe and Lee [5] completely described commuting Toeplitz operators with two general analytic symbols. Meanwhile, they also studied harmonic symbols, either of which is a polynomial. Some of these results were subsequently extended to the unit ball case in [10] and the polydisk case in [7]. However, the corresponding problem for weighted harmonic Bergman space has not been studied so far and we shall investigate it in what follows.

Our article is organized as follows. In Section 2, we characterize commuting Toeplitz operators with analytic symbols (see Theorem 2). We also consider harmonic symbols one of which is the sum of a polynomial and a conjugate-analytic function (see Theorem 4 and Corollary 1). In Section 3, we characterize normal Toeplitz operators with analytic or harmonic symbols (see Theorem 5 and Theorem 6). In Section 4, we characterize semi-commuting Toeplitz operators, that is, when the semi-commutator $T_{uv} - T_u T_v$ is zero (see Theorem 7 and Corollary 2). Our results show that in many respects, Toeplitz operators on the harmonic Bergman spaces behave very differently from the ones defined on the Bergman spaces.

2. Commuting Toeplitz operators

LEMMA 1. *Let n,m be nonnegative integers, then*

$$P_\alpha(w^n\bar{w}^m)(z) = \begin{cases} 0 & n < m, \\ \frac{n!\Gamma(n-m+2+\alpha)}{(n-m)!\Gamma(n+2+\alpha)} z^{n-m} & n \geqslant m. \end{cases}$$

Proof. This is easily verified by a straightforward calculation. $\square$

LEMMA 2. *Let $f,g \in A^2_\alpha$. Then $P_\alpha(\bar{f}P_\alpha(\bar{w}g)) = P_\alpha(\bar{f}\bar{w}g)$. Moreover, if $f(0) = 0$, then $P_\alpha(w\bar{f})(z) = \overline{P_\alpha(\bar{w}f)(0)}$ for all $z \in \mathbb{D}$.*

Proof. By the reproducing formula (1), for $z \in \mathbb{D}$,

$$\begin{aligned} P_\alpha(\bar{f}P_\alpha(\bar{w}g))(z) &= \langle \bar{f}P_\alpha(\bar{w}g), K_z^{(\alpha)}\rangle_\alpha = \langle P_\alpha(\bar{w}g), fK_z^{(\alpha)}\rangle_\alpha \\ &= \langle \bar{w}g, fK_z^{(\alpha)}\rangle_\alpha = P_\alpha(\bar{f}\bar{w}g)(z), \end{aligned}$$

so we have $P_\alpha(\bar{f}P_\alpha(\bar{w}g)) = P_\alpha(\bar{f}\bar{w}g)$. For the second assertion, we let $f(z) = \sum_{n=1}^\infty a_n z^n$ be its Taylor series. Then the series converges to f in A^2_α (see [16]). By Lemma 1,

$$P_\alpha(w\bar{f})(z) = P_\alpha(\bar{a}_1 w\bar{w})(z) = \frac{\bar{a}_1}{2+\alpha}.$$

Similarly,

$$P_\alpha(\bar{w}f)(z) = \sum_{n\geqslant 1} P_\alpha(a_n\bar{w}w^n)(z) = \sum_{n\geqslant 1} \frac{a_n n}{n+1+\alpha} z^{n-1}. \tag{3}$$

Therefore, $P_\alpha(w\bar{f})(z) = \overline{P_\alpha(\bar{w}f)(0)} = \bar{a}_1/(2+\alpha)$. $\square$

LEMMA 3. *$\{\frac{k}{k+1+\alpha}\}_{k\geqslant 1}$ is a strictly monotone increasing sequence.*

Proof. Note that the function $x/(x+1+\alpha)$ is a strictly monotone increasing function on $[0,+\infty)$. $\square$

THEOREM 1. *Suppose $u = f + \bar{g}, v = h + \bar{k}$ are bounded harmonic functions in $\mathbb{D}$ such that T_u commutes with T_v on b^2_α. If f and h are not constant, then f, h and 1 are linearly dependent.*

Proof. For simplicity and without loss of generality, we assume that $u(0) = v(0) = 0$. Otherwise, we can replace u and v with $u - u(0)$ and $v - v(0)$ respectively. It is clear that the functions f, g, h and k are in A^2_α by the boundedness of the projection P_α on L^2_α. For convenience, we may assume that 0 is a zero point of these four functions. In the following, we will prove the existence of some constant λ satisfying $h = \lambda f$.

Firstly,

$$\begin{aligned} T_h(\bar{w}) &= Q_\alpha(\bar{w}h) \\ &= P_\alpha(\bar{w}h) + \overline{P_\alpha(w\bar{h})} - P_\alpha(\bar{w}h)(0) \\ &= P_\alpha(\bar{w}h) + P_\alpha(\bar{w}h)(0) - P_\alpha(\bar{w}h)(0) \\ &= P_\alpha(\bar{w}h). \end{aligned} \tag{4}$$

The third equality above follows from Lemma 2. Consequently,

$$T_f T_h(\bar{w}) = f P_\alpha(\bar{w}h).$$

Secondly,

$$\begin{aligned} T_f T_{\bar{k}}(\bar{w}) &= Q_\alpha[f Q_\alpha(\bar{w}\bar{k})] = Q_\alpha(f\bar{w}\bar{k}) \\ &= P_\alpha(f\bar{w}\bar{k}) + \overline{P_\alpha(\bar{f}wk)} - P_\alpha(f\bar{w}\bar{k})(0). \end{aligned}$$

Thirdly, combining (4) with Lemma 2 gives

$$\begin{aligned} T_{\bar{g}} T_h(\bar{w}) &= Q_\alpha[\bar{g} P_\alpha(\bar{w}h)] \\ &= P_\alpha[\bar{g} P_\alpha(\bar{w}h)] + \overline{P_\alpha[g\overline{P_\alpha(\bar{w}h)}]} - P_\alpha[\bar{g} P_\alpha(\bar{w}h)](0) \\ &= P_\alpha(\bar{g}\bar{w}h) + \overline{P_\alpha[g\overline{P_\alpha(\bar{w}h)}]} - P_\alpha(\bar{g}\bar{w}h)(0). \end{aligned}$$

Finally,

$$T_{\bar{g}} T_{\bar{k}}(\bar{w}) = Q_\alpha(\bar{g} Q_\alpha(\bar{k}\bar{w})) = Q_\alpha(\bar{g}\bar{k}\bar{w}) = \bar{g}\bar{k}\bar{w}.$$

By the computations above, we have

$$\begin{aligned} T_{f+\bar{g}} T_{h+\bar{k}}(\bar{w}) &= T_f T_h(\bar{w}) + T_f T_{\bar{k}}(\bar{w}) + T_{\bar{g}} T_h(\bar{w}) + T_{\bar{g}} T_{\bar{k}}(\bar{w}) \\ &= f P_\alpha(\bar{w}h) + P_\alpha(f\bar{w}\bar{k}) + \overline{P_\alpha(\bar{f}wk)} - P_\alpha(f\bar{w}\bar{k})(0) \\ &\quad + P_\alpha(\bar{g}\bar{w}h) + \overline{P_\alpha[g\overline{P_\alpha(\bar{w}h)}]} - P_\alpha(\bar{g}\bar{w}h)(0) + \bar{g}\bar{k}\bar{w}. \end{aligned}$$

Similarly,

$$\begin{aligned} T_{h+\bar{k}} T_{f+\bar{g}}(\bar{w}) &= h P_\alpha(\bar{w}f) + P_\alpha(h\bar{w}\bar{g}) + \overline{P_\alpha(\bar{h}wg)} - P_\alpha(h\bar{w}\bar{g})(0) \\ &\quad + P_\alpha(\bar{k}\bar{w}f) + \overline{P_\alpha[k\overline{P_\alpha(\bar{w}f)}]} - P_\alpha(\bar{k}\bar{w}f)(0) + \bar{k}\bar{g}\bar{w}. \end{aligned}$$

Since $T_{f+\bar{g}}T_{h+\bar{k}} = T_{h+\bar{k}}T_{f+\bar{g}}$ on b^2_α, particularly we have

$$fP_\alpha(\bar{w}h) + \overline{P_\alpha(\bar{f}wk)} + \overline{P_\alpha[g\overline{P_\alpha(\bar{w}h)}]} = hP_\alpha(\bar{w}f) + \overline{P_\alpha(\bar{h}wg)} + \overline{P_\alpha[k\overline{P_\alpha(\bar{w}f)}]}. \tag{5}$$

Taking the analytic part on both sides and noting that $f(0) = h(0) = 0$, we get

$$fP_\alpha(\bar{w}h) = hP_\alpha(\bar{w}f).$$

Let $h(z) = \sum_{n=1}^{\infty} a_n z^n$, $f(z) = \sum_{n=1}^{\infty} b_n z^n$ be the Taylor series expansion of h and f, respectively. By (3) and the preceding equality, we have

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} \frac{ka_k b_{n-k}}{k+1+\alpha} z^{n-1} = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} \frac{kb_k a_{n-k}}{k+1+\alpha} z^{n-1},$$

which implies

$$\sum_{k=1}^{n-1} \frac{ka_k b_{n-k}}{k+1+\alpha} = \sum_{k=1}^{n-1} \frac{kb_k a_{n-k}}{k+1+\alpha} \tag{6}$$

for every $n \geqslant 2$. Now we consider the following three cases for the Taylor coefficients a_1 and b_1.

Case 1. If $b_1 \neq 0$, put $\frac{a_1}{b_1} = \lambda$. Let $n = 3$ in (6), we have

$$\frac{a_1 b_2}{2+\alpha} + \frac{2a_2 b_1}{3+\alpha} = \frac{b_1 a_2}{2+\alpha} + \frac{2b_2 a_1}{3+\alpha}.$$

Then Lemma 3 implies $a_1 b_2 = a_2 b_1$, or $a_2 = \lambda b_2$. Assume $a_i = \lambda b_i$ for $1 \leqslant i \leqslant n_0$ and let $n = n_0 + 2$ in (6), we obtain

$$\frac{a_1 b_{n_0+1}}{2+\alpha} + \frac{(n_0+1)a_{n_0+1}b_1}{n_0+2+\alpha} = \frac{b_1 a_{n_0+1}}{2+\alpha} + \frac{(n_0+1)b_{n_0+1}a_1}{n_0+2+\alpha}.$$

Again by Lemma 3, we have $a_1 b_{n_0+1} = b_1 a_{n_0+1}$, or $a_{n_0+1} = \lambda b_{n_0+1}$. By mathematical induction, $a_i = \lambda b_i$ for all $i \geqslant 1$, which implies $h = \lambda f$.

Case 2. If $a_1 \neq 0$, put $\frac{b_1}{a_1} = \lambda$. By exactly the same way, we obtain $f = \lambda h$.

Case 3. If $a_1 = b_1 = 0$, then the equation (6) turns into

$$\sum_{k=2}^{n-2} \frac{ka_k b_{n-k}}{k+1+\alpha} = \sum_{k=2}^{n-2} \frac{kb_k a_{n-k}}{k+1+\alpha}$$

for every $n \geqslant 4$. By the same argument as above with replacing a_1, b_1 by a_2, b_2 respectively, we find also that f and h are linearly dependent unless $a_2 = b_2 = 0$.

Repeat this process until there exists some n_0 such that $a_{n_0} \neq$ or $b_{n_0} \neq 0$. Since we assume f and h are not constant, the integer n_0 indeed exists. Now replace a_1, b_1 by a_{n_0}, b_{n_0} respectively in Case 1 and Case 2, we conclude that f and h must be linearly dependent. The proof is completed. □

If we further assume that the symbols are analytic, then we can give a characterization of commuting Toeplitz operators as follows.

THEOREM 2. *Suppose f, g are nonconstant functions in A^2_α, then T_f and T_g commute on b^2_α if and only if f, g and 1 are linearly dependent.*

Proof. The necessity follows from Theorem 1 and the sufficiency is obvious. □

REMARK 1. On the analytic Bergman space, Toeplitz operators with analytic symbols always commute, as proved in [2] and [9]. This is very different from the case on the harmonic Bergman space.

Next we focus on Toeplitz operators with harmonic symbols. Intuitively, Theorem 2 suggests that Toeplitz operators with harmonic symbols should only commute in the trivial case as well. In this direction, we obtain the following results.

THEOREM 3. *Let $f, g \in A^2_\alpha$ and suppose one of them is a polynomial. Then $T_f T_{\bar{g}} = T_{\bar{g}} T_f$ on b^2_α if and only if either f or g is constant.*

Proof. We only need to prove the necessity. For simplicity and without loss of generality, we assume $f(0) = g(0) = 0$. Otherwise, we can replace f and g with $f - f(0)$ and $g - g(0)$ respectively. We also assume f is a polynomial with positive degree, or else we could take the adjoints. In the following, we prove g to be constant.

If g is nonconstant, then 0 is a zero point (with finite multiplicities) of g and hence $g = w^k h$ for some positive integer k and $h \in A^2_\alpha$ with $h(0) \neq 0$. Let $f(z) = \sum_{m=1}^n b_m z^m$, $g(z) = \sum_{l=k}^\infty a_l z^l$ be their Taylor series, where b_n and a_k are nonzero. Now we consider the following two cases by comparison of k and n.

Case 1. $k \leqslant n$. It follows from (5) that

$$P_\alpha[f\overline{P_\alpha(g\bar{w})}] = P_\alpha(f\bar{g}w).$$

Taking inner product with w^{n-k+1} on both sides of the preceding identity gives

$$\int_{\mathbb{D}} f(w)\overline{P_\alpha(g\bar{w})}\bar{w}^{n-k+1} dA_\alpha(w) = \int_{\mathbb{D}} f(w)w\overline{g(w)}\bar{w}^{n-k+1} dA_\alpha(w).$$

By (3) we have

$$b_n \bar{a}_k \frac{k}{k+1+\alpha} = b_n \bar{a}_k \frac{n+1}{n+2+\alpha},$$

which implies that $b_n \bar{a}_k = 0$ by Lemma 3, a contradiction.

Case 2. $k > n$. Since $f(z) = \sum_{m=1}^n b_m z^m$, it follows from Lemma 1 that

$$P_\alpha(f\bar{g}\bar{w}^{n+1}) = P_\alpha(f\bar{w}^{n+1}) = 0.$$

Hence we have

$$\begin{aligned} T_f T_{\bar{g}}(\bar{w}^{n+1}) &= T_f(\bar{g}\bar{w}^{n+1}) \\ &= P_\alpha(f\bar{g}\bar{w}^{n+1}) + \overline{P_\alpha(\bar{f}gw^{n+1})} - P_\alpha(f\bar{g}\bar{w}^{n+1})(0) \\ &= \overline{P_\alpha(\bar{f}gw^{n+1})}, \end{aligned}$$

and

$$\begin{aligned} T_{\bar g}T_f(\bar w^{n+1}) &= T_{\bar g}Q_\alpha(f\bar w^{n+1}) \\ &= T_{\bar g}[P_\alpha(f\bar w^{n+1}) + \overline{P_\alpha(\bar f w^{n+1})} - P_\alpha(f\bar w^{n+1})(0)] \\ &= T_{\bar g}[\overline{P_\alpha(\bar f w^{n+1})}] = \overline{gP_\alpha(\bar f w^{n+1})}. \end{aligned}$$

Then the equality $T_fT_{\bar g}(\bar w^{n+1}) = T_{\bar g}T_f(\bar w^{n+1})$ implies

$$P_\alpha(\bar f g w^{n+1}) = gP_\alpha(\bar f w^{n+1}).$$

Consequently,

$$\langle P_\alpha(\bar f g w^{n+1}), w^{k+1}\rangle_\alpha = \langle gP_\alpha(\bar f w^{n+1}), w^{k+1}\rangle_\alpha.$$

For the left hand side,

$$\begin{aligned} \langle P_\alpha(\bar f g w^{n+1}), w^{k+1}\rangle_\alpha &= \langle \bar f g w^{n+1}, w^{k+1}\rangle_\alpha = \langle h w^{n+k+1}, f w^{k+1}\rangle_\alpha \\ &= h(0)\bar b_n \int_{\mathbb D} |w|^{2(n+k+1)} dA_\alpha(w) \\ &= h(0)\bar b_n \frac{(n+k+1)!\Gamma(2+\alpha)}{\Gamma(n+k+3+\alpha)}. \end{aligned}$$

For the right hand side, since

$$\begin{aligned} P_\alpha(\bar g w^{k+1}) &= P_\alpha(\bar h \bar w^k w^{k+1}) \\ &= \overline{h(0)}P_\alpha(\bar w^k w^{k+1}) + \overline{h'(0)}P_\alpha(|w|^{2(k+1)}) \\ &= \overline{h(0)}\frac{(k+1)!\Gamma(3+\alpha)}{\Gamma(k+3+\alpha)}w + \overline{h'(0)}\frac{(k+1)!\Gamma(2+\alpha)}{\Gamma(k+3+\alpha)}, \end{aligned}$$

we have

$$\begin{aligned} \langle gP_\alpha(\bar f w^{n+1}), w^{k+1}\rangle_\alpha &= \langle P_\alpha(\bar f w^{n+1}), \bar g w^{k+1}\rangle_\alpha = \langle \bar f w^{n+1}, P_\alpha(\bar g w^{k+1})\rangle_\alpha \\ &= h(0)\bar b_n \frac{(k+1)!\Gamma(3+\alpha)}{\Gamma(k+3+\alpha)}\frac{(n+1)!\Gamma(2+\alpha)}{\Gamma(n+3+\alpha)}. \end{aligned}$$

Finally, we obtain

$$\frac{(n+k+1)!}{\Gamma(n+k+3+\alpha)} = \frac{\Gamma(3+\alpha)(k+1)!(n+1)!}{\Gamma(k+3+\alpha)\Gamma(n+3+\alpha)},$$

which implies

$$\frac{n+k+1}{n+k+2+\alpha}\cdots\frac{k+2}{k+3+\alpha} = \frac{n+1}{n+2+\alpha}\cdots\frac{2}{3+\alpha}.$$

Note that there are n factors in the products on both sides of the preceding equality, by Lemma 3, this is also a contradiction. The proof is completed. □

THEOREM 4. *Suppose $u = p + \bar g, v$ are bounded harmonic functions in $\mathbb D$, where p is a polynomial of z with positive degree. Then $T_uT_v = T_vT_u$ on b^2_α if and only if u, v and 1 are linearly dependent.*

Proof. We only need to consider v as nonconstant. Let $v = h + \overline{k}$, where $h, k \in A^2_\alpha$, at least one of which is nonconstant. We will discuss the following six cases.

Case 1. If g, k are constant, then by Theorem 2, p, h and 1 are linearly dependent, so that u, v and 1 are linearly dependent.

Case 2. If g, h are constant, then by Theorem 3, k is constant, which yields v is constant, a contradiction.

Case 3. If g is constant, h, k are nonconstant, then by Theorem 1, there exists some nonzero constant λ such that $h = \lambda p$. The assumption $T_u T_v = T_v T_u$ implies $T_p T_{\overline{k}} = T_{\overline{k}} T_p$. Then it follows from Theorem 3 that k is constant, a contradiction.

Case 4. If g, h, k are nonconstant, then by Theorem 1, there exists some constants λ, μ such that $h = \lambda p$ and $\overline{k} = \mu \overline{g}$. The assumption $T_u T_v = T_v T_u$ then implies that $(\lambda - \mu)(T_p T_{\overline{g}} - T_{\overline{g}} T_p) = 0$. Again by Theorem 3, we have $\lambda = \mu$ and so $v = \lambda u$.

Case 5. If g is nonconstant, h is constant, then by Theorem 1, there exists some nonzero constant λ such that $\overline{k} = \lambda \overline{g}$. A discussion similar to Case 3 shows that g is constant, a contradiction.

Case 6. If g is nonconstant, k is constant, then by Theorem 1, there exists some nonzero constant λ such that $h = \lambda p$. The discussion in Case 3 again shows that g is constant, a contradiction. $\square$

COROLLARY 1. *Let $u = f + \overline{g}, v$ are bounded harmonic functions, either of f and g is a nonconstant polynomial. Then $T_u T_v = T_v T_u$ on b^2_α if and only if u, v and 1 are linearly dependent.*

3. Normal Toeplitz operators with harmonic symbols

Normal Toeplitz operators with harmonic symbols on A^2_α were completely characterized in [2] ($\alpha = 0$) and [9] ($\alpha > -1$). In this section, we investigate the same problem on b^2_α. Unlike the commuting problem, the result for normal Toeplitz operators on b^2_α coincides with that on A^2_α. Before doing this, we first consider Toeplitz operators with analytic symbols.

THEOREM 5. *Let $f \in A^2_\alpha$. Then T_f is a normal operator if and only if f is constant.*

Proof. We only need to prove the necessity. Without loss of generality, we assume $f(0) = 0$. Otherwise, we can replace f with $f - f(0)$. If T_f is normal, then T_f commutes with $T_{\overline{f}}$. It follows from (5) that $P_\alpha[f\overline{P_\alpha(f\overline{w})}] = P_\alpha(|f|^2 w)$. Particularly we have

$$\langle f\overline{P_\alpha(f\overline{w})}, w\rangle_\alpha = \langle |f|^2 w, w\rangle_\alpha. \tag{7}$$

Let $f(z)=\sum_{n=1}^{\infty}a_nz^n$ be its Taylor series. Then by (3), the left hand side of (7) equals

$$\int_{\mathbb{D}} f(w)\overline{P_\alpha(f\bar{w})}\bar{w}dA_\alpha(w) = \int_{\mathbb{D}}\sum_{n=1}^{\infty}a_nw^n\sum_{m=1}^{\infty}\bar{a}_m\frac{m}{m+1+\alpha}\bar{w}^{m-1}\bar{w}dA_\alpha(w)$$
$$=\sum_{n=1}^{\infty}|a_n|^2\frac{n}{n+1+\alpha}\frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}.$$

On the other hand, the right hand side of (7) equals

$$\int_{\mathbb{D}}|f(w)|^2|w|^2dA_\alpha(w) = \int_{\mathbb{D}}\sum_{n=1}^{\infty}a_nw^n\sum_{m=1}^{\infty}\bar{a}_m\bar{w}^m|w|^2dA_\alpha(w)$$
$$=\sum_{n=1}^{\infty}|a_n|^2\frac{(n+1)!\Gamma(2+\alpha)}{\Gamma(n+3+\alpha)}.$$

By the equality (7), we have

$$\sum_{n=1}^{\infty}|a_n|^2\frac{n}{n+1+\alpha}\frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} = \sum_{n=1}^{\infty}|a_n|^2\frac{(n+1)!\Gamma(2+\alpha)}{\Gamma(n+3+\alpha)}$$
$$=\sum_{n=1}^{\infty}|a_n|^2\frac{n+1}{n+2+\alpha}\frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}.$$

It follows from Lemma 3 that $a_n=0$ for all $n\geqslant 1$, which implies $f=0$. This completes the proof. $\square$

As a consequence of Theorem 5, we obtain the following characterization of normal Toeplitz operators with harmonic symbols.

THEOREM 6. *Suppose u is a bounded harmonic function in $\mathbb{D}$. Then T_u is a normal operator on b_α^2 if and only if the range of u lies in a straight line in $\mathbb{C}$.*

Proof. Assume $u(0)=0$ and $u=f+\bar{g}$, then $f,g\in A_\alpha^2$ and $f(0)=g(0)=0$. If f or g is constant, then the proof is completed by Theorem 5. So we just need to consider the case f,g are nonconstant. If T_u is normal, then T_u commutes with $T_{\bar{u}}$. By Theorem 1, there exists some constant λ such that $g=\lambda f$. Then the equation $T_{f+\bar{g}}T_{\bar{f}+g}=T_{\bar{f}+g}T_{f+\bar{g}}$ yields

$$(1-|\lambda|^2)(T_fT_{\bar{f}}-T_{\bar{f}}T_f)=0.$$

Since f is not constant, Theorem 5 implies that T_f is not normal. Therefore, we must have $|\lambda|=1$, and it follows that

$$u=f+\bar{g}=f+\overline{\lambda f}=\bar{\lambda}(\lambda f+\bar{f})=\bar{\lambda}\bar{u}.$$

This implies that $\sqrt{\lambda}u$ is real-valued and hence $u(\mathbb{D})$ lies in a straight line.

Conversely, if $u(\mathbb{D})$ lies in a straight line, then there must be a unimodular constant μ such that μu is a real-valued harmonic function, so $T_{\mu u}$ is self-adjoint on b_α^2. Therefore, $T_u=\frac{T_{\mu u}}{\mu}$ is normal on b_α^2. $\square$

4. Semi-commuting Toeplitz operators

In this section, we characterize when the semi-commutator of two Toeplitz operators on b^2_α is zero. To give a necessary condition for harmonic symbols, we prove the following lemma.

LEMMA 4. *Let $f, g \in A^2_\alpha$. Then*

$$P_\alpha(fg\overline{w}) = fP_\alpha(g\overline{w})$$

if and only if f is constant or $g = 0$.

Proof. Assume f is not constant, we prove g must be zero. Let $f(z) = \sum_{n=0}^\infty a_n z^n$, $g(z) = \sum_{n=0}^\infty b_n z^n$ be the Taylor series of f and g. Then we have

$$\begin{aligned} P_\alpha(fg\overline{w})(z) &= \int_{\mathbb{D}} f(w)g(w)\overline{w}\overline{K_z^{(\alpha)}(w)}dA_\alpha(w) \\ &= \int_{\mathbb{D}} \sum_{n=0}^\infty a_n w^n \cdot \sum_{m=0}^\infty b_m w^m \cdot \overline{w} \cdot \sum_{k=0}^\infty \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} z^k \overline{w}^k dA_\alpha(w) \\ &= \sum_{n+m\geqslant 1} a_n b_m \frac{n+m}{n+m+1+\alpha} z^{n+m-1}. \end{aligned}$$

By (3) we also have

$$P_\alpha(g\overline{w})(z) = \sum_{m\geqslant 1} b_m \frac{m}{m+1+\alpha} z^{m-1}.$$

Hence

$$f(z)P_\alpha(g\overline{w})(z) = \sum_{n=0}\sum_{m\geqslant 1} a_n b_m \frac{m}{m+1+\alpha} z^{n+m-1}.$$

By the condition $P_\alpha(fg\overline{w}) = fP_\alpha(g\overline{w})$, we obtain

$$\sum_{n+m\geqslant 1} a_n b_m \frac{n+m}{n+m+1+\alpha} z^{n+m-1} = \sum_{n=0}\sum_{m\geqslant 1} a_n b_m \frac{m}{m+1+\alpha} z^{n+m-1}. \tag{8}$$

Compare the constant terms of the series on both sides of (8) , we get

$$\frac{a_0 b_1 + a_1 b_0}{2+\alpha} = \frac{a_0 b_1}{2+\alpha}$$

which implies $a_1 b_0 = 0$, so that $a_1 = 0$ or $b_0 = 0$. We will discuss two cases as follows:

Case 1. If $a_1 \neq 0$, then $b_0 = 0$. Comparing the coefficients of z in (8) gives

$$a_0 b_2 \frac{2}{3+\alpha} + a_1 b_1 \frac{2}{3+\alpha} = a_0 b_2 \frac{2}{3+\alpha} + a_1 b_1 \frac{1}{2+\alpha},$$

so that $b_1 = 0$ by Lemma 3. Comparing the coefficients of z^2 gives

$$a_0 b_3 \frac{3}{4+\alpha} + a_1 b_2 \frac{3}{4+\alpha} = a_0 b_3 \frac{3}{4+\alpha} + a_1 b_2 \frac{2}{3+\alpha},$$

and so $b_2 = 0$. Consequently, $b_m = 0$ for all $m \geqslant 0$ by induction. Therefore $g = 0$.

Case 2. If $a_1 = 0$. Comparing the coefficients of z implies $a_2 b_0 = 0$. If $a_2 \neq 0$, then $b_0 = 0$. Compare the coefficients of $z^k, k \geqslant 2$ as in Case 1, we will also get $g = 0$. If $a_2 = 0$, comparing the coefficients of z^2 implies $a_3 b_0 = 0$. Since we assume f is not constant, let a_l be the first nonzero coefficient of the Taylor series of f. Repeat the preceding process l times, we obtain $g = 0$ or $a_l b_0 = 0$. The latter case implies $b_0 = 0$. A discussion similar to Case 1 then shows that $g = 0$. The proof is completed. $\square$

THEOREM 7. *Let $u = f + \bar{g}$, $v = h + \bar{k}$ be bounded harmonic functions such that $T_{uv} = T_u T_v$ on b^2_α, then at least one of f and h is constant, and at least one of g and k is constant.*

Proof. Without loss of generality we may assume $f(0) = g(0) = h(0) = k(0) = 0$, or else we can replace f,g,h,k with $f - f(0)$, $g - g(0)$, $h - h(0)$, $k - k(0)$ respectively. By Lemma 2,

$$\begin{aligned} T_{uv}(\bar{w}) &= Q_\alpha[(fh + f\bar{k} + \bar{g}h + \bar{g}\bar{k})\bar{w}] \\ &= P_\alpha(fh\bar{w}) + P_\alpha(f\bar{k}\bar{w}) + \overline{P_\alpha(\bar{f}kw)} - P_\alpha(f\bar{k}\bar{w})(0) \\ &\quad + P_\alpha(\bar{g}h\bar{w}) + \overline{P_\alpha(g\bar{h}w)} - P_\alpha(\bar{g}h\bar{w})(0) + \bar{g}\bar{k}\bar{w}, \end{aligned}$$

and

$$\begin{aligned} T_u T_v(\bar{w}) &= fP_\alpha(h\bar{w}) + P_\alpha(f\bar{k}\bar{w}) + \overline{P_\alpha(\bar{f}kw)} - P_\alpha(f\bar{k}\bar{w})(0) \\ &\quad + P_\alpha(\bar{g}h\bar{w}) + \overline{P_\alpha(g\overline{P_\alpha(h\bar{w})})} - P_\alpha(\bar{g}h\bar{w})(0) + \bar{g}\bar{k}\bar{w}. \end{aligned}$$

By the condition $T_{uv} = T_u T_v$, we have

$$P_\alpha(fh\bar{w}) + \overline{P_\alpha(g\bar{h}w)} = fP_\alpha(h\bar{w}) + \overline{P_\alpha(g\overline{P_\alpha(h\bar{w})})}. \tag{9}$$

Note that

$$P_\alpha(fh\bar{w})(0) = f(0)P_\alpha(h\bar{w})(0) = 0$$

by our assumption, so taking the analytic part on both sides of (9) gives

$$P_\alpha(fh\bar{w}) = fP_\alpha(h\bar{w}).$$

Now it follows from Lemma 4 that either f or h is 0.

For the second assertion, just take the adjoints of both sides of the equality $T_{uv} = T_u T_v$. $\square$

As a corollary of the preceding theorem, we characterize semi-commuting analytic Toeplitz operators on b^2_α.

COROLLARY 2. *Let $f,g \in A^2_\alpha$. Then $T_{fg} = T_f T_g$ if and only if either f or g is constant.*

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