Operators and Matrices
Volume 15, Number 1 (2021), 175–193 doi:10.7153/oam-2021-15-13

LVOV–KAPLANSKY CONJECTURE ON UT_m^+ WITH THE TRANSPOSE INVOLUTION

RONALD ISMAEL QUISPE URURE* AND WILLIAN FRANCA

(*Communicated by I. Klep*)

Abstract. Let UT_m be the algebra of all $m\times m$ upper matrices with entries in a field F. Let us consider UT_m equipped with the transpose involution $*$. Under a mild technical assumption on F, we will show that the image of any multilinear Jordan polynomial in three variables evaluated on $UT_m^+ = \{U \in UT_m \mid U^* = U\}$ is a vector space. In particular, we will determine a basis for such image. As an application, we will describe the set of values of some multilinear Jordan polynomials in four variables.

1. Introduction

The following question is known as the Lvov-Kaplansky conjecture:

Let $f(x_1,\dots,x_n)$ be a multilinear polynomial over a field F. Is the set of values of f on the matrix algebra $M_m(F)$ a vector space?

A major breakthrough in this direction was made by Kanel-Belov, Malev and Rowen [8, 12]. They have provided a positive answer for this question when $n=2$. Later, they also have obtained significant results for 3×3 matrices [9], but the complete problem for matrices of order $\geqslant 3$ is still open.

This conjecture has motivated many different studies related to other algebras and other types of polynomials. The reader is referred to [13] for more information about recent and important results on this subject. In the present work, we are interested in the Lvov-Kaplansky conjecture for Jordan algebras, and we refer to [7] for basic properties about the Jordan theory.

Let x,y,z be three non-associative and commutative variables. The polynomial $(xy)z-x(yz)$ is called the associator of x,y,z. In 1974, S. R. Gordon [6] presented a result for a finite dimensional simple Jordan algebra J over a field F which is algebraically closed. Gordon proved that the image of the associator on J is the subspace formed by all elements of zero trace in J. Associators are important polynomials in the

Mathematics subject classification (2010): 15A54, 16S50, 16W10, 17C05.
Keywords and phrases: Image of polynomials, Jordan polynomials, upper triangular matrices, transpose involution.
The first autor was financed by the Coordenação de Aperfeiçoamento de Pessoal de Nıvel Superior – Brasil (CAPES) – Finance Code 001.

* Corresponding author.

free Jordan algebra. In the special free Jordan algebra, associators coincide with commutators of length three. In Lemmas 1, 2 and 3, we obtain, in terms of commutators, a characterization for multilinear Jordan polynomials in three and four variables.

In 2015 A. Ma and J. Oliva [11] proved that the image of any multilinear Jordan polynomial in three variables evaluated on the Jordan algebra formed by the real (resp. complex) symmetric matrices is a vector space, namely, the set of all real (resp. complex) symmetric matrices of zero trace. In the same year, C. Li and M. C. Tsui [10] published a result for finite dimensional central simple algebras over fields of characteristic zero. They showed that for a suitable element γ in the field, the image of the polynomial $[[z,y],x]+\gamma[[x,y],z]$ is the vector space formed by all zero trace elements of the algebra.

Let us denote by UT_m the algebra of all of $m\times m$ upper triangular matrices over a field F. In 2019, P. S. Fagundes [3] investigated the image of an noncommutative and associative multilinear polynomial evaluated on the set of all strictly upper triangular matrices, and he obtained as image a nilpotent subalgebra of UT_m (this subalgebra only depends on the number of the variables of the polynomial). Later, P. S. Fagundes and T. C. de Mello [4] studied the same type of polynomials, but now, evaluated on UT_m and obtain a result for polynomials with at most four variables.

In the present work, we will be considering the algebra UT_m equiped with an involution. The involutions of the first type on UT_m have a very good description as we can see in [2]. The only involutions of the first type on UT_m (up to a $*$-isomorphism) are the transpose and symplectic involutions (the symplectic involution on UT_m occurs only when m is even). We are particularly interested in UT_m^+, the Jordan algebra formed by all symmetric elements of UT_m, with the transpose involution.

In this paper, we give a positive answer to Lvov-Kaplansky conjecture for a multilinear Jordan polynomial in three variables defined on UT_m^+. Our proof will be divided in two cases. For m odd, we will see in Section 3 that the image of such polynomial is exactly the nilpotent algebra formed by all elements of UT_m^+ with zero diagonal. For m even, we will obtain as the set of values, another vector space which is a proper subset of the nilpotent algebra formed by all elements of UT_m^+ with zero diagonal (Section 4).

At the last section, as an application, we will describe the image of certain types of multilinear Jordan polynomials in four variables.

2. Preliminaries

Let F be a field of characteristic different from 2. Let R be an unital (associative) algebra over F with multiplicative identity 1. We write $Z(R)$ to designate the center of R. Note that $F=F\cdot 1$. Thus, $F\subseteq Z(R)$.

A map $*:R\to R$ is called an *involution* on R if

$$(x+y)^*=x^*+y^*,\ \ (xy)^*=y^*x^* \ \text{ and } \ (x^*)^*=x$$

for all $x,y\in R$. Involutions which leave the center elementwise invariant are called involutions of the first kind. Otherwise, we say that the involution is of the second kind.

From now on, we only consider involutions of the first kind. In this case, $(\alpha x)^* = \alpha x^*$ for all $\alpha \in F$ and $x \in R$. We say that an element $x \in R$ is *symmetric* (resp. *skew-symmetric*) if $x^* = x$ (resp. $x^* = -x$). We set $R^+ = \{x \in R|\ x^* = x\}$ and $R^- = \{x \in R \mid x^* = -x\}$. Hence, if $x \in R$, we see that

$$x = (1/2)(x + x^*) + (1/2)(x - x^*).$$

Therefore, $R = R^+ \oplus R^-$.

For each $x, y, z \in R$, we define $[x, y] = xy - yx$. Every associative algebra can be regarded as a Lie algebra under the operation $[x, y]$ which is called the *additive commutator* of x and y. Similarly, the *cirle* operation $x \circ y = xy + yx$ turns R into a Jordan algebra. Some properties of the Lie and Jordan algebras can be found in [1, 7]. A vector subspace V of R is called a Jordan subalgebra when $a \circ b \in V$ for all $a, b \in V$.

In the next lemma, we will prove some identities regarding the circle and the bracket operations in R.

LEMMA 1. *Let x, y, z be elements of an associative algebra R. We set $[x, y, z] = [[x, y], z]$. Then, the following identities hold:*

i) $(x \circ y) \circ z = x \circ (y \circ z) + [z, x, y]$.

ii) $x \circ (y \circ z) = y \circ (x \circ z) + [x, y, z]$.

iii) $[z, x \circ y] = x \circ [z, y] + y \circ [z, x]$.

Proof. $i)$ By definition of the circle operation, $(x \circ y) \circ z = xyz + yxz + zxy + zyx$ and $x \circ (y \circ z) = (z \circ y) \circ x = zyx + yzx + xzy + xyz$. After subtracting these last two equalities, we have that $yxz + zxy - yzx - xzy = [z, x]y - y[z, x] = [z, x, y]$.

$ii)$ Using $i)$ and the commutativity of the circle operation, we have that

$$x \circ (y \circ z) + [z, x, y] = (x \circ y) \circ z = (y \circ x) \circ z = y \circ (x \circ z) + [z, y, x].$$

Now, the Jacobi identity $[z, y, x] - [z, x, y] = [x, y, z]$ finishes the proof.

$iii)$ The proof in this case it is similar to $i)$. □

Let $\{y_i \mid i \in \mathbb{N}\}$ be a countable set of variables. We write $F\langle y_1, y_2, \ldots\rangle$ to denote the unital, associative and non-commutative algebra over F, which is freely generated by the set $\{y_i \mid i \in \mathbb{N}\}$. The elements of $F\langle y_1, y_2, \ldots\rangle$ are polynomials in the non-commutative variables y_i with scalars in F, where $i \in \mathbb{N}$.

An element $f \in F\langle y_1, y_2, \ldots\rangle$ is a *multilinear polynomial* in the variables y_{i_1}, $\ldots, y_{i_n}$, $i_1 < \ldots < i_n$, when f can be written in the following way:

$$f = \sum_{\sigma \in S_\zeta} \alpha_\sigma y_{\sigma(i_1)} \cdots y_{\sigma(i_n)},$$

where S_ζ denotes the permutation group of the set $\zeta = \{i_1, i_2, \ldots, i_n\}$ and $\alpha_\sigma \in F$ for each $\sigma \in S_\zeta$.

An element g of $F\langle y_1,y_2,\dots\rangle$ is called a *Jordan polynomial* when g belongs to the Jordan subalgebra of $F\langle y_1,y_2,\dots\rangle$ generated by the set $\{y_i \mid i\in\mathbb{N}\}$. Observe that the polynomial $f(y_1,y_2)=y_1y_2$ is not a Jordan polynomial. In the present work, we are interested in multilinear Jordan polynomials, and for this reason, we believe that it is appropriate to list some examples of polynomials of this type:

$y_i, y_{i_1}\circ y_{i_2}$ (polynomials in 1 and 2 variables respectively);
$y_{j_1}\circ(y_{j_2}\circ y_{j_3})$ (polynomial in 3 variables);
$y_{j_1}\circ(y_{j_2}\circ(y_{j_3}\circ y_{j_4})), (y_{j_1}\circ y_{j_2})\circ(y_{j_3}\circ y_{j_4})$ (polynomials in 4 variables);
$y_{k_1}\circ(y_{k_2}\circ(y_{k_3}\circ(y_{k_4}\circ y_{k_5}))), y_{k_1}\circ((y_{k_2}\circ y_{k_3})\circ(y_{k_4}\circ y_{k_5}))$ (polynomials in 5 variables).

In what follows, all the polynomials belong to the algebra $F\langle y_1,y_2,\dots\rangle$. Now, we will prove a couple of results about multilinear Jordan polynomials.

LEMMA 2. *Let f be a multilinear Jordan polynomial in the variables $\{y_{i_1},y_{i_2},y_{i_3} \mid i_1<i_2<i_3\}$. Then, f can be written in the form*

$$\alpha y_{i_1}\circ(y_{i_2}\circ y_{i_3})+\beta[y_{i_2},y_{i_1},y_{i_3}]+\gamma[y_{i_3},y_{i_1},y_{i_2}],$$

where $\alpha,\beta,\gamma\in F$.

Proof. We can assume without loss of generality that $(i_1,i_2,i_3)=(1,2,3)$. Let f be a multilinear Jordan polynomial in the variables y_1,y_2,y_3. Since the circle operation is commutative, we may assume that f is a linear combination of $y_1\circ(y_2\circ y_3)$, $y_2\circ(y_1\circ y_3)$ and $y_3\circ(y_1\circ y_2)$. Now, using Lemma 1 (item *ii*)), we have $y_2\circ(y_1\circ y_3)=y_1\circ(y_2\circ y_3)+[y_2,y_1,y_3]$ and $y_3\circ(y_1\circ y_2)=y_1\circ(y_2\circ y_3)+[y_3,y_1,y_2]$. Thus, f is a linear combination of $y_1\circ(y_2\circ y_3)$, $[y_2,y_1,y_3]$ and $[y_3,y_1,y_2]$. □

LEMMA 3. *Consider the following sets:*

$$J=\{y_i\circ[y_j,y_k,y_l] \mid \{i,j,k,l\}=\{1,2,3,4\}, j>k<l\}$$

$$K=\{[y_2,y_1]\circ[y_3,y_4],\ [y_3,y_1]\circ[y_2,y_4],\ [y_4,y_1]\circ[y_2,y_3]\}.$$

Let f be a multilinear Jordan polynomial in the variables $\{y_1,y_2,y_3,y_4\}$. Then, f can be written as a linear combination of the set $B=\{y_1\circ(y_2\circ(y_3\circ y_4))\}\cup J\cup K$.

Proof. Let f be a multilinear Jordan polynomial in the variables $\{y_1,y_2,y_3,y_4\}$, and let S be the permutation group of the set $\{1,2,3,4\}$. We can write $f=f_1+f_2$, where

$$f_1=\sum_{\sigma\in S}\beta_\sigma(y_{\sigma(1)}\circ y_{\sigma(2)})\circ(y_{\sigma(3)}\circ y_{\sigma(4)}),$$

$$f_2=\sum_{\sigma\in S}\alpha_\sigma y_{\sigma(1)}\circ(y_{\sigma(2)}\circ(y_{\sigma(3)}\circ y_{\sigma(4)})).$$

The equality $(y_i \circ y_j) \circ (y_k \circ y_1) = (y_k \circ y_1) \circ (y_i \circ y_j) = (y_1 \circ y_k) \circ (y_i \circ y_j)$ can be used to rewrite f_1 in the form

$$f_1 = \beta_2 (y_1 \circ y_2) \circ (y_3 \circ y_4) + \beta_3 (y_1 \circ y_3) \circ (y_2 \circ y_4) + \beta_4 (y_1 \circ y_4) \circ (y_2 \circ y_3).$$

It suffices to show that each of the $g_i = (y_1 \circ y_i) \circ (y_j \circ y_k) \in Span(B)$, where $\{i, j, k\} = \{2, 3, 4\}$. By Lemma 1(item i)), we have

$$g_i = y_1 \circ (y_i \circ (y_j \circ y_k)) + [y_j \circ y_k, y_1, y_i].$$

Since the term $y_1 \circ (y_i \circ (y_j \circ y_k))$ appears in f_2, we can suppose that $g_i = [y_j \circ y_k, y_1, y_i]$. By Lemma 1(item iii)) we see that $[y_j \circ y_k, y_1] = y_j \circ [y_k, y_1] + y_k \circ [y_j, y_1]$. Using Lemma 1(item iii)) one more time, we have

$$\begin{aligned} g_i &= [y_j \circ [y_k, y_1], y_i] + [y_k \circ [y_j, y_1], y_i] \\ &= y_j \circ [y_k, y_1, y_i] + [y_k, y_1] \circ [y_j, y_i] + y_k \circ [y_j, y_1, y_i] + [y_j, y_1] \circ [y_k, y_i]. \end{aligned}$$

Thus, $g_i \in Span(B)$. Consequently, $f_1 \in Span(B)$.

Now, we will show the result for $f_2 = y_i \circ (y_j \circ (y_k \circ y_l))$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. In this part, We will divide the proof in 2 cases:

Case 1) $i = 1$.

Lemma 2 guarantees that there exist $\alpha, \beta, \gamma \in F$ such that

$$y_j \circ (y_k \circ y_l) = \alpha y_2 \circ (y_3 \circ y_4) + \beta [y_3, y_2, y_4] + \gamma [y_4, y_2, y_3],$$

since $\{j, k, l\} = \{2, 3, 4\}$. Thus, $f_2 = \alpha y_1 \circ (y_2 \circ (y_3 \circ y_4)) + \beta y_1 \circ [y_3, y_2, y_4] + \gamma y_1 \circ [y_4, y_2, y_3] \in Span(B)$.

Case 2) $i > 1$.

In this case, $1 \in \{j, k, l\}$. Once again, Lemma 2 guarantees that there exist $\alpha, \beta, \gamma \in F$ such that

$$y_j \circ (y_k \circ y_l) = \alpha y_1 \circ (y_r \circ y_s) + \beta [y_r, y_1, y_s] + \gamma [y_s, y_1, y_r],$$

where $\{1, r, s\} = \{j, k, l\}$. Since $y_i \circ [y_r, y_1, y_s]$, $y_i \circ [y_s, y_1, y_r] \in J$, it is enough to consider the case when $f_2 = y_i \circ (y_1 \circ (y_r \circ y_s))$. By Lemma 1(item ii)), we have

$$f_2 = y_1 \circ (y_i \circ (y_r \circ y_s)) + [y_i, y_1, (y_r \circ y_s)].$$

Since $y_1 \circ (y_i \circ (y_r \circ y_s))$ has the same form as in Case 1), we can suppose that $f_2 = [y_i, y_1, (y_r \circ y_s)]$. After applying Lemma 1(item iii)), we obtain

$$f_2 = y_r \circ [[y_i, y_1], y_s] + y_s \circ [[y_i, y_1], y_r] = y_r \circ [y_i, y_1, y_s] + y_s \circ [y_i, y_1, y_r].$$

And this completes the proof. $\square$

For each $m \in \mathbb{N}$, let M_m be the algebra of all $m \times m$ matrices with entries in F, and UT_m the subalgebra of all $m \times m$ upper triangular matrices. We define $* : M_m \to M_m$

by $U^* = JU^tJ$, where U^t is the transpose of the matrix U, and J is the following permutation matrix

$$J = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

The map above is an involution of the first kind called the *transpose involution*. For each $U \in UT_m$, the matrix U^* is obtained by reflecting U along its secondary diagonal. Let e_{ij} be the standard matrix unit of M_m. Thus, we can write $J = \sum_{i=1}^{m} e_{m+1-i,i}$. Hence, $e_{ij}^* = Je_{ji}J = e_{m+1-j,m+1-i}$. So, we see that the subalgebra UT_m is closed under this involution.

For each $U \in UT_m$, we set $\overline{U} = \mathbf{s}(U) = U + U^*$ and $\tilde{U} = \mathbf{a}(U) = U - U^*$. The elements $\overline{U}$ and $\tilde{U}$ are respectively, symmetric and skew-symmetric elements of M_m. With this notation in mind, we see that a linear basis for UT_m^+ is given by the elements of the form $\overline{e}_{i,i+t}$, where i,t are integers such that $i \geqslant 1$, $t \geqslant 0$ and $2i+t \leqslant m+1$.

Let $f = f(y_1,\ldots,y_n)$ be a multilinear Jordan polynomial. The image of f evaluated on UT_m^+ is defined by

$$Im_m(f) = \{f(Y_1,\ldots,Y_n) \mid Y_1,\ldots,Y_n \in UT_m^+\}.$$

From a direct inspection, we can see that UT_m^+ is a Jordan subalgebra of UT_m, because $(w \circ v)^* = w \circ v$ whenever $w, v \in UT_m^+$. Therefore, UT_m^+ is invariant by f, since f is a Jordan polynomial. In other words, $Im_m(f)$ is a subset of UT_m^+.

Now, let us discuss some possible images for f. Note that if $\alpha \neq 0$, then $Im_m(\alpha f)$ $= Im_m(f)$. Besides, it is not difficult to see that a multilinear Jordan polynomial in the variable $\{y_1\}$ has the form αy_1, where $\alpha \in F$. Thus, its image evaluated on UT_m^+ is either UT_m^+ or $\{0\}$.

A multilinear Jordan polynomial in the variables $\{y_1,y_2\}$ has the form $\alpha(y_1 \circ y_2)$, where $\alpha \in F$. For a given $W \in UT_m^+$, we have that $W \circ 1_m = 2W$, where 1_m denote the identity matrix of M_m. Thus, we conclude that the image of $(y_1 \circ y_2)$ evaluated on UT_m^+ is UT_m^+ if $\alpha \neq 0$. For a multilinear Jordan polynomial in three variables (evaluated on UT_m^+), we will see that it is possible to obtain nontrivial images.

Now, let f be a multilinear Jordan polynomial in the variables $\{y_1,y_2,y_3\}$. By Lemma 2, we can assume that $f = \alpha y_1 \circ (y_2 \circ y_3) + \beta[y_2,y_1,y_3] + \gamma[y_3,y_1,y_2]$. Note that $f(1_m,1_m,W) = 4\alpha W$ for all $W \in UT_m^+$. Then, $Im_m(f) = UT_m^+$ when $\alpha \neq 0$. For $\alpha = 0$, observe that $f(Y_1,Y_2,Y_3) = \beta[Y_2,Y_1,Y_3] + \gamma[Y_3,Y_1,Y_2]$ is a element of UT_m^+ with null diagonal for all $Y_1,Y_2,Y_3 \in UT_m^+$. Thus,

$$Im_m(f) \subseteq (UT_m^+)_0,$$

where $(UT_m^+)_0$ denotes the subspace of UT_m^+ consisting of all matrices with null diagonal. At this point, we can suppose that either β or γ is nonzero (otherwise $f = 0$ and therefore $Im_m(f) = \{0\}$). Without loss of generality, we may assume that $\beta \neq 0$. So, the image of f on UT_m^+ is equal to image on UT_m^+ of

$$[y_2,y_1,y_3] + \beta^{-1}\gamma[y_3,y_1,y_2]. \tag{1}$$

In the next two sections, we will study the image of (1) evaluated on UT_m^+. From now on, we set $\delta_i = (-1)^i$ for all $i \in \mathbb{Z}$.

3. m is odd

Let k be a positive integer. In this section, we will prove the following theorem.

THEOREM 1. *Let $\gamma \in F$. The image of the Jordan polynomial $f = [y_2, y_1, y_3] + \gamma[y_3, y_1, y_2]$ evaluated on UT_{2k+1}^+ is $(UT_{2k+1}^+)_0$.*

In order to prove Theorem 1, we will need some technical results. And for convenience, we establish the following convention:

CONVENTION 1. $e_{0,t} = e_{t,2k+2} = 0$ for all $t \geqslant 0$. Therefore, $\overline{e}_{0,t} = 0$ for all $t \geqslant 0$.

LEMMA 4. *Let $g : UT_{2k+1}^+ \longrightarrow UT_{2k+1}$ be the linear map defined by*

$$g(W) = \left[\sum_{j=2}^{2k+1} \delta_j e_{j-1,j}\,,\ W\right].$$

Then, $g(\overline{e}_{i,i+t}) = \delta_i \overline{e}_{i-1,i+t} + \delta_{i+t} \overline{e}_{i,i+t+1}$ for all integers $i \geqslant 1$ and $t \geqslant 0$ where $i+t \leqslant 2k$.

Proof. Let i,t integers such that $i \geqslant 1$, $t \geqslant 0$ where $i+t \leqslant 2k$. Set $v = 2k+2-(i+t)$ and $w = 2k+2-i$ then $e_{i,i+t}^* = e_{vw}$. Then,

$$\begin{aligned} g(\overline{e}_{i,i+t}) &= [e_{12} - e_{23} + \ldots + e_{2k-1,2k} - e_{2k,2k+1}\,,\ e_{i,i+t} + e_{vw}] \\ &= \delta_i e_{i-1,i} e_{i,i+t} + \delta_v e_{v-1,v} e_{vw} - \delta_{i+t+1} e_{i,i+t} e_{i+t,i+t+1} - \delta_{w+1} e_{vw} e_{w,w+1} \\ &= \delta_i e_{i-1,i+t} + \delta_{i+t} e_{v-1,w} + \delta_{i+t} e_{i,i+t+1} + \delta_i e_{v,w+1}. \end{aligned}$$

It is easy to see that $e_{v-1,w} = e_{i,i+t+1}^*$ and $e_{v,w+1} = e_{i-1,i+t}^*$. Thus, this proof is complete. $\square$

COROLLARY 1. *Let r,t integers such that $r \geqslant 1$, $t \geqslant 0$, $r+t \leqslant 2k$. If t is even (resp. odd) then $\sum_{i=1}^r g(\overline{e}_{i,i+t}) = \delta_r \overline{e}_{r,r+t+1}$ (resp. $\sum_{i=1}^r \delta_{i+1} g(\overline{e}_{i,i+t}) = \overline{e}_{r,r+t+1}$).*

Proof. Let us suppose that t is even. By Lemma 4 we have $\sum_{i=1}^r g(\overline{e}_{i,i+t}) = \sum_{i=2}^r \delta_i \overline{e}_{i-1,i+t} + \sum_{i=1}^r \delta_{i+t} \overline{e}_{i,i+t+1} = \delta_{r+t} \overline{e}_{r,r+t+1}$. Now, suppose that t is odd. Let $1 \leqslant i \leqslant r$. By Lemma 4 again, we have $\delta_{i+1} g(\overline{e}_{i,i+t}) = \delta_{2i+1} \overline{e}_{i-1,i+t} + \delta_{2i+t+1} \overline{e}_{i,i+t+1} = -\overline{e}_{i-1,i+t} + \overline{e}_{i,i+t+1}$. Thus, $\sum_{i=1}^r \delta_{i+1} g(\overline{e}_{i,i+t}) = \overline{e}_{r,r+t+1}$. $\square$

LEMMA 5. *Let $h : UT_{2k+1}^+ \to UT_{2k+1}$ be the linear map defined by*

$$h(W) = \left[W\,,\ \sum_{l=1}^{k} e_{2l,2l}\,,\ \sum_{j=1}^{2k} e_{j,j+1}\right].$$

Let i,t be non-negative integers such that $1 \leqslant i$ and $i+t \leqslant 2k$. If t is odd then $h(\overline{e}_{i,i+t}) = g(\overline{e}_{i,i+t})$. Otherwise, $h(\overline{e}_{i,i+t}) = 0$.

Proof. Set $v = 2k+2-(i+t)$ and $w = 2k+2-i$, then $e^*_{i,i+t} = e_{vw}$. Let us suppose that t is odd. If i is even then $i+t$ and v are odd, and w is even. So,

$$[e_{i,i+t} + e_{vw} \ , \ \sum_{i=1}^{k} e_{2l,2l}] = e_{vw} - e_{i,i+t} = e^*_{i,i+t} - e_{i,i+t}.$$

If i is odd then w is odd, and $i+t$ and v are even. Then,

$$[e_{i,i+t} + e_{vw} \ , \ \sum_{i=1}^{k} e_{2l,2l}] = e_{i,i+t} - e_{vw} = e_{i,i+t} - e^*_{i,i+t}.$$

In summation, $[\overline{e}_{i,i+t} \ , \ \sum_{i=1}^{k} e_{2l,2l}] = \delta_{i+1}(e_{i,i+t} - e^*_{i,i+t})$. Therefore,

$$\begin{aligned} h(\overline{e}_{i,i+t}) &= \delta_{i+1}[e_{i,i+t} - e_{vw} \ , \ \sum_{j=1}^{2k} e_{j,j+1}] \\ &= \delta_{i+1}(e_{i,i+t+1} - e_{v,w+1} - e_{i-1,i+t} + e_{v-1,w}) \\ &= \delta_{i+1}(e_{i,i+t+1} - e^*_{i-1,i+t} - e_{i-1,i+t} + e^*_{i,i+t+1}) \\ &= \delta_{i+1}(\overline{e}_{i,i+t+1} - \overline{e}_{i-1,i+t}) = g(\overline{e}_{i,i+t}). \end{aligned}$$

Finally, suppose that t is even. If i is even then $i+t$, v and w are even. So, $[e_{i,i+t} + e_{vw} \ , \ \sum_{i=1}^{k} e_{2l,2l}] = e_{i,i+t} + e_{vw} - e_{i,i+t} - e_{vw} = 0$. If i is odd then $i+t$, v and w are odd. So, $[e_{i,i+t} + e_{vw} \ , \ \sum_{i=1}^{k} e_{2l,2l}] = 0$ as desired. □

Proof of Theorem 1. The elements $Y_1 = \sum_{l=1}^{k} e_{2l,2l}$ and $Y_2 = \sum_{i=1}^{2k} e_{i,i+1}$ are in UT^+_{2k+1}. It is not difficult to verify that $[Y_2, Y_1] = \sum_{i=1}^{2k} \delta_{i+1} e_{i,i+1}$. Then, for all $W \in UT^+_{2k+1}$, we have $f(Y_1, Y_2, W) = g(W) + \gamma h(W)$, where g and h are in accordance Lemmas 4 and 5. Thus, it is enough to show that the map $W \mapsto g(W) + \gamma h(W)$ is a surjective linear transformation from UT^+_{2k+1} onto $(UT^+_{2k+1})_0$.

A linear basis for $(UT^+_m)_0$ is given by the elements $\overline{e}_{r,r+t+1}$ where r,t are integers such that $r \geqslant 1$, $t \geqslant 0$ and $r+t \leqslant 2k$. We will to show that each $\overline{e}_{r,r+t+1}$ belongs to the image of the map $W \mapsto g(W) + \gamma h(W)$. Indeed, let r,t non-negative integers such that $1 \leqslant r$, $r+t \leqslant 2k$. Consider the following two elements of UT^+_{2k+1}: $W_0 = \sum_{i=1}^{r} \overline{e}_{i,i+t}$ and $W_1 = \sum_{i=1}^{r} \delta_{i+1} \overline{e}_{i,i+t}$. If t is even (resp. t is odd) then $h(W_0) = \sum_{i=1}^{r} h(\overline{e}_{i,i+t}) = 0$ (resp. $h(W_1) = \sum_{i=1}^{r} \delta_{i+1} h(\overline{e}_{i,i+t}) = g(W_1)$) by Lemma 5, and consequently $f(Y_1, Y_2, W_0) = g(W_0) = \delta_r \, \overline{e}_{r,r+t+1}$ (resp. $f(Y_1, Y_2, W_1) = (1+\gamma) g(W_1) = (1+\gamma)\overline{e}_{r,r+t+1}$)) by Corollary 1.

If $1 + \gamma \neq 0$ the proof is complete. Otherwise, we have that $f = [y_2, y_1, y_3] - [y_3, y_1, y_2] = [y_2, y_3, y_1]$. The same argument can be repeated with $\gamma = 0$. □

4. m is even

Let k be a positive integer. Each element $U \in UT_{2k}$ can be written in the following way:

$$U = \begin{pmatrix} A & \Psi \\ 0 & B \end{pmatrix},$$

where $A, B \in UT_k$ and $\Psi \in M_k$. Thus, the transpose involution of U can be written in the following way

$$U^* = \begin{pmatrix} B^* & \Psi^* \\ 0 & A^* \end{pmatrix},$$

where A^*, B^*, Ψ^* denotes, respectively, the transpose involution of A, B and Ψ on M_k.

If $U \in UT_{2k}^+$, then $B = A^*$ and $\Psi \in M_k^+$. In the case that $U \in UT_{2k}^-$, we see that $B = -A^*$ and $\Psi \in M_k^-$. Besides, we can conclude that the elements in M_k^- have all the entries equal to zero in the secondary diagonal, that is, if $\Psi = \sum_{i,j=1}^{k} \Psi_{ij} e_{ij} \in M_k^-$, then $\Psi_{k1} = \Psi_{k-1,2} = \ldots = \Psi_{1k} = 0$. For more details about these and other properties regarding this involution, we recommend [2].

In order to avoid ambiguity, we denote by d_{ij}, e_{ij} and c_{ij} the standard matrices unit for M_k, M_{2k} and M_{2k+2}, respectively.

Let us recall that the polynomial $[y_1, y_2]$ is an identity for UT_2^+ (see [2]), that is, $[Y_1, Y_2] = 0$ for all $Y_1, Y_2 \in UT_2^+$. Thus, the image of (1) evaluated on UT_2^+ is $\{0\}$. For $k \geqslant 2$, we have the following necessary condition for an element $W \in UT_{2k}^+$ to lie in the image of (1):

PROPOSITION 1. *Let $f = [y_2, y_1, y_3] + \gamma[y_3, y_1, y_2]$, where $\gamma \in F$ and let $k \geqslant 2$. If $W \in Im_{2k}(f)$, then $e_{kk} W e_{k+1,k+1} = 0$.*

Proof. It suffices to show that $e_{kk}[Y_1, Y_2, Y_3]e_{k+1,k+1} = 0$ for all Y_1, Y_2, Y_3 in UT_{2k}^+. For this, since $[Y_1, Y_2] \in UT_{2k}^-$, we can write

$$[Y_1, Y_2] = \begin{pmatrix} C & \Lambda \\ 0 & -C^* \end{pmatrix},$$

where $C \in UT_k$ and $\Lambda \in M_k^-$. Moreover, C is an element with null diagonal, since $[Y_1, Y_2]$ has null diagonal. Now, write Y_3 in the form $Y_3 = \begin{pmatrix} A & \Gamma \\ 0 & A^* \end{pmatrix}$, where $A \in UT_k$, $\Gamma \in M_k^+$. Thus,

$$\begin{aligned} [Y_1, Y_2, Y_3] &= \begin{pmatrix} C & \Lambda \\ 0 & -C^* \end{pmatrix} \begin{pmatrix} A & \Gamma \\ 0 & A^* \end{pmatrix} - \begin{pmatrix} A & \Gamma \\ 0 & A^* \end{pmatrix} \begin{pmatrix} C & \Lambda \\ 0 & -C^* \end{pmatrix} \\ &= \begin{pmatrix} [C,A] & C\Gamma + \Lambda A^* - A\Lambda + \Gamma C^* \\ 0 & [C,A]^* \end{pmatrix} = \begin{pmatrix} [C,A] & \Phi \\ 0 & [C,A]^* \end{pmatrix}. \end{aligned}$$

We can also write $e_{kk} = \begin{pmatrix} d_{kk} & 0 \\ 0 & 0 \end{pmatrix}$ and $e_{k+1,k+1} = \begin{pmatrix} 0 & 0 \\ 0 & d_{11} \end{pmatrix}$. Then,

$$\begin{pmatrix} d_{kk} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [C,A] & \Phi \\ 0 & [C,A]^* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & d_{11} \end{pmatrix} = \begin{pmatrix} d_{kk}[C,A] & d_{kk}\Phi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & d_{11} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & d_{kk}\Phi d_{11} \\ 0 & 0 \end{pmatrix}.$$

Note that in M_k, $\Gamma C^* = (C\Gamma)^*$, $\Lambda A^* = -(A\Lambda)^*$ and $d_{11}^* = d_{kk}$. Hence, it is enough to show that $d_{kk}C\Gamma d_{11} = d_{kk}A\Lambda d_{11} = 0$. Indeed, since $C \in (UT_k)_0$ we can write $C = \sum_{i,j=1}^k \alpha_{ij} d_{ij}$ where each $\alpha_{ij} \in F$ with $\alpha_{ij} = 0$ if $i \geqslant j$. Thus, $d_{kk}C = \sum_{j=1}^k \alpha_{kj} d_{kj} = 0$.

Finally, let $A = \sum_{1 \leqslant i \leqslant j \leqslant k} \beta_{ij} d_{ij}$ with $\beta_{ij} \in F$. Then, $d_{kk}A = \beta_{kk} d_{kk}$. Therefore,

$$d_{kk}A\Lambda d_{11} = \beta_{kk} d_{kk} \Lambda d_{11}.$$

It follows immediately that $d_{kk}\Lambda d_{11} = 0$, since $\Lambda \in M_k^-$ has $(k,1)$-entry equal to zero. Thus, the proof is complete. $\square$

As in the odd case, we will fix two convenient symmetric elements. For each $k \geqslant 2$, we define the following elements in UT_{2k}^+

$$Y_2 = \begin{pmatrix} J_k & \overline{d}_{k-1,1} \\ 0 & J_k \end{pmatrix} \quad \text{and} \quad Y_1 = \begin{pmatrix} B_k & 0 \\ 0 & B_k^* \end{pmatrix}, \tag{2}$$

where $J_k = \sum_{i=2}^k d_{i-1,i}$, and $B_k = \begin{cases} d_{11} + d_{33} + \cdots + d_{kk} & \text{if } k \text{ is odd} \\ d_{22} + d_{44} + \cdots + d_{kk} & \text{if } k \text{ is even} \end{cases}$. For this choice of Y_1 and Y_2, we have: $[Y_2, Y_1] = \begin{pmatrix} [J_k, B_k] & \theta \\ 0 & -[J_k, B_k]^* \end{pmatrix}$, where

$$\theta = \overline{d}_{k-1,1} B_k^* - B_k \overline{d}_{k-1,1} = (B_k \overline{d}_{k-1,1})^* - B_k \overline{d}_{k-1,1} = -\mathbf{a}(B_k \overline{d}_{k-1,1})$$
$$= -\mathbf{a}(B_k(d_{k-1,1} + d_{k2})) = -\mathbf{a}(B_k d_{k2}) = -\mathbf{a}(d_{k2}) = \tilde{d}_{k-1,1}.$$

On one hand, for k odd, we have $[J_k, B_k] = [\sum_{i=2}^k d_{i-1,i}, d_{11} + d_{33} + \cdots + d_{kk}] = -d_{12} + d_{23} - \ldots + d_{k-1,k}$. On the other hand, for k even, then $[J_k, B_k] = [\sum_{i=2}^k d_{i-1,i}, d_{22} + d_{44} + \cdots + d_{kk}] = d_{12} - d_{23} + \ldots + d_{k-1,k}$. Therefore, for all $k \geqslant 2$

$$[J_k, B_k] = \delta_k D_k, \text{ where } D_k = \sum_{i=2}^k \delta_i d_{i-1,i}.$$

And, because $D_k^* = \delta_k D_k$ and $\delta_k^2 = 1$, we conclude that

$$[Y_2, Y_1] = \begin{pmatrix} \delta_k D_k & \tilde{d}_{k-1,1} \\ 0 & -\delta_k D_k^* \end{pmatrix} = \begin{pmatrix} \delta_k D_k & \tilde{d}_{k-1,1} \\ 0 & -D_k \end{pmatrix} = \delta_k \left(\sum_{i=2}^k \delta_i \tilde{e}_{i-1,i} \right) + \tilde{e}_{k-1,k+1}. \tag{3}$$

In the sequence, we will show that the converse of Proposition 1 holds. More precisely, we will prove the following theorem.

THEOREM 2. *Let $k \geqslant 2$ and let $\gamma \in F$. The image of the Jordan polynomial $f = [y_2, y_1, y_3] + \gamma[y_3, y_1, y_2]$ evaluated on UT_{2k}^+ is a linear space with basis $\overline{e}_{ij}$, where $1 \leqslant i < j \leqslant 2k$, $i + j \leqslant 2k+1$ and $(i, j) \neq (k, k+1)$.*

Proof. As in the proof of the Theorem 1, we can suppose, without loss of generality, that $\gamma \neq -1$. Let V_k be the linear subspace of $(UT_{2k})_0^+$ with basis $\overline{e}_{ij}$, where $1 \leqslant i < j \leqslant 2k$, $i + j \leqslant 2k+1$ and $(i, j) \neq (k, k+1)$. Let us define the linear map $f_k : UT_{2k}^+ \longrightarrow V_k$ by $f_k(W) = f(Y_1, Y_2, W) = [Y_2, Y_1, W] + \gamma[W, Y_1, Y_2]$. The map f_k is well-defined by Proposition 1. Note that if f_k is surjective then $Im_{2k}(f) = V_k$. Our goal, it will be to show the surjectivity of the linear map f_k. The proof of this theorem will be divided in two lemmas. □

LEMMA 6. *Let k be an integer $\geqslant 2$. Then $\overline{e}_{1,t+1} \in f_k(UT_{2k}^+)$ for all t such that $1 \leqslant t < 2k$.*

Proof. We start proving 4 facts regarding the map f_k. Set $\varepsilon_z = \frac{1+(-1)^z}{2}$ for all $z \in Z$.

Fact 1) $f_k(\overline{e}_{2k}) = (1 + \gamma\varepsilon_{k+1})(\delta_k \overline{e}_{1k} + \overline{e}_{2,k+2})$.

First of all, we write $\overline{e}_{2k}$ in blocks: $\overline{e}_{2k} = \begin{pmatrix} d_{2k} & 0 \\ 0 & d_{2k}^* \end{pmatrix}$. Thus, using the block notation part of (3), we have

$$[Y_2, Y_1, \overline{e}_{2k}] = \begin{pmatrix} \delta_k[D_k, d_{2k}] & \theta \\ 0 & \delta_k[D_k, d_{2k}]^* \end{pmatrix},$$

where

$$\begin{aligned} \theta &= \tilde{d}_{k-1,1} d_{2k}^* - d_{2k}\tilde{d}_{k-1,1} = -\mathbf{s}(d_{2k}\tilde{d}_{k-1,1}) \\ &= -\mathbf{s}(d_{2k}(d_{k-1,1} - d_{k2})) = -\mathbf{s}(-d_{22}) = \overline{d}_{22}. \end{aligned}$$

From the definition of D_k, we see that

$$\delta_k[D_k, d_{2k}] = \delta_k[\sum_{i=2}^{k} \delta_i d_{i-1,i}\ , d_{2k}] = \delta_k d_{1k}.$$

Therefore,

$$[Y_2, Y_1, \overline{e}_{2k}] = \begin{pmatrix} \delta_k d_{1k} & \overline{d}_{22} \\ 0 & \delta_k d_{1k}^* \end{pmatrix} = \delta_k \overline{e}_{1k} + \overline{e}_{2,k+2}. \tag{4}$$

On the other hand, using the definition of Y_1 in (2), we have

$$[\overline{e}_{2k}\ , Y_1] = \begin{pmatrix} [d_{2k}, B_k] & 0 \\ 0 & -[d_{2k}, B_k]^* \end{pmatrix}.$$

For k odd, we obtain directly from the definition of B_k that

$$[d_{2k}, B_k] = [d_{2k}, d_{11} + d_{33} + \cdots + d_{kk}] = d_{2k} = \varepsilon_{k+1} d_{2k}.$$

In the same fashion, for k even, $[d_{2k}, B_k] = [d_{2k}, d_{22} + d_{44} + \cdots + d_{kk}] = d_{2k} - d_{2k} = 0$. Thus, $[d_{2k}, B_k] = \varepsilon_{k+1} d_{2k}$ for all $k \geqslant 2$. Combining this last identity with the definition of Y_2 in (2), and the fact that J_k is symmetric, we arrive at

$$[\overline{e}_{2k}, Y_1, Y_2] = \varepsilon_{k+1} \begin{pmatrix} [d_{2k}, J_k] & \phi \\ 0 & -[d_{2k}, J_k]^* \end{pmatrix},$$

where $\phi = d_{2k}\overline{d}_{k-1,1} + \overline{d}_{k-1,1} d_{2k}^* = d_{2k}\overline{d}_{k-1,1} + (d_{2k}\overline{d}_{k-1,1})^* = \mathbf{s}(d_{2k}\overline{d}_{k-1,1}) =$ $= \mathbf{s}(d_{2k}(d_{k-1,1} + d_{k2})) = \overline{d}_{22}$, and $[d_{2k}, J_k] = [d_{2k}, \sum_{i=2}^{k} d_{i-1,i}] = -d_{1k}$. Thus,

$$\begin{aligned}[\overline{e}_{2k}, Y_1, Y_2] &= \varepsilon_{k+1} \begin{pmatrix} -d_{1k} & \overline{d}_{22} \\ 0 & -d_{1k}^* \end{pmatrix} = \varepsilon_{k+1}(-\overline{e}_{1k} + \overline{e}_{2,k+2}) \\ &= \varepsilon_{k+1}(\delta_k \overline{e}_{1k} + \overline{e}_{2,k+2}), \text{ since } -\varepsilon_{k+1} = \varepsilon_{k+1}\delta_k.\end{aligned}$$

Now, the result follows from the equality above combined with (4).

Fact 2) $f_k(\overline{e}_{2,k+1}) = (1 + \gamma\varepsilon_{k+1})(\delta_k \overline{e}_{1,k+1} + \overline{e}_{2,k+2})$.

Note that $\overline{e}_{2,k+1} = e_{2,k+1} + e_{2,k+1}^* = e_{2,k+1} + e_{k,2k-1}$. So, in blocks, we have $\overline{e}_{2,k+1} = \begin{pmatrix} 0 & \overline{d}_{21} \\ 0 & 0 \end{pmatrix}$. By the block notation part of (3), we obtain

$$[Y_2, Y_1, \overline{e}_{2,k+1}] = \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}\psi &= \delta_k D_k \overline{d}_{21} + \overline{d}_{21} D_k = \delta_k (D_k \overline{d}_{21} + (D_k \overline{d}_{21})^*) = \delta_k \mathbf{s}(D_k \overline{d}_{21}) \\ &= \delta_k \mathbf{s}\left(\sum_{i=2}^{k} \delta_i d_{i-1,i}(d_{21} + d_{k,k-1})\right) = \delta_k \mathbf{s}(d_{11} + \delta_k d_{k-1,k-1})) = \delta_k \overline{d}_{11} + \overline{d}_{22}.\end{aligned}$$

Thus,

$$[Y_2, Y_1, \overline{e}_{2,k+1}] = \begin{pmatrix} 0 & \delta_k \overline{d}_{11} + \overline{d}_{22} \\ 0 & 0 \end{pmatrix} = \delta_k \overline{e}_{1,k+1} + \overline{e}_{2,k+2}. \tag{5}$$

Using the definition of Y_1 in (2), it is straightforward to verify that

$$[\overline{e}_{2,k+1}, Y_1] = \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix},$$

where $\theta = \overline{d}_{21} B_k^* - B_k \overline{d}_{21} = (B_k \overline{d}_{21})^* - B_k \overline{d}_{21} = -\mathbf{a}(B_k \overline{d}_{21})$.

When k is odd, $B_k \overline{d}_{21} = (d_{11} + d_{33} + \cdots + d_{kk})(d_{21} + d_{k,k-1}) = d_{k,k-1}$. So, $\theta = -\tilde{d}_{k,k-1} = \tilde{d}_{21}$. Similarly, for k even,

$$B_k \overline{d}_{21} = (d_{22} + d_{44} + \cdots + d_{kk})(d_{21} + d_{k,k-1}) = d_{21} + d_{k,k-1} = \overline{d}_{21}.$$

Then, $\theta = -\mathbf{a}(\overline{d}_{21}) = 0$. Therefore, $\theta = \varepsilon_{k+1}\tilde{d}_{21}$ for all $k \geqslant 2$.

Thus, it follows from the previous computations and the definition of Y_2 in (2), and the fact that J_k and $\tilde{d}_{21}$ are, respectively, symmetric and skew-symmetric elements that

$$[\overline{e}_{2,k+1}\ , Y_1, Y_2] = \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}\psi = [\theta, J_k] &= \varepsilon_{k+1}(\tilde{d}_{21}J_k - J_k\tilde{d}_{21}) = \varepsilon_{k+1}(\tilde{d}_{21}J_k + (\tilde{d}_{21}J_k)^*) = \varepsilon_{k+1}\mathbf{s}(\tilde{d}_{21}J_k) \\ &= \varepsilon_{k+1}\mathbf{s}\left((d_{21} - d_{k,k-1})\sum_{i=2}^{k} d_{i-1,i}\right) = \varepsilon_{k+1}\mathbf{s}(d_{22} - d_{kk}) = \varepsilon_{k+1}(\overline{d}_{22} - \overline{d}_{11}) \\ &= \varepsilon_{k+1}(\overline{d}_{22} + \delta_k\overline{d}_{11}),\ \text{since}\ \varepsilon_{k+1}\delta_k = -\varepsilon_{k+1}.\end{aligned}$$

Thus,

$$[\overline{e}_{2,k+1}\ , Y_1, Y_2] = \varepsilon_{k+1}\begin{pmatrix} 0 & \overline{d}_{22} + \delta_k\overline{d}_{11} \\ 0 & 0 \end{pmatrix} = \varepsilon_{k+1}(\overline{e}_{2,k+2} + \delta_k\overline{e}_{1,k+1}).$$

The desired result can be obtained, from the identity above together with (5).

Fact 3) Let t such that $k < t < 2k$. Then, $f_k(\overline{e}_{1t}) = \delta_{1-t+k}(1+\gamma\varepsilon_{1-t})\overline{e}_{1,t+1}$.

Set $t = k+s$. So, $1 \leqslant s \leqslant k-1$. By definition, $\overline{e}_{1,k+s} = e_{1,k+s} + e^*_{1,k+s} =$ $= e_{1,k+s} + e_{k+1-s,2k}$. Hence, $\overline{e}_{1,k+s} = \begin{pmatrix} 0 & \overline{d}_{1s} \\ 0 & 0 \end{pmatrix}$. Once again, using the block notation part of (3), we see

$$[Y_2, Y_1, \overline{e}_{1,k+s}] = \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}\theta &= \delta_k D_k \overline{d}_{1s} + \overline{d}_{1s} D_k \\ &= \delta_k \sum_{i=2}^{k} \delta_i d_{i-1,i}(d_{1s} + d_{k+1-s,k}) + \sum_{i=2}^{k} \delta_i (d_{1s} + d_{k+1-s,k}) d_{i-1,i} \\ &= \delta_k\delta_{k+1-s}d_{k-s,k} + \delta_{s+1}d_{1,s+1} = \delta_{1-s}(d_{k-s,k} + d_{1,s+1}) = \delta_{1-s}\overline{d}_{1,s+1}.\end{aligned}$$

So,

$$[Y_2, Y_1, \overline{e}_{1,k+s}] = \delta_{1-s}\overline{e}_{1,k+s+1}. \tag{6}$$

According the definition of Y_1 in (2), we have

$$[\overline{e}_{1,k+s}\ , Y_1] = \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix},$$

where $\varphi = \overline{d}_{1s}B_k^* - B_k\overline{d}_{1s} = (B_k\overline{d}_{1s})^* - B_k\overline{d}_{1s} = -\mathbf{a}(B_k\overline{d}_{1s})$.

First, we will suppose that $(k+1-s)$ is even. For k even, we have $B_k\overline{d}_{1s} = (d_{22}+d_{44}+\cdots+d_{kk})(d_{1s}+d_{k+1-s,k}) = d_{k+1-s,k}$. So, $\varphi = -\tilde{d}_{k+1-s,k} = \tilde{d}_{1s}$. If k is odd, we see that $B_k\overline{d}_{1s} = (d_{11}+d_{33}+\cdots+d_{kk})(d_{1s}+d_{k+1-s,k}) = d_{1s}$. Then, $\varphi = -\tilde{d}_{1s}$. Hence,

$$\varphi = \delta_k\tilde{d}_{1s} = \delta_{1-s}\tilde{d}_{1s}, \text{ when } (k+1-s) \text{ is even.}$$

Secondly, let us suppose that $(k+1-s)$ is odd. For k odd, we have $B_k\overline{d}_{1s} = (d_{11}+d_{33}+\cdots+d_{kk})(d_{1s}+d_{k+1-s,k}) = d_{1s}+d_{k+1-s,k} = \overline{d}_{1s}$. Consequently, $\varphi = \overline{d}_{1s}^* - \overline{d}_{1s} = 0$. When k is even, we have $B_k\overline{d}_{1s} = (d_{22}+d_{44}+\cdots+d_{kk})(d_{1s}+d_{k+1-s,k}) = 0$. So,

$$\varphi = 0, \text{ when } (k+1-s) \text{ is odd.}$$

Therefore,

$$\varphi = \delta_{1-s}\varepsilon_{k+1-s}\tilde{d}_{1s} \text{ for all } k \geqslant 2, \text{ where } t = k+s \text{ and } k < t < 2k.$$

From the previous discussion and the definition of Y_2 in (2), and the fact that J_k and φ are, respectively, symmetric and skew-symmetric elements, we conclude

$$[\overline{e}_{1,k+s}, Y_1, Y_2] = \begin{pmatrix} 0 & \varphi J_k - J_k\varphi \\ 0 & 0 \end{pmatrix},$$

where $[\varphi, J_k] = \varphi J_k - J_k\varphi = \mathbf{s}(\varphi J_k) = \delta_{1-s}\varepsilon_{k+1-s}\mathbf{s}(\tilde{d}_{1s}J_k)$. And since

$$\tilde{d}_{1s}J_k = (d_{1s} - d_{k+1-s,k})\sum_{i=2}^{k} d_{i-1,i} = d_{1,s+1},$$

we see that $[\varphi, J_k] = \delta_{1-s}\varepsilon_{k+1-s}\overline{d}_{1,s+1}$. Thus,

$$[\overline{e}_{1,k+s}, Y_1, Y_2] = \delta_{1-s}\varepsilon_{k+1-s}\overline{e}_{1,k+s+1}.$$

The conlusion follows from the equality above combined with (6), and using that $s = t-k$.

Fact 4) Let t such that $1 \leqslant t \leqslant k-1$. Then,

$$f_k(\overline{e}_{1t}) = \begin{cases} -(1+\gamma\varepsilon_{k-1})(\overline{e}_{1k} + \overline{e}_{1,k+1}) & \text{if } t = k-1 \\ \delta_{k+t}(1+\gamma\varepsilon_t)\overline{e}_{1,t+1} & \text{if } t < k-1 \end{cases}.$$

Writing $\overline{e}_{1t}$ in blocks, we obtain $\overline{e}_{1t} = \begin{pmatrix} d_{1t} & 0 \\ 0 & d_{1t}^* \end{pmatrix}$. By the block part of (3), we see

$$[Y_2, Y_1, \overline{e}_{1t}] = \begin{pmatrix} \delta_k[D_k, d_{1t}] & \theta \\ 0 & \delta_k[D_k, d_{1t}]^* \end{pmatrix},$$

where

$$\theta = \tilde{d}_{k-1,1}d_{1t}^* - d_{1t}\tilde{d}_{k-1,1} = -(d_{1t}\tilde{d}_{k-1,1})^* - d_{1t}\tilde{d}_{k-1,1}.$$

Note that $d_{1t}\tilde{d}_{k-1,1} = d_{1t}(d_{k-1,1} - d_{k2}) = d_{11}$ if $t = k-1$ and $d_{1t}\tilde{d}_{k-1,1} = 0$ otherwise. Thus,

$$\theta = \begin{cases} -\overline{d}_{11} & \text{if } t = k-1 \\ 0 & \text{if } t < k-1 \end{cases}.$$

On the other hand,

$$[D_k, d_{1t}] = [\sum_{i=2}^{k} \delta_i d_{i-1,i}\ , d_{1t}] = \delta_t d_{1,t+1}.$$

In particular, if $t = k-1$ then $\delta_k[D_k, d_{1,k-1}] = \delta_k\delta_{k-1}d_{1k} = -d_{1k}$. Thus,

$$[Y_2, Y_1, \overline{e}_{1t}] = \begin{cases} \delta_{k+t}\overline{e}_{1,t+1} & \text{if } t < k-1 \\ -\overline{e}_{1k} - \overline{e}_{1,k+1} & \text{if } t = k-1 \end{cases}. \tag{7}$$

Using, (2), it is immediate to check that

$$[\overline{e}_{1t}\ , Y_1] = \begin{pmatrix} [d_{1t}, B_k] & 0 \\ 0 & -[d_{1t}, B_k]^* \end{pmatrix}.$$

When k is even, we see that $[d_{1t}, B_k] = [d_{1t}, d_{22} + d_{44} + \cdots + d_{kk}] = \varepsilon_t d_{1t}$. In the same way, if k is odd, we have $[d_{1t}, B_k] = [d_{1t}, d_{11} + d_{33} + \cdots + d_{kk}] = -\varepsilon_t d_{1t}$.

Therefore,

$$[d_{1t}, B_k] = \delta_k\varepsilon_t d_{1t}.$$

Thus, using the previous computations and the definition of Y_2 in (2), and the fact that J_k and $\overline{d}_{k-1,1}$ are symmetric, we see that

$$[\overline{e}_{1t}\ , Y_1, Y_2] = \delta_k\varepsilon_t \begin{pmatrix} [d_{1t}, J_k] & \Gamma \\ 0 & [d_{1t}, J_k]^* \end{pmatrix},$$

where

$$\Gamma = d_{1t}\overline{d}_{k-1,1} + \overline{d}_{k-1,1}d_{1t}^* = \mathbf{s}(d_{1t}\overline{d}_{k-1,1}) = \mathbf{s}(d_{1t}d_{k-1,1}) = \begin{cases} \overline{d}_{11} & \text{if } t = k-1 \\ 0 & \text{if } t < k-1 \end{cases}.$$

Now, observe that $[d_{1t}, J_k] = [d_{1t}\ , \sum_{i=2}^{k} d_{i-1,i}] = d_{1,t+1}$. Hence, for $t < k-1$,

$$[\overline{e}_{1t}\ , Y_1, Y_2] = \delta_k\varepsilon_t \begin{pmatrix} d_{1,t+1} & 0 \\ 0 & d_{1,t+1}^* \end{pmatrix} = \delta_k\varepsilon_t\overline{e}_{1,t+1} = \delta_{k+t}\varepsilon_t\overline{e}_{1,t+1}.$$

And, for $t=k-1$,

$$[\overline{e}_{1,k-1}\ ,Y_1,Y_2]=\delta_k\varepsilon_{k-1}\begin{pmatrix} d_{1k} & \overline{d}_{11} \\ 0 & d^*_{1k}\end{pmatrix}=-\varepsilon_{k-1}\begin{pmatrix} d_{1k} & \overline{d}_{11} \\ 0 & d^*_{1k}\end{pmatrix}=-\varepsilon_{k-1}(\overline{e}_{1k}+\overline{e}_{1,k+1}).$$

The desired result follows from (7) and the last two equalities above.

So far, we have proved that if $\gamma\neq -1$, then the following elements belong to the image of f_k:

$$\begin{aligned}
&\overline{e}_{12},\ldots,\overline{e}_{1,k-1} \text{ (By Fact 4)},\\
&\overline{e}_{1,k+2},\ldots,\overline{e}_{1,2k} \text{ (By Fact 3)},\\
&x_1=\overline{e}_{1k}+\overline{e}_{1,k+1} \text{ (By Fact 4)},\\
&x_2=\delta_k\overline{e}_{1k}+\overline{e}_{2,k+2} \text{ (By Fact 1)},\\
&x_3=\delta_k\overline{e}_{1,k+1}+\overline{e}_{2,k+2} \text{ (By Fact 2)}.
\end{aligned}$$

Note that $\delta_k(x_2-x_3)+x_1=2\overline{e}_{1k}\in f_k(UT^+_{2k})$, because f_k is linear. In particular, $\overline{e}_{1,k+1}\in f_k(UT^+_{2k})$. And this completes the proof. □

Before proving the next lemma, we will make some considerations. Let us consider the following embedding of algebras $\varphi: UT_{2k}\to UT_{2k+2}$ given by

$$\varphi(U)=\begin{pmatrix} 0 & 0 & 0\\ 0 & U & 0\\ 0 & 0 & 0\end{pmatrix},\ \text{where}\quad U\in UT_{2k}.$$

It follows directly from the definition of the map φ that $\varphi(e_{ij})=c_{i+1,j+1}$ for all i,j such that $1\leqslant i\leqslant j\leqslant 2k$. In particular, we see that $\varphi(\overline{e}_{ij})=\overline{c}_{i+1,j+1}$ and $\varphi(\tilde{e}_{ij})=\tilde{c}_{i+1,j+1}$, and that the map φ preserves involution, i.e., $\varphi(U)^*=\varphi(U^*)$ for all $U\in UT_{2k}$. Consequently, $\varphi(UT^+_{2k})\subseteq UT^+_{2k+2}$.

For convenience, let us rewrite the elements defined in (2) as

$$Y_2=\overline{e}_{k-1,k+1}+\sum_{i=2}^{k}\overline{e}_{i-1,i}\ ,\quad Y_1=\begin{cases}\overline{e}_{11}+\overline{e}_{33}+\ldots+\overline{e}_{kk} & \text{if } k \text{ is odd},\\ \overline{e}_{22}+\overline{e}_{44}+\ldots+\overline{e}_{kk} & \text{if } k \text{ is even},\end{cases}$$

and let us define two new elements that lie in UT^+_{2k+2} as below

$$Y_2'=\overline{c}_{k,k+2}+\sum_{i=2}^{k+1}\overline{c}_{i-1,i}\ ,\quad Y_1'=\begin{cases}\overline{c}_{11}+\overline{c}_{33}+\ldots+\overline{c}_{k+1,k+1} & \text{if } k+1 \text{ is odd},\\ \overline{c}_{22}+\overline{c}_{44}+\ldots+\overline{c}_{k+1,k+1} & \text{if } k+1 \text{ is even}.\end{cases}$$

Then, by (3)

$$[Y_2,Y_1]=\delta_k\left(\sum_{i=2}^{k}\delta_i\tilde{e}_{i-1,i}\right)+\tilde{e}_{k-1,k+1}\ ,\quad [Y_2',Y_1']=\delta_{k+1}\left(\sum_{i=2}^{k+1}\delta_i\tilde{c}_{i-1,i}\right)+\tilde{c}_{k,k+2}.$$

Now, for a given element $W\in UT_{2k}$, by definition of f_{k+1}, we have that

$$f_{k+1}(\varphi(W))=[Y_2',Y_1',\varphi(W)]+\gamma[\varphi(W),Y_1',Y_2'].$$

Note that,

$$\varphi(Y_2) = \varphi(\overline{e}_{k-1,k+1}) + \varphi(\sum_{i=2}^{k} \overline{e}_{i-1,i}) = \overline{c}_{k,k+2} + \sum_{i=2}^{k} \overline{c}_{i,i+1} = Y_2' - \overline{c}_{12},$$

and

$$\begin{aligned}\varphi([Y_2,Y_1]) &= \delta_k \left(\sum_{i=2}^{k} \delta_i \varphi(\tilde{e}_{i-1,i}) \right) + \varphi(\tilde{e}_{k-1,k+1}) = \delta_k \left(\sum_{i=2}^{k} \delta_i \tilde{c}_{i,i+1} \right) + \tilde{c}_{k,k+2} \\ &= \delta_{k+1} \left(\sum_{j=3}^{k+1} \delta_j \tilde{c}_{j-1,j} \right) + \tilde{c}_{k,k+2} = [Y_2', Y_1'] - \delta_{k+1}\tilde{c}_{12}.\end{aligned}$$

If k is odd, then

$$\varphi(Y_1) = \varphi(\overline{e}_{11} + \overline{e}_{33} + \cdots + \overline{e}_{kk}) = \overline{c}_{22} + \overline{c}_{44} + \cdots + \overline{c}_{k+1,k+1} = Y_1'.$$

For k even, we have

$$\varphi(Y_1) = \varphi(\overline{e}_{22} + \overline{e}_{44} + \cdots + \overline{e}_{kk}) = \overline{c}_{33} + \overline{c}_{55} + \cdots + \overline{c}_{k+1,k+1} = Y_1' - \overline{c}_{11}.$$

Since $\varphi(W) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we see that $[\varphi(W), \overline{c}_{11}] = 0$. Hence,

$$[\varphi(W), \varphi(Y_1)] = [\varphi(W), Y_1'].$$

After all this, we see, at last, that

$$\begin{aligned}\varphi(f_k(W)) &= [\varphi([Y_2,Y_1]), \varphi(W)] + \gamma[\varphi(W), \varphi(Y_1), \varphi(Y_2)] \\ &= [Y_2', Y_1', \varphi(W)] - \delta_{k+1}[\tilde{c}_{12}, \varphi(W)] + \gamma[\varphi(W), Y_1', Y_2'] - \gamma[\varphi(W), Y_1', \overline{c}_{12}] \\ &= f_{k+1}(\varphi(W)) - \delta_{k+1}[\tilde{c}_{12}, \varphi(W)] - \gamma[\varphi(W), Y_1', \overline{c}_{12}]. \qquad (8)\end{aligned}$$

LEMMA 7. *Let $k \geqslant 2$. Then, $\overline{e}_{ij}$ belongs to $f_k(UT_{2k}^+)$ for all i,j such that $1 \leqslant i < j \leqslant 2k$, $i+j \leqslant 2k+1$ and $(i,j) \neq (k,k+1)$.*

Proof. We will proceed by induction on k. The base case $k=2$ was coverded by Lemma 6, since we have that $\overline{e}_{12}, \overline{e}_{13}, \overline{e}_{14} \in f_2(UT_4^+)$. Let us assume that the result holds for k. We will show that the result is valid for $k+1$. Once again, Lemma 6 tells us that it suffices to show that $\overline{c}_{rs} \in f_{k+1}(UT_{2k+2}^+)$ for all (r,s) such that $2 \leqslant r < s \leqslant 2k+1$, $r+s \leqslant 2k+3$ and $(r,s) \neq (k+1,k+2)$. Indeed, take (r,s) as above. By induction hypothesis, there exists $W \in UT_{2k}^+$ such that $f_k(W) = \overline{e}_{r-1,s-1}$. Then, $\varphi(f_k(W)) = \varphi(\overline{e}_{r-1,s-1}) = \overline{c}_{rs}$.

By (8),

$$\overline{c}_{rs} = \varphi(f_k(W)) = f_{k+1}(\varphi(W)) - \delta_{k+1}[\tilde{c}_{12}, \varphi(W)] - \gamma[\varphi([W,Y_1]), \overline{c}_{12}].$$

It remains to show that $[\tilde{c}_{12},\varphi(W)]$ and $[\overline{c}_{12},\varphi([W,Y_1])]\in f_{k+1}(UT^+_{2k+2})$, since $\varphi(W)\in\varphi(UT^+_{2k})\subseteq UT^+_{2k+2}$. From the fact that W and $Y_1\in UT^+_{2k}$, we see that $[W,Y_1]$ is skew-symmetric. Thus, we can write

$$W=\sum_{(i,j)\in\Lambda}\alpha_{ij}\overline{e}_{ij}\quad\text{and}\quad[W,Y_1]=\sum_{(i,j)\in\Lambda,i<j}\beta_{ij}\tilde{e}_{ij},$$

where $\alpha_{ij},\beta_{ij}\in F$ and $\Lambda=\{(i,j)\mid 1\leqslant i\leqslant j\leqslant 2k,\ i+j\leqslant 2k+1\}$.

Note that $\varphi(W)=\sum_{(i,j)\in\Lambda}\alpha_{ij}\overline{c}_{i+1,j+1}$. And this yields,

$$[\tilde{c}_{12},\varphi(W)]=\sum_{i,j\in\Lambda}\alpha_{ij}[\tilde{c}_{12},\overline{c}_{i+1,j+1}].$$

Fix $(i,j)\in\Lambda$. If $i\neq 1$ then $j\neq 2k$. Thus, since $i\leqslant 2k$, we see

$$[\tilde{c}_{12},\overline{c}_{i+1,j+1}]=[c_{12}-c_{2k+1,2k+2}\ ,\ c_{i+1,j+1}+c_{2k+2-j,2k+2-i}]=0.$$

Therefore, by Lemma 6, we have

$$[\tilde{c}_{12},\varphi(W)]=\sum_{j=1}^{2k}\alpha_{1j}[\tilde{c}_{12},\overline{c}_{2,j+1}]=\sum_{j=1}^{2k-1}\alpha_{1j}\overline{c}_{1,j+1}+2\alpha_{1,2k}\overline{c}_{1,2k+1}\in f_{k+1}(UT^+_{2k+2}).$$

Similarly, we can show that $[\overline{c}_{12},\varphi([W,Y_1])]\in f_{k+1}(UT^+_{2k+2})$. $\square$

5. Application

As an application of the last two sections, we will characterize the image of some multilinear Jordan polynomials in the variables $\{y_1,y_2,y_3,y_4\}$, namely, we will find the image of polynomials in the following form:

$$f(y_1,y_2,y_3,y_4)=\alpha y_1\circ(y_2\circ(y_3\circ y_4))+g(y_1,y_2,y_3,y_4),$$

where J is in accordance with Lemma 3, $g\in Span(J)$ and $\alpha\in F$. Note that

$$f(Y,1_m,1_m,1_m)=\alpha Y\circ(1_m\circ(1_m\circ 1_m))=8\alpha Y$$

for all $Y\in UT^+_m$. Thus, $Im_m(f)=UT^+_m$ for all nonzero α in F. When $\alpha=0$, we have the following result.

THEOREM 3. *Let g be a nonzero element of $Span(J)$. Then $Im_m(g)=(UT^+_m)_0$ if m is odd. When m is even, $Im_m(g)=V_{m/2}$ where $V_{m/2}$ is the linear space with basis $\overline{e}_{ij}$ where $1\leqslant i<j\leqslant m$, $i+j\leqslant m+1$ and $(i,j)\neq(m/2,m/2+1)$.*

Proof. We can write g in the form

$$g=\alpha_1y_1\circ[y_3,y_2,y_4]+\alpha_1'y_1\circ[y_4,y_2,y_3]+\sum_{r=2}^{4}(\alpha_ry_r\circ[y_{j_r},y_1,y_{k_r}]+\alpha_r'y_r\circ[y_{k_r},y_1,y_{j_r}]),$$

where $j_r, k_r \in \{2,3,4\} \setminus \{r\}$ with $j_r < k_r$, and $\alpha_r, \alpha_r' \in F$. We can suppose, without less of generality, that either α_1 or α_1' is nonzero. Let $Y_1 = 1_m$ and Y_2, Y_3, Y_4 be three arbitrary elements of UT_m^+. Then,

$$g(1_m, Y_2, Y_3, Y_4) = 2\alpha_1[Y_3, Y_2, Y_4] + 2\alpha_1'[Y_4, Y_2, Y_3] = 2p(Y_2, Y_3, Y_4),$$

where $p(w_1, w_2, w_3) = \alpha_1[w_2, w_1, w_3] + \alpha_1'[w_3, w_1, w_2]$ is a multilinear Jordan polynomial in three variables. Thus, $Im_m(p) \subseteq Im_m(g)$. Hence, by Theorems 1 and 2 the result follows. □

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(Received December 20, 2019)

Ronald Ismael Quispe Urure
Department of Mathematics
Federal University of Juiz de Fora
36036-900, Juiz de Fora – MG, Brazil
e-mail: urure6@gmail.com

Willian Franca
Department of Mathematics
Federal University of Juiz de Fora
36036-900, Juiz de Fora – MG, Brazil
e-mail: wilian.franca@ufjf.edu.br