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ON THE DRAZIN INVERSE AND M–P INVERSE FOR SUM OF MATRICES

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Abstract. Drazin inverse and M-P inverse have many important applications in the aspects of theoretic research of operator and statistics. In this article, we will exhibit under suitable conditions a neat relationship between the Drazin inverse of $A+B$ and the Drazin inverses of the individual terms A and B. Furthermore, with the same thread, we will give an expression of the M-P inverse of $A+B$ in terms of only the M-P inverses of matrices A and B.

1. Introduction

Throughout this article, $C^{m\times n}$ denotes the set of all $m\times n$ matrices over the complex field C, I_k denotes the identity matrix of order K and $O_{m\times n}$ is the $m\times n$ matrix of all zero entries (if no confusion occurs, we will drop the subscript). For a matrix $A\in C^{m\times n}$, A^*, $R(A)$ and $r(A)$ denote the conjugate transpose, the range space and the rank of the matrix A, respectively.

If A is an $n\times n$ complex matrix, then the Drazin inverse of A denoted by A^D, is the unique matrix X satisfying the relations [1]

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA,$$

where $k = Ind(A)$, the index of A, is the smallest nonnegative integer for which $r(A^k) = r(A^{k+1})$. In particular, when $Ind(A) = 1$, the matrix X is called the group inverse of A, and is denoted by $X = A^g$. If A is nonsingular, then it is easily seen that $Ind(A) = 0$ and $A^D = A^{-1}$.

Let $A\in C^{m\times n}$, the M-P inverse, denoted by $X = A^\dagger$ of A is defined to be the unique solution of the following four Penrose equations [2]

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA.$$

In particular, when A is nonsingular, then it is easily seen that $A^\dagger = A^{-1}$. We refer the reader to [3, 4] for basic results on the Drazin inverse and the M-P inverse of matrices.

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The concepts of Drazin inverse and M-P inverse were shown to be very useful in various applied mathematical settings. For example, applications to singular differential or difference equations, Markov chains, cryptography, iterative method or multibody system dynamics, and so on, which can be found in [3, 4, 5, 6, 7, 8].

Although the theories of Drazin inverse and M-P inverse had have a substantial development over the past several decades, there are lots of fundamental problems on these inverses of matrices that need further investigation. One such problem is concerned with the Drazin inverse or the M-P inverse for sum of matrices. Suppose A and B are a pair of matrices with the same size. In many situations, one wants to know the expressions of $(A+B)^D$ and $(A+B)^\dagger$ and its properties. For example, under what conditions do the following equations hold?

$$(A+B)^D = A^D + B^D, \quad (A+B)^g = A^g + B^g, \quad (A+B)^\dagger = A^\dagger + B^\dagger.$$

The Drazin inverse or the M-P inverse for sum of matrices was introduced by Penrose [2]. It has quite important applications in numerical linear algebra and applied fields [3, 4, 8], such as linear control theory [9], matrix theory [10, 11], statistics [12], projection algorithms [13] and perturbation analysis of matrix [14]. Moreover, as one of the fundamental research problems in matrix theory, the Drazin inverse or the M-P inverse for sum of matrices is a very useful tool in many algorithms for the computation of the generalized parallel sum of A and B. The generalized parallel sum originally arose in an attempt to generalize a network synthesis procedure of Duffin [15] and has been studied in the scalar case by Erickson [16]. Suppose that $A,B \in C^{m\times n}$, then we define the generalized parallel sum of A and B by the formula $A(A+B)^D B$ or $A(A+B)^\dagger B$. One such problem concerns with the Drazin inverse or the M-P inverse of $A+B$.

The Drazin inverse or the M-P inverse for sum of matrices yields a class of interesting problems that are fundamental in statistic and matrix theory, see [3, 4, 8]. They have attracted considerable attentions and some interesting results have been obtained by Cline [17], Radoslaw and Krezsztof [18], Minamide [19], Tian [20], Xiong and Qin [21, 22] and others see [23, 24, 25]. In this article, we provide a complete solution to the problem of relationship between the Drazin inverse of $A+B$ and the Drazin inverses of A and B, and present a equivalent condition for equation $(A+B)^D = A^D + B^D$. The same question is also discussed for the M-P inverses of A, B and $A+B$. To our knowledge, there is no article discussed these in the literature.

As the main tools in our discussion, we first mention the following three lemmas, which will be used in this paper.

LEMMA 1.1. [3] *Let $A \in C^{n\times n}$ and $Ind(A) = k$. Then*

$$A^D = A^k (A^{2k+1})^\dagger A^k \ \ \text{and} \ \ r(A^l) = r(A^k),\ l \geqslant k. \tag{1.1}$$

LEMMA 1.2. [26] *Let $A \in C^{m\times n}$ has the block representations $A = \big(U,\, V\big)$. Then*

$$A^\dagger = \big(U,\, V\big)^\dagger = \begin{pmatrix} KU^\dagger(I - VC^\dagger) \\ \rho^* KU^\dagger(I - VC^\dagger) + C^\dagger \end{pmatrix}, \tag{1.2}$$

where $C = (I - UU^{\dagger})V$, $\rho = U^{\dagger}V(I - C^{\dagger}C)$ *and* $K = (I + \rho\rho^{*})^{-1}$.

LEMMA 1.3. [27] *Suppose matrices* A, B, C *and* D *satisfy the following conditions:*

$$R(B) \subseteq R(A) \ \text{ and } \ R(C^{*}) \subseteq R(A^{*}) \tag{1.3}$$

or

$$R(C) \subseteq R(D) \ \text{ and } \ R(B^{*}) \subseteq R(D^{*}). \tag{1.4}$$

Then

$$r(A) + r(D - CA^{\dagger}B) = r\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.5}$$

or

$$r(D) + r(A - BD^{\dagger}C) = r\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1.6}$$

LEMMA 1.4. [28] *Let* $A \in C^{m\times n}$, $B \in C^{n\times k}$ *and* $C \in C^{k\times l}$. *Then the reverse order law* $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$ *holds if and only if* A, B *and* C *satisfy the following rank equality:*

$$r\begin{pmatrix} BB^{*}B & O & BC \\ O & -DD^{*}D & DC^{*}C \\ AB & AA^{*}D & O \end{pmatrix} = r(D) + r(B), \tag{1.7}$$

where $D = ABC$.

LEMMA 1.5. [28] *Suppose* $A_i \in C^{s_i\times l_i}$, $i = 1,2,\cdots,n$ *and* $B_i \in C^{s_i\times l_{i+1}}$, $i = 1,2,\cdots,n-1$, *satisfy*

$$B_i = A_iX_iA_{i+1}, \ \ i = 1,2,\cdots,n-1 \ \text{ for some } X_i. \tag{1.8}$$

Then

$$R(B_i) \subseteq R(A_i), \ \ R(B_i^{*}) \subseteq R(A_{i+1}^{*}), \ \ i = 1,2,\cdots,n-1, \tag{1.9}$$

and the M-P inverse of the $n\times n$ *block matrix*

$$J_n = \begin{pmatrix} O & O & \cdots & \cdots & O & A_n \\ O & O & \cdots & O & A_{n-1} & B_{n-1} \\ \vdots & \vdots & / & / & / & O \\ \vdots & O & / & / & / & \vdots \\ O & A_2 & B_2 & O & \cdots & O \\ A_1 & B_1 & O & O & \cdots & O \end{pmatrix},$$

may be repressed as

$$J_n^{\dagger} = \begin{pmatrix} F(1,n) & F(1,n-1) & \cdots & F(1,2) & F(1,1) \\ F(2,n) & F(2,n-1) & \cdots & F(2,2) & O \\ \vdots & \vdots & \diagup & \diagup & \vdots \\ F(n-1,n) & F(n-1,n-1) & O & \cdots & O \\ F(n,n) & O & O & \cdots & O \end{pmatrix}, \tag{1.10}$$

where

$$F(i,i) = A_i^{\dagger},\ i = 1,2,\cdots,n,$$

$$F(i,j) = (-1)^{j-i} A_i^{\dagger} B_i A_{i+1}^{\dagger} B_{i+1} \cdots A_{j-1}^{\dagger} B_{j-1} A_j^{\dagger},\ 1 \leqslant i \leqslant j \leqslant n.$$

2. Drazin inverse for sum of matrices

In this section, we will present some conditions for the equality of $(A+B)^D = A^D + B^D$. The relative results are included in the following three lemmas.

LEMMA 2.1. *Suppose $A_i \in C^{m\times m}$, $i=1,2,3$ and $k = \max\{Ind(A_i), 1, Ind(A_1A_2A_3)\}$, $i=1,2,3$. Then the M-P inverse of the 5×5 block matrix*

$$M = \begin{pmatrix} O & O & O & O & I_m \\ O & O & O & A_3^{2k+1} & A_3^k \\ O & O & A_2^{2k+1} & A_2^k A_3^k & O \\ O & A_1^{2k+1} & A_1^k A_2^k & O & O \\ I_m & A_1^k & O & O & O \end{pmatrix}, \tag{2.1}$$

may be repressed as

$$M^{\dagger} = \begin{pmatrix} M(1,5) & M(1,4) & M(1,3) & M(1,2) & M(1,1) \\ M(2,5) & M(2,4) & M(2,3) & M(2,2) & O \\ M(3,5) & M(3,4) & M(3,3) & O & O \\ M(4,5) & M(4,4) & O & O & O \\ M(5,5) & O & O & O & O \end{pmatrix}, \tag{2.2}$$

where $M(i,j)$ in (2.2) can be expressed as the formulas (1.10) in Lemma 1.5. In particular,

$$\begin{aligned} M(1,5) &= PM^{\dagger}Q = (-1)^{5-1}(I_m)^{\dagger} A_1^k (A_1^{2k+1})^{\dagger} A_1^k A_2^k (A_2^{2k+1})^{\dagger} A_2^k A_3^k (A_3^{2k+1})^{\dagger} A_3^k (I_m)^{\dagger} \\ &= A_1^D A_2^D A_3^D, \end{aligned} \tag{2.3}$$

where $P = (I_m,\ O,\ O,\ O,\ O)$ and $Q = (I_m,\ O,\ O,\ O,\ O)^*$.

Proof. From (2.1) and the definition of Drazin inverse, we have

$$A_1^k = I_m(A_1^D)^{k+1}A_1^{2k+1},\ \ R(A_1^k) \subseteq R(I_m),\ \ R((A_1^k)^*) \subseteq R((A_1^{2k+1})^*), \tag{2.4}$$

$$\begin{aligned} A_1^kA_2^k &= A_1^{2k+1}(A_1^D)^{k+1}(A_2^D)^{k+1}A_2^{2k+1},\ R(A_1^kA_2^k) \subseteq R(A_1^{2k+1}), \\ &\quad R((A_1^kA_2^k)^*) \subseteq R((A_2^{2k+1})^*), \end{aligned} \tag{2.5}$$

$$\begin{aligned} A_2^kA_3^k &= A_2^{2k+1}(A_2^D)^{k+1}(A_3^D)^{k+1}A_3^{2k+1},\ R(A_2^kA_3^k) \subseteq R(A_2^{2k+1}), \\ &\quad R((A_2^kA_3^k)^*) \subseteq R((A_3^{2k+1})^*), \end{aligned} \tag{2.6}$$

$$A_3^k = A_3^{2k+1}(A_3^D)^{k+1}I_m,\ \ R(A_3^k) \subseteq R(A_3^{2k+1}),\ \ R((A_3^k)^*) \subseteq R(I_m^*). \tag{2.7}$$

Combining the formulas (2.4)–(2.7), with the formulas (1.8)–(1.9) in Lemma 1.5, we have the results in Lemma 2.1. In particular, from Lemma 1.1, we have

$$A_i^D = A_i^k(A_i^{2k+1})^\dagger A_i^k, \quad i = 1,2,3,$$

then the last equality in (2.3) holds. □

From the structure of M in (2.1) and the formula (1.1) in Lemma 1.1, we have

$$r(A_i^k) = r((A_i^D)^{k+1}A_i^{2k+1}) \leqslant r(A_i^{2k+1}) \leqslant r(A_i^k),\ \ i = 1,2,3. \tag{2.8}$$

By the formulas (2.8) and the structure of M in (2.1) ,we at once see that it has the following simple properties, which will be used in the sequel.

LEMMA 2.2. *Let M, P and Q be given as in Lemma 2.1 and let $k = \max\{Ind(A_i),$ $1,\ Ind(A)\}$, $A = A_1A_2A_3$. Then*

$$r(M) = 2m + r(A_1^k) + r(A_2^k) + r(A_3^k), \tag{2.9}$$

$$R(Q) \subseteq R(M)\ \ \text{and}\ \ R(P^*) \subseteq R(M^*), \tag{2.10}$$

$$R(QA) \subseteq R(M)\ \ \text{and}\ \ R(P^*A^*) \subseteq R(M^*). \tag{2.11}$$

Proof. Let

$$D_1 = \begin{pmatrix} I_m & -A_1^k & O & O & O \\ O & I_m & O & O & O \\ O & O & I_m & O & O \\ O & O & O & I_m & O \\ O & O & O & O & I_m \end{pmatrix},\ D_2 = \begin{pmatrix} I_m & O & O & O & O \\ O & I_m & -(A_1^D)^{k+1}(A_2^D)^{k+1}A_2^{2k+1} & O & O \\ O & O & I_m & O & O \\ O & O & O & I_m & O \\ O & O & O & O & I_m \end{pmatrix},$$

$$D_3 = \begin{pmatrix} I_m & O & O & O & O \\ O & I_m & O & O & O \\ O & O & I_m & -(A_2^D)^{k+1}(A_3^D)^{k+1}A_3^{2k+1} & O \\ O & O & O & I_m & O \\ O & O & O & O & I_m \end{pmatrix},$$

$$D_4=\begin{pmatrix} I_m & O & O & O & O \\ O & I_m & O & O & O \\ O & O & I_m & O & O \\ O & O & O & I_m & -(A_3^D)^{k+1} \\ O & O & O & O & I_m \end{pmatrix},\quad D_5=\begin{pmatrix} O \\ O \\ O \\ O \\ I_m \end{pmatrix}, \tag{2.12}$$

we have

$$MD_1D_2D_3D_4=\begin{pmatrix} O & O & O & O & I_m \\ O & O & O & A_3^{2k+1} & O \\ O & O & A_2^{2k+1} & O & O \\ O & A_1^{2k+1} & O & O & O \\ I_m & O & O & O & O \end{pmatrix} \text{ and } MD_1D_2D_3D_4D_5=\begin{pmatrix} I_m \\ O \\ O \\ O \\ O \end{pmatrix}=Q \tag{2.13}$$

Since $D_i,\ i=1,2,3,4$ are nonsingular, then combining the formulas (2.8) with (2.13), we have

$$r(M)=r(MD_1D_2D_3D_4)=2m+r(A_1^k)+r(A_2^k)+r(A_3^k), \tag{2.14}$$

and

$$R(QA)\subseteq R(Q)=R(MD_1D_2D_3D_4D_5)\subseteq R(M). \tag{2.15}$$

On the other hand, let

$$S_1=\begin{pmatrix} I_m & O & O & O & O \\ -A_3^k & I_m & O & O & O \\ O & O & I_m & O & O \\ O & O & O & I_m & O \\ O & O & O & O & I_m \end{pmatrix},\quad S_2=\begin{pmatrix} I_m & O & O & O & O \\ O & I_m & O & O & O \\ O & -A_2^{2k+1}(A_2^D)^{k+1}(A_3^D)^{k+1} & I_m & O & O \\ O & O & O & I_m & O \\ O & O & O & O & I_m \end{pmatrix},$$

$$S_3=\begin{pmatrix} I_m & O & O & O & O \\ O & I_m & O & O & O \\ O & O & I_m & O & O \\ O & O & -A_1^{2k+1}(A_1^D)^{k+1}(A_2^D)^{k+1} & I_m & O \\ O & O & O & O & I_m \end{pmatrix},$$

$$S_4=\begin{pmatrix} I_m & O & O & O & O \\ O & I_m & O & O & O \\ O & O & I_m & O & O \\ O & O & O & I_m & O \\ O & O & O & -(A_1^D)^{k+1} & I_m \end{pmatrix},\quad S_5=\left(O,\, O,\, O,\, O,\, I_m\right), \tag{2.16}$$

we have

$$S_4S_3S_2S_1M=\begin{pmatrix} O & O & O & O & I_m \\ O & O & O & A_3^{2k+1} & O \\ O & O & A_2^{2k+1} & O & O \\ O & A_1^{2k+1} & O & O & O \\ I_m & O & O & O & O \end{pmatrix} \text{ and } S_5S_4S_3S_2S_1M=P. \tag{2.17}$$

From formulas (2.17) , we have

$$R(P^*A^*) \subseteq R(P^*) = R((S_5S_4S_3S_2S_1M)^*) = R(M^*S_1^*S_2^*S_3^*S_4^*S_5^*) \subseteq R(M^*). \quad (2.18)$$

Combining the formulas (2.14), (2.15) with (2.18), we have the results in Lemma 2.2. □

From Lemma 2.1 and Lemma 2.2, a necessary and sufficient condition can be given for the forward order law $(A_1A_2A_3)^D = A_1^DA_2^DA_3^D$.

LEMMA 2.3. *Suppose $A_i \in C^{m\times m}$, $i = 1,2,3$, $A = A_1A_2A_3$, $k = \max\{Ind(A_i),\ 1,\ Ind(A)\}$ and $X = A_1^DA_2^DA_3^D$. Then the following statements are equivalent:*

(1) $A^D = (A_1A_2A_3)^D = A_1^DA_2^DA_3^D = X$;

(2) $r\begin{pmatrix} -A^{2k+1} & A^kE_1 \\ E_2A^k & N \end{pmatrix} = r(A^k) + r(A_1^k) + r(A_2^k) + r(A_3^k)$;

where $E_1 = (O,\ O,\ O,\ I_m)$, $E_2 = (O,\ O,\ O,\ I_m)^*$ and

$$N = \begin{pmatrix} O & O & A_3^{2k+1} & A_3^k \\ O & A_2^{2k+1} & A_2^kA_3^k & O \\ A_1^{2k+1} & A_1^kA_2^k & O & O \\ A_1^k & O & O & O \end{pmatrix}.$$

Proof. From the formulas (2.1)–(2.3) in Lemma 2.1, we have

$$X = A_1^DA_2^DA_3^D = PM^\dagger Q \tag{2.19}$$

and

$$M = \begin{pmatrix} O & E_1 \\ E_2 & N \end{pmatrix}. \tag{2.20}$$

It is obvious that $A^D = (A_1A_2A_3)^D = A_1^DA_2^DA_3^D = X$ holds if and only if

$$0 = r(A^D - X) = r(A^D - PM^\dagger Q). \tag{2.21}$$

Now using the matrices in (2.21), we construct a 3×3 block matrix as follows:

$$G = \begin{pmatrix} M & O & Q \\ O & -A^{2k+1} & A^k \\ P & A^k & O \end{pmatrix}. \tag{2.22}$$

It follows from Lemma 2.2 that

$$R\begin{pmatrix} Q \\ A^k \end{pmatrix} \subseteq R\begin{pmatrix} M & O \\ O & -A^{2k+1} \end{pmatrix}, \tag{2.23}$$

$$R\left[\left(P,\ A^k\right)^*\right] \subseteq R\begin{pmatrix} M^* & O \\ O & (-A^{2k+1})^* \end{pmatrix}. \tag{2.24}$$

Hence by the rank formulas in Lemma 1.3, we have

$$\begin{aligned} r(G) &= r\begin{pmatrix} M & O \\ O & -A^{2k+1} \end{pmatrix} + r\left[(P,\, A^k) \begin{pmatrix} M & O \\ O & -A^{2k+1} \end{pmatrix}^{\dagger} \begin{pmatrix} Q \\ A^k \end{pmatrix} \right] \\ &= r(M) + r(A^{2k+1}) + r(PM^{\dagger}Q - A^k(A^{2k+1})^{\dagger}A^k). \end{aligned} \tag{2.25}$$

Combining (2.25), Lemma 1.1 with Lemma 2.2, we have

$$A^D = A^k(A^{2k+1})^{\dagger}A^k \text{ and } r(A^k) = r((A^D)^{k+1}A^{2k+1}) \leqslant r(A^{2k+1}) \leqslant r(A^k) \tag{2.26}$$

and

$$r(G) = 2m + r(A_1^k) + r(A_2^k) + r(A_3^k) + r(A^k) + r(A^D - PM^{\dagger}Q). \tag{2.27}$$

On the other hand, substituting the complete expression of M in (2.12) and then calculating the rank of G will produce the following result

$$\begin{aligned} r(G) &= r\begin{pmatrix} O & E_1 & O & I_m \\ E_2 & N & O & O \\ O & O & -A^{2k+1} & A^k \\ I_m & O & A^k & O \end{pmatrix} = r\begin{pmatrix} O & O & O & I_m \\ O & N & -E_2A^k & O \\ O & -A^kE_1 & -A^{2k+1} & O \\ I_m & O & O & O \end{pmatrix} \\ &= 2m + r\begin{pmatrix} -A^{2k+1} & A^kE_1 \\ E_2A^k & N \end{pmatrix}. \end{aligned} \tag{2.28}$$

Combining (2.21), (2.27) and (2.28) will yield the results in Lemma 2.3. □

According to Lemma 2.1, Lemma 2.2 and Lemma 2.3, we immediately obtain the following key result in this article.

THEOREM 2.1. *Suppose $A \in C^{m\times m}$, $B \in C^{m\times m}$ and $k = \max\{Ind(A),\, Ind(B),\, 1,\, Ind(A+B)\}$ satisfy the following condition*

$$r\begin{pmatrix} -(A+B)^{2k+1} & O & O & (A+B)^k \\ O & A^{2k+1} & O & A^k \\ O & O & B^{2k+1} & B^k \\ (A+B)^k & A^k & B^k & O \end{pmatrix} = r[(A+B)^k] + r(A^k) + r(B^k). \tag{2.29}$$

Then

$$(A+B)^D = A^D + B^D.$$

Proof. It is obvious that

$$(A+B)^D = (I_m,\, O)\left[\begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix} \right]^D \begin{pmatrix} I_m \\ O \end{pmatrix}. \tag{2.30}$$

By Lemma 2.3 with $\left(A_1 = \begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix},\ A_2 = \begin{pmatrix} A & O \\ O & B \end{pmatrix},\ A_3 = \begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}\right.$ and $\left. A = A_1A_2A_3\right)$, we have

$$\left[\begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix}\begin{pmatrix} A & O \\ O & B \end{pmatrix}\begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}\right]^D = \begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix}^D\begin{pmatrix} A & O \\ O & B \end{pmatrix}^D\begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}^D \tag{2.31}$$

if and only if

$$\begin{aligned}
& r\begin{pmatrix}
-(A+B)^{2k+1} & O & O & O & O & O & O & O & (A+B)^k & O \\
O & O & O & O & O & O & O & O & O & O \\
O & O & O & O & O & O & I_m & O & I_m & O \\
O & O & O & O & O & O & I_m & O & I_m & O \\
O & O & O & O & A^{2k+1} & O & A^k & O & O & O \\
O & O & O & O & O & B^{2k+1} & B^k & O & O & O \\
O & O & I_m & I_m & A^k & B^k & O & O & O & O \\
O & O & O & O & O & O & O & O & O & O \\
(A+B)^k & O & I_m & I_m & O & O & O & O & O & O \\
O & O & O & O & O & O & O & O & O & O
\end{pmatrix} \\
& = 2m + r[(A+B)^k] + r(A^k) + r(B^k) \\
\Leftrightarrow\ & r\begin{pmatrix}
-(A+B)^{2k+1} & O & O & -(A+B)^k \\
O & A^{2k+1} & O & A^k \\
O & O & B^{2k+1} & B^k \\
-(A+B)^k & A^k & B^k & O
\end{pmatrix} = r[(A+B)^k] + r(A^k) + r(B^k). \\
\Leftrightarrow\ & r\begin{pmatrix}
-(A+B)^{2k+1} & O & O & (A+B)^k \\
O & A^{2k+1} & O & A^k \\
O & O & B^{2k+1} & B^k \\
(A+B)^k & A^k & B^k & O
\end{pmatrix} = r[(A+B)^k] + r(A^k) + r(B^k).
\end{aligned} \tag{2.32}$$

On the other hand, it is obvious that

$$\begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix}^D = \begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix}, \tag{2.33}$$

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}^D = \begin{pmatrix} A^D & O \\ O & B^D \end{pmatrix} \tag{2.34}$$

and

$$\begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}^D = \begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}. \tag{2.35}$$

Finally from (2.30)–(2.35), we obtain that if the rank equality (2.29) holds, then

$$\begin{aligned}
(A+B)^D &= \left(I_m,\, O\right)\left[\begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix}\begin{pmatrix} A & O \\ O & B \end{pmatrix}\begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}\right]^D \begin{pmatrix} I_m \\ O \end{pmatrix} \\
&= \left(I_m,\, O\right)\begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix}^D \begin{pmatrix} A & O \\ O & B \end{pmatrix}^D \begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}^D \begin{pmatrix} I_m \\ O \end{pmatrix} \\
&= \left(I_m,\, O\right)\begin{pmatrix} I_m & I_m \\ O & O \end{pmatrix}\begin{pmatrix} A^D & O \\ O & B^D \end{pmatrix}\begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}\begin{pmatrix} I_m \\ O \end{pmatrix} \\
&= \left(I_m,\, O\right)\begin{pmatrix} A^D & B^D \\ O & O \end{pmatrix}\begin{pmatrix} I_m & O \\ I_m & O \end{pmatrix}\begin{pmatrix} I_m \\ O \end{pmatrix} \\
&= \left(I_m,\, O\right)\begin{pmatrix} A^D+B^D & O \\ O & O \end{pmatrix}\begin{pmatrix} I_m \\ O \end{pmatrix} \\
&= A^D+B^D. \quad \square
\end{aligned}$$

COROLLARY 2.1. *Suppose $A \in C^{m\times m}$ and $B \in C^{m\times m}$ satisfy the following condition*

$$r\begin{pmatrix} -(A+B)^3 & O & O & A+B \\ O & A^3 & O & A \\ O & O & B^3 & B \\ A+B & A & B & O \end{pmatrix} = r(A+B)+r(A)+r(B).$$

Then

$$(A+B)^g = A^g + B^g.$$

3. M-P inverse for sum of matrices

In this section, we will present a relationship between the M-P inverse $(A+B)^\dagger$ and the M-P inverses $A^\dagger$ and $B^\dagger$. The relative results are included in the following theorem and corollaries.

THEOREM 3.1. *Let $A \in C^{m\times n}$, $B \in C^{m\times n}$ and $D = A+B$. Then the following statements are equivalent:*

(1) $(A+B)^\dagger = \frac{1}{4}A^\dagger + \frac{1}{4}B^\dagger$;

(2) $r\begin{pmatrix} AA^*A & O & O & A \\ O & BB^*B & O & B \\ O & O & -\frac{1}{4}DD^*D & D \\ A & B & D & O \end{pmatrix} = r(A)+r(B)+r(D).$

Proof. With I for an identity matrix of appropriate size, it is easy to verify that

$$(A+B)^\dagger = \left[\left(I,\, I\right)\begin{pmatrix} A & O \\ O & B \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}\right]^\dagger. \tag{3.1}$$

From Lemma 1.4, we know that the following reverse order law:

$$\left[(I,\,I)\begin{pmatrix} A & O \\ O & B \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}\right]^{\dagger} = \begin{pmatrix} I \\ I \end{pmatrix}^{\dagger}\begin{pmatrix} A & O \\ O & B \end{pmatrix}^{\dagger}(I,\,I)^{\dagger} \tag{3.2}$$

holds if and only if

$$r\begin{pmatrix} AA^*A & O & O & A \\ O & BB^*B & O & B \\ O & O & -DD^*D & 2D \\ A & B & 2D & O \end{pmatrix} = r(D)+r(A)+r(B). \tag{3.3}$$

On the other hand, from Lemma 1.2 and the definition of M-P inverse, we have

$$\begin{pmatrix} I \\ I \end{pmatrix}^{\dagger} = \left(\tfrac{1}{2}I,\ \tfrac{1}{2}I\right), \tag{3.4}$$

$$(I,\,I)^{\dagger} = \begin{pmatrix} \frac{1}{2}I \\ \frac{1}{2}I \end{pmatrix} \tag{3.5}$$

and

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}^{\dagger} = \begin{pmatrix} A^{\dagger} & O \\ O & B^{\dagger} \end{pmatrix}. \tag{3.6}$$

Combining the formulas (3.1)–(3.6), we have

$$(A+B)^{\dagger} = \left[(I,\,I)\begin{pmatrix} A & O \\ O & B \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}\right]^{\dagger} = \begin{pmatrix} I \\ I \end{pmatrix}^{\dagger}\begin{pmatrix} A & O \\ O & B \end{pmatrix}^{\dagger}(I,\,I)^{\dagger} = \frac{1}{4}A^{\dagger}+\frac{1}{4}B^{\dagger}. \tag{3.7}$$

if and only if

$$r\begin{pmatrix} AA^*A & O & O & A \\ O & BB^*B & O & B \\ O & O & -\frac{1}{4}DD^*D & D \\ A & B & D & O \end{pmatrix} = r(A)+r(B)+r(D). \quad \square \tag{3.8}$$

REMARK 1. Although the conditions in Theorem 3.1 seems very special, such examples do exist in matrices. For example, let $A=-B$, we can verify that the results in Theorem 3.1 is correct.

COROLLARY 3.1. *Let* $A\in C^{m\times n}$, $B\in C^{m\times n}$ *and* $D=A+B$. *Then the following statements are equivalent:*

(1) $(\frac{1}{2}A+\frac{1}{2}B)^{\dagger} = \frac{1}{2}A^{\dagger}+\frac{1}{2}B^{\dagger}$;

(2) $r\begin{pmatrix} AA^*A & O & O & A \\ O & BB^*B & O & B \\ O & O & -\frac{1}{4}DD^*D & D \\ A & B & D & O \end{pmatrix} = r(A)+r(B)+r(D).$

Proof. By Theorem 3.1 with $(A = \frac{1}{2}A$ and $B = \frac{1}{2}B)$, we know that the following equalities

$$\left(\frac{1}{2}A+\frac{1}{2}B\right)^{\dagger} = \frac{1}{4}\left(\frac{1}{2}A\right)^{\dagger} + \frac{1}{4}\left(\frac{1}{2}B\right)^{\dagger} = \frac{1}{4}(2A^{\dagger}) + \frac{1}{4}(2B^{\dagger}) = \frac{1}{2}A^{\dagger} + \frac{1}{2}B^{\dagger} \tag{3.9}$$

hold, if and only if

$$r\begin{pmatrix} (\frac{1}{2}A)(\frac{1}{2}A)^*(\frac{1}{2}A) & O & O & \frac{1}{2}A \\ O & (\frac{1}{2}B)(\frac{1}{2}B)^*(\frac{1}{2}B) & O & \frac{1}{2}B \\ O & O & -\frac{1}{4}(\frac{1}{2}(A+B))(\frac{1}{2}(A+B))^*(\frac{1}{2}(A+B)) & \frac{1}{2}(A+B) \\ \frac{1}{2}A & \frac{1}{2}B & \frac{1}{2}(A+B) & O \end{pmatrix}$$

$$= r\left(\frac{1}{2}A\right) + r\left(\frac{1}{2}B\right) + r\left(\frac{1}{2}(A+B)\right)$$

$$\Leftrightarrow r\begin{pmatrix} AA^*A & O & O & A \\ O & BB^*B & O & B \\ O & O & -\frac{1}{4}DD^*D & D \\ A & B & D & O \end{pmatrix} = r(A)+r(B)+r(D). \tag{3.10}$$

□

COROLLARY 3.2. *Let $A \in C^{m\times n}$, $B \in C^{m\times n}$ and $D = A+B$. Then the following statements are equivalent:*

(1) $(\frac{1}{4}A+\frac{1}{4}B)^{\dagger} = A^{\dagger}+B^{\dagger}$;

(2) $r\begin{pmatrix} AA^*A & O & O & A \\ O & BB^*B & O & B \\ O & O & -\frac{1}{4}DD^*D & D \\ A & B & D & O \end{pmatrix} = r(A)+r(B)+r(D).$

Proof. By Theorem 3.1 with $(A = \frac{1}{4}A$ and $B = \frac{1}{4}B)$, we know that the following equalities

$$\left(\frac{1}{4}A+\frac{1}{4}B\right)^{\dagger} = \frac{1}{4}\left(\frac{1}{4}A\right)^{\dagger} + \frac{1}{4}\left(\frac{1}{4}B\right)^{\dagger} = \frac{1}{4}(4A^{\dagger}) + \frac{1}{4}(4B^{\dagger}) = A^{\dagger} + B^{\dagger} \tag{3.11}$$

hold, if and only if

$$r\begin{pmatrix} (\frac{1}{4}A)(\frac{1}{4}A)^*(\frac{1}{4}A) & O & O & \frac{1}{4}A \\ O & (\frac{1}{4}B)(\frac{1}{4}B)^*(\frac{1}{4}B) & O & \frac{1}{4}B \\ O & O & -\frac{1}{4}(\frac{1}{4}(A+B))(\frac{1}{4}(A+B))^*(\frac{1}{4}(A+B)) & \frac{1}{4}(A+B) \\ \frac{1}{4}A & \frac{1}{4}B & \frac{1}{4}(A+B) & O \end{pmatrix}$$

$$= r\left(\frac{1}{4}A\right) + r\left(\frac{1}{4}B\right) + r\left(\frac{1}{4}(A+B)\right)$$

$$\Leftrightarrow r\begin{pmatrix} AA^*A & O & O & A \\ O & BB^*B & O & B \\ O & O & -\frac{1}{4}DD^*D & D \\ A & B & D & O \end{pmatrix} = r(A)+r(B)+r(D). \tag{3.12}$$

□

COROLLARY 3.3. *Let $A \in C^{m\times n}$ and $B \in C^{m\times k}$. Then the following statements are equivalent:*

(1) $\begin{pmatrix} A, B \end{pmatrix}^\dagger = \begin{pmatrix} \frac{1}{4}A^\dagger \\ \frac{1}{4}B^\dagger \end{pmatrix}$;

(2) $r\begin{pmatrix} AA^*A & BB^*B & BB^*A & AA^*B \\ A & B & 3A & 3B \end{pmatrix} = r\begin{pmatrix} A, B \end{pmatrix}$.

Proof. By Theorem 3.1 with $(A = \begin{pmatrix} A, O \end{pmatrix}$ and $B = \begin{pmatrix} O, B \end{pmatrix}$, we know that the following equalities

$$\begin{aligned} \begin{pmatrix} A, B \end{pmatrix}^\dagger &= (\begin{pmatrix} A, O \end{pmatrix} + \begin{pmatrix} O, B \end{pmatrix})^\dagger = \frac{1}{4}\begin{pmatrix} A, O \end{pmatrix}^\dagger + \frac{1}{4}\begin{pmatrix} O, B \end{pmatrix}^\dagger \\ &= \frac{1}{4}\begin{pmatrix} A^\dagger \\ O \end{pmatrix} + \frac{1}{4}\begin{pmatrix} O \\ B^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{4}A^\dagger \\ \frac{1}{4}B^\dagger \end{pmatrix} \end{aligned} \tag{3.13}$$

hold, if and only if

$$\begin{aligned} &r\begin{pmatrix} AA^*A & O & O & O & O & O & A & O \\ O & O & O & BB^*B & O & O & O & B \\ O & O & O & O & -\frac{1}{4}(AA^*A + BB^*A) & -\frac{1}{4}(AA^*B + BB^*B) & A & B \\ A & O & O & B & A & B & O & O \end{pmatrix} \\ &= r\begin{pmatrix} A, O \end{pmatrix} + r\begin{pmatrix} O, B \end{pmatrix} + \begin{pmatrix} A, B \end{pmatrix}. \end{aligned} \tag{3.14}$$

Let I for some identity matrix of appropriate size and

$$\begin{aligned} &T = \begin{pmatrix} AA^*A & O & O & O & A & O \\ O & BB^*B & O & O & O & B \\ O & O & -\frac{1}{4}(AA^*A + BB^*A) & -\frac{1}{4}(AA^*B + BB^*B) & A & B \\ A & B & A & B & O & O \end{pmatrix}, \\ &\mu_1 = \begin{pmatrix} I & O & O & O & O & O \\ O & I & O & O & O & O \\ O & O & I & O & O & O \\ O & O & O & I & O & O \\ O & O & O & O & I & O \\ O & -B^*B & O & O & O & I \end{pmatrix}, \ \mu_2 = \begin{pmatrix} I & O & O & O & O \\ O & I & O & O & O \\ O & O & I & O & O \\ O & O & O & I & O \\ -A^*A & O & O & O & I \end{pmatrix}, \\ &\mu_3 = \begin{pmatrix} I & O & -I & O \\ O & I & O & O \\ O & O & I & O \\ O & O & O & I \end{pmatrix}, \ \mu_4 = \begin{pmatrix} I & O & O & O \\ O & I & O & -I \\ O & O & I & O \\ O & O & O & I \end{pmatrix}. \end{aligned} \tag{3.15}$$

Then

$$
\begin{aligned}
& r\begin{pmatrix} AA^*A & O & O & O & O & O & A & O \\ O & O & O & BB^*B & O & O & O & B \\ O & O & O & O & -\frac{1}{4}(AA^*A+BB^*A) & -\frac{1}{4}(AA^*B+BB^*B) & A & B \\ A & O & O & B & A & B & O & O \end{pmatrix} \\
&= r(T\mu_1)+r(B) \\
&= r(\begin{pmatrix} AA^*A & O & O & O & A \\ O & -BB^*B & -\frac{1}{4}(AA^*A+BB^*A) & -\frac{1}{4}(AA^*B+BB^*B) & A \\ A & B & A & B & O \end{pmatrix}\mu_2)+r(B) \\
&= r\begin{pmatrix} -AA^*A & -BB^*B & -\frac{1}{4}(AA^*A+BB^*A) & -\frac{1}{4}(AA^*B+BB^*B) \\ A & B & A & B \end{pmatrix}+r(B)+r(A) \\
&= r\begin{pmatrix} AA^*A & BB^*B & AA^*A+BB^*A & AA^*B+BB^*B \\ A & B & 4A & 4B \end{pmatrix}+r(B)+r(A) \\
&= r(\begin{pmatrix} AA^*A & BB^*B & AA^*A+BB^*A & AA^*B+BB^*B \\ A & B & 4A & 4B \end{pmatrix}\mu_3\mu_4)+r(B)+r(A) \\
&= r\begin{pmatrix} AA^*A & BB^*B & BB^*A & AA^*B \\ A & B & 3A & 3B \end{pmatrix}+r(B)+r(A). \qquad (3.16)
\end{aligned}
$$

From the formulas (3.13), (3.14) and (3.16), we have the results in Corollary 3.3. □

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