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THE GENERALIZED CROFOOT TRANSFORM

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Abstract. We introduce a generalized Crofoot transform between the model spaces corresponding to matrix-valued inner functions. As an application, we obtain results about matrix-valued truncated Toeplitz operators.

1. Introduction

The theory of completely nonunitary contractions on a Hilbert space, as developed in [12], provides functional models for arbitrary completely nonunitary contractions. In the particular case when the dimensions of the defect spaces of the contraction (to be defined below) is 1 and the contraction is stable, the model space is the function space $H^2 \ominus \theta H^2$, where H^2 is the Hardy–Hilbert space and θ is an inner function. These spaces are often called shortly *model spaces* and have been the object of extensive study in the last decades. In particular, a direction of study initiated in [11] deals with the so-called *truncated Toeplitz operators*, which are compression to model spaces of multiplication operators. The Crofoot transform, introduced in [6], is a useful tool for transferring properties between model spaces and between the associated spaces of truncated Toeplitz operators.

A more general type of model space is obtained when the scalar inner function is replaced by a matrix-valued inner function Θ. Then the space $K_\Theta = H^2(E) \ominus \Theta H^2(E)$, with E a finite dimensional Hilbert space. In this context, matrix valued truncated Toeplitz operators and their properties has been formally introduced in [9].

The current paper introduces the generalization of the Crofoot transform to the model spaces associated to matrix-valued inner functions. As an application, we investigate the behaviour of the space of matrix-valued truncated Toeplitz operators with respect to this transformation.

The structure of the paper is the following. After a section of general preliminaries about spaces of vector and matrix valued functions, we give a primer of the properties of the vector-valued model spaces and models operators. The generalized Crofoot transformation and its link to matrix valued truncated Toeplitz operators is defined in Section 3. In Section 4 we investigate the case when the matrix-valued inner function is complex symmetric.

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One should note that the generalized Crofoot transform that we introduce is related to the study of perturbations of contractions as appearing in [1, 2, 3, 7]. However, we work here in a concrete framework and we obtain explicit results for all the transformations involved.

2. Preliminaries

Let $\mathbb{C}$ denote the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. Throughout the paper $\mathbb{C}^d$ will denote d dimensional complex Hilbert space, and $\mathscr{L}(\mathbb{C}^d)$ the algebra of bounded linear operators on $\mathbb{C}^d$, which may be identified with $d \times d$ matrices.

The space $L^2(\mathbb{C}^d)$ is defined, as usual, by

$$L^2(\mathbb{C}) = \left\{ f : \mathbb{T} \to \mathbb{C}^d : f(e^{it}) = \sum_{-\infty}^{\infty} a_n e^{int} : a_n \in \mathbb{C}^d, \quad \sum_{-\infty}^{\infty} \|a_n\|^2 < \infty \right\},$$

endowed with the inner product

$$\langle f, g \rangle_{L^2(\mathbb{C}^d)} = \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{it}), g(e^{it}) \rangle_{\mathbb{C}^d} \, dt.$$

The Hardy space $H^2(\mathbb{C}^d)$ is the subspace of $L^2(\mathbb{C}^d)$ formed by the functions with vanishing negative Fourier coefficients; it can be identified with a space of $\mathbb{C}^d$-valued functions analytic in $\mathbb{D}$, from which the boundary values can be recovered almost everywhere through radial limits.

Let S denote the forward shift operator $(Sf)(z) = zf(z)$ on $H^2(\mathbb{C}^d)$; it is the restriction of M_z, the multiplication with the variable z, to $H^2(\mathbb{C}^d)$. Its adjoint (the backward shift) is the operator

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}.$$

An *inner function* will be an element $\Theta \in H^2(\mathscr{L}(\mathbb{C}^d)$ whose boundary values are almost everywhere unitary operators (equivalently, isometries or coisometries) in $\mathscr{L}(\mathbb{C}^d)$. All niner functions in the sequel are assumed to be pure, that is $\|\Theta(0)\| < 1$.

The *model space* associated to Θ, denoted by K_Θ, is defined by $K_\Theta = H^2(\mathbb{C}^d) \ominus \Theta H^2(\mathbb{C}^d)$; the orthogonal projection onto K_Θ will be denoted by P_Θ. The properties of the model space are familiar to many analysts in the scalar case. On the other hand, the vector valued version is less widely known (despite playing an important role in the Sz.-Nagy–Foias theory of contractions [12]).

The model space K_Θ is a vector valued reproducing kernel Hilbert space; its reproducing kernel function, which takes values in $\mathscr{L}(\mathbb{C}^d)$, is

$$k_\lambda^\Theta(z) = \frac{1}{1 - \overline{\lambda} z} (I - \Theta(z)\Theta(\lambda)^*).$$

This means that for any $x \in \mathbb{C}^d$ we have $k_\lambda^\Theta x \in K_\Theta$, and, if $f \in K_\Theta$, then

$$\langle f, k_\lambda^\Theta x \rangle_{K_\Theta} = \langle f(\lambda), x \rangle_{\mathbb{C}^d}.$$

We will also have the occasion to consider a related family of functions, namely

$$\widetilde{k_\lambda^\Theta}(z) = \frac{1}{z-\lambda}(\Theta(z) - \Theta(\lambda)).$$

The *model operator* $S_\Theta \in \mathscr{L}(K_\Theta)$ is defined by the formula

$$(S_\Theta f)(z) = P_\Theta(zf), \quad f \in K_\Theta. \tag{2.1}$$

The adjoint of S_Θ is given by

$$(S_\Theta^* f)(z) = \frac{f(z) - f(0)}{z};$$

it is the restriction of the left shift in $H^2(\mathbb{C}^d)$ to the S^*-invariant subspace K_Θ. The action of S_Θ is more precisely described if we introduce the following subspaces of K_Θ (the defect spaces of S_Θ in the terminology of [12]):

$$\begin{aligned} \mathscr{D}_* &= \left\{ \frac{1}{z}(\Theta(z) - \Theta(0))x : x \in \mathbb{C}^d \right\} \\ \mathscr{D} &= \{(I - \Theta(z)\Theta(0)^*)x : x \in \mathbb{C}^d\}. \end{aligned} \tag{2.2}$$

The action of S_Θ on $\mathscr{D}^\perp$, $\mathscr{D}$ and of S_Θ^* on $\mathscr{D}_*^\perp$, $\mathscr{D}_*$, are given by the formula's below:

$$\begin{aligned} (S_\Theta^* f)(z) &= \begin{cases} \frac{f(z)}{z} & \text{for } f \in D^\perp, \\ -\frac{1}{z}\big(\Theta(z) - \Theta(0)\big)\Theta(0)^* x & \text{for } f = \big(I - \Theta(z)\Theta(0)^*\big)x \in D; \end{cases} \\ (S_\Theta f)(z) &= \begin{cases} z f(z) & \text{for } f \in D_*^\perp, \\ -\big(I - \Theta(z)\Theta(0)^*\big)\Theta(0)x & \text{for } f = \frac{1}{z}\big(\Theta(z) - \Theta(0)\big)x \in D_*. \end{cases} \end{aligned} \tag{2.3}$$

We will use the following standard notation. If $T \in \mathscr{L}(E)$ is a contraction, then the operators $D_T = (I - T^*T)^{\frac{1}{2}}$ and $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ are called the defect operators and $\mathscr{D}_T = \overline{D_T E}$ and $\mathscr{D}_{T^*} = \overline{D_{T^*} E}$ are called the defect spaces of T.

3. Generalized Crofoot transform

Let $\Theta(\lambda) : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ be a pure inner function and W a fixed strict contraction acting on $\mathbb{C}^d$.

PROPOSITION 3.1. *The function Θ' defined in terms of inner function Θ and strict contraction W given by*

$$\Theta^{'}(\lambda) = -W + D_{W^*}(I - \Theta(\lambda)W^*)^{-1}\Theta(\lambda)D_W \tag{3.1}$$

is a pure inner function.

Proof. Consider

$$
\begin{aligned}
\Theta'(e^{it})\Theta'^{*}(e^{it}) &= [-W + D_{W^*}(I-\Theta(e^{it})W^*)^{-1}\Theta(e^{it})D_W] \\
&\quad [-W^* + D_W\Theta(e^{it})^*(I-W\Theta(e^{it})^*)^{-1}D_{W^*}] \\
&= WW^* - WD_W\Theta^*(I-W\Theta^*)^{-1}D_{W^*} - D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W W^* \\
&\quad + D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W^2\Theta^*(I-W\Theta^*)^{-1}D_{W^*} \\
&= WW^* - D_{W^*}W\Theta^*(I-W\Theta^*)^{-1}D_{W^*} - D_{W^*}(I-\Theta W^*)^{-1}\Theta W^* D_{W^*} \\
&\quad + D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W^2\Theta^*(I-W\Theta^*)^{-1}D_{W^*} \\
&= WW^* - D_{W^*}[W\Theta^*(I-W\Theta^*)^{-1} - (I-\Theta W^*)^{-1}\Theta W^* \\
&\quad + (I-\Theta W^*)^{-1}\Theta D_W^2\Theta^*(I-W\Theta^*)^{-1}]D_{W^*},
\end{aligned}
$$

We have

$$
\begin{aligned}
&W\Theta^*(I-W\Theta^*)^{-1} - (I-\Theta W^*)^{-1}\Theta W^* + (I-\Theta W^*)^{-1}\Theta D_W^2\Theta^*(I-W\Theta^*)^{-1} \\
&\quad = W\Theta^*(I-W\Theta^*)^{-1} - (I-\Theta W^*)^{-1}\Theta W^* \\
&\qquad + (I-\Theta W^*)^{-1}\Theta(I-W^*W)\Theta^*(I-W\Theta^*)^{-1} \\
&\quad = W\Theta^*(I-W\Theta^*)^{-1} - (I-\Theta W^*)^{-1}\Theta W^* \\
&\qquad + (I-\Theta W^*)^{-1}(I-W\Theta^*)^{-1} + (I-\Theta W^*)^{-1}\Theta W^*W\Theta^*(I-W\Theta^*)^{-1} \\
&\quad = W\Theta^*(I-W\Theta^*)^{-1} - (I-\Theta W^*)^{-1}(I-W\Theta^*)^{-1} \\
&\qquad + (I-\Theta W^*)^{-1}\Theta W^* + (I-\Theta W^*)^{-1}\Theta W^*W\Theta^*(I-W\Theta^*)^{-1} \\
&\quad = [W\Theta^* - (I-\Theta W^*)^{-1}](I-W\Theta^*)^{-1} \\
&\qquad + (I-\Theta W^*)^{-1}\Theta W^*(I-W\Theta^*)(I-W\Theta^*)^{-1} \\
&\qquad + (I-\Theta W^*)^{-1}\Theta W^*W\Theta^*(I-W\Theta^*)^{-1} \\
&\quad = [W\Theta^* - (I-\Theta W^*)^{-1}](I-W\Theta^*)^{-1} \\
&\qquad + (I-\Theta W^*)^{-1}[\Theta W^*(I-W\Theta^*) + \Theta W^*W\Theta^*](I-W\Theta^*)^{-1} \\
&\quad = [W\Theta^* - (I-\Theta W^*)^{-1}](I-W\Theta^*)^{-1} \\
&\qquad + (I-\Theta W^*)^{-1}\Theta W^*(I-W\Theta^*)^{-1} \\
&\quad = [W\Theta^* - (I-\Theta W^*)^{-1} + (I-\Theta W^*)^{-1}\Theta W^*](I-W\Theta^*)^{-1} \\
&\quad = [W\Theta^* - (I-\Theta W^*)^{-1}(I-\Theta W^*)](I-W\Theta^*)^{-1} \\
&\quad = (W\Theta^* - I)(I-W\Theta^*)^{-1} = -(I-W\Theta^*)(I-W\Theta^*)^{-1} = -I.
\end{aligned}
$$

Therefore

$$\Theta'(e^{it})\Theta'^{*}(e^{it}) = WW^* + D_{W^*}^2 = I,$$

and so Θ' is inner. We leave to the reader to check that Θ' is pure. □

REMARK 3.2. The function Θ can be obtained from Θ' as

$$\Theta(\lambda) = W + D_{W^*}(I+\Theta'(\lambda)W^*)^{-1}\Theta'(\lambda)D_W.$$

Let K_Θ be the model space corresponding to inner function Θ and $K_{\Theta'}$ be model space corresponding to Θ'. We introduce now the generalized Crofoot transformation between these spaces.

THEOREM 3.3. (Generalized Crofoot transformation) *Let W be a strict contraction, Θ a pure inner function, and suppose Θ' is defined by (3.1). Then the map J_W defined by*

$$J_W f = D_{W^*}(I-\Theta(\lambda)W^*)^{-1} f$$

is a unitary operator from K_Θ to $K_{\Theta'}$.

To prove Theorem 3.3 we first prove the following proposition:

PROPOSITION 3.4. *Let $y \in E$ and $\lambda \in \mathbb{D}$, then*

$$J_W(k_\lambda^\Theta (I-W\Theta(\lambda)^*)^{-1} D_{W^*} y) = k_\lambda^{\Theta'} y, \quad J_W(\widetilde{k_\lambda^\Theta}(I-W^*\Theta(\lambda))^{-1} D_W y) = \widetilde{k_\lambda^{\Theta'}} y. \tag{3.2}$$

Proof. We have

$$\begin{aligned}
&(I-\Theta'(z)\Theta'(\lambda)^*)y \\
&\quad = y - (-W + D_{W^*}(I-\Theta(z)W^*)^{-1}\Theta(z)D_W) \\
&\qquad\quad (-W^* + D_W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}D_{W^*})y \\
&\quad = (I-WW^*)y + WD_W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}D_{W^*}y \\
&\qquad + D_{W^*}(I-\Theta(z)W^*)^{-1}\Theta(z)D_W W^* y \\
&\qquad - D_{W^*}(I-\Theta(z)W^*)^{-1}\Theta(z)D_W^2\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}D_{W^*}y \\
&\quad = D_{W^*}^2 y + D_{W^*}W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}D_{W^*}y \\
&\qquad + D_{W^*}(I-\Theta(z)W^*)^{-1}\Theta(z)W^*D_{W^*}y \\
&\qquad - D_{W^*}(I-\Theta(z)W^*)^{-1}\Theta(z)D_W^2\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}D_{W^*}y \\
&\quad = D_{W^*}[I + W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1} + (I-\Theta(z)W^*)^{-1}\Theta(z)W^* \\
&\qquad - (I-\Theta(z)W^*)^{-1}\Theta(z)D_W^2\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}]D_{W^*}y \\
&\quad = D_{W^*}(I-\Theta(z)W^*)^{-1}[(I-\Theta(z)W^*) \\
&\qquad + (I-\Theta(z)W^*)W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1} \\
&\qquad + \Theta(z)W^* - \Theta(z)D_W^2\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}]D_{W^*}y \\
&\quad = D_{W^*}(I-\Theta(z)W^*)^{-1}[I + (I-\Theta(z)W^*)W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1} \\
&\qquad - \Theta(z)D_W^2\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}]D_{W^*}y \\
&\quad = D_{W^*}(I-\Theta(z)W^*)^{-1}[I + (I-\Theta(z)W^*)W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1} \\
&\qquad - \Theta(z)(I-WW^*)\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}]D_{W^*}y \\
&\quad = D_{W^*}(I-\Theta(z)W^*)^{-1}[I + W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1} \\
&\qquad - \Theta(z)W^*W\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1} - \Theta(z)\Theta(\lambda)^*(I-W\Theta(\lambda)^*)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \Theta(z)W^*W\Theta(\lambda)^*(I - W\Theta(\lambda)^*)^{-1}]D_{W^*}y \\
&= D_{W^*}(I - \Theta(z)W^*)^{-1}[I - W\Theta(\lambda)^* + W\Theta(\lambda)^* \\
&\quad - \Theta(z)\Theta(\lambda)^*](I - W\Theta(\lambda)^*)^{-1}D_{W^*}y \\
&= D_{W^*}(I - \Theta(z)W^*)^{-1}[I - \Theta(z)\Theta(\lambda)^*](I - W\Theta(\lambda)^*)^{-1}D_{W^*}y \\
&= J_W(I - \Theta(z)\Theta(\lambda)^*)(I - W\Theta(\lambda)^*)^{-1}D_{W^*}y.
\end{aligned}
$$

It follows that $J_W(k_\lambda^\Theta(I - W\Theta(\lambda)^*)^{-1}D_{W^*}y) = k_\lambda^{\Theta'}y$.

For the other equality, we have

$$
\begin{aligned}
\widetilde{k_\lambda^{\Theta'}}y &= \frac{1}{z-\lambda}(\Theta'(z) - \Theta'(\lambda))y \\
&= \frac{1}{z-\lambda}[-W + D_{W^*}(I - \Theta(z)W^*)^{-1}\Theta(z)D_W + W \\
&\quad - D_{W^*}(I - \Theta(\lambda)W^*)^{-1}\Theta(\lambda)D_W]y \\
&= \frac{1}{z-\lambda}[D_{W^*}(I - \Theta(z)W^*)^{-1}\Theta(z)D_W - D_{W^*}(I - \Theta(\lambda)W^*)^{-1}\Theta(\lambda)D_W]y \\
&= \frac{1}{z-\lambda}D_{W^*}[(I - \Theta(z)W^*)^{-1}\Theta(z) - (I - \Theta(\lambda)W^*)^{-1}\Theta(\lambda)]D_Wy \\
&= \frac{1}{z-\lambda}D_{W^*}[(I - \Theta(z)W^*)^{-1}\Theta(z) - (I - \Theta(z)W^*)^{-1}\Theta(\lambda) + (I - \Theta(z)W^*)^{-1}\Theta(\lambda) \\
&\quad - (I - \Theta(\lambda)W^*)^{-1}\Theta(\lambda)]D_Wy \\
&= \frac{1}{z-\lambda}D_{W^*}(I - \Theta(z)W^*)^{-1}[\Theta(z) - (I - \Theta(z)W^*)(I - \Theta(\lambda)W^*)^{-1}\Theta(\lambda)]D_Wy \\
&= \frac{1}{z-\lambda}D_{W^*}(I - \Theta(z)W^*)^{-1}[\Theta(z) - (I - \Theta(\lambda)W^*)^{-1}\Theta(\lambda) \\
&\quad + \Theta(z)W^*(I - \Theta(\lambda)W^*)^{-1}\Theta(\lambda)]D_Wy \\
&= \frac{1}{z-\lambda}D_{W^*}(I - \Theta(z)W^*)^{-1}[\Theta(z) - \Theta(\lambda)(I - W^*\Theta(\lambda))^{-1} \\
&\quad + \Theta(z)W^*\Theta(\lambda)(I - W^*\Theta(\lambda))^{-1}]D_Wy \\
&= \frac{1}{z-\lambda}D_{W^*}(I - \Theta(z)W^*)^{-1}[\Theta(z)(I - W^*\Theta(\lambda)) - \Theta(\lambda) \\
&\quad + \Theta(z)W^*\Theta(\lambda)](I - W^*\Theta(\lambda))^{-1}D_Wy \\
&= \frac{1}{z-\lambda}D_{W^*}(I - \Theta(z)W^*)^{-1}[\Theta(z) - \Theta(\lambda)](I - W^*\Theta(\lambda))^{-1}D_Wy \\
&= D_{W^*}(I - \Theta(z)W^*)^{-1}\left(\frac{1}{z-\lambda}(\Theta(z) - \Theta(\lambda))\right)(I - W^*\Theta(\lambda))^{-1}D_Wy \\
&= J_W(\widetilde{k_\lambda^\Theta}(I - W^*\Theta(\lambda))^{-1}D_Wy. \quad \square
\end{aligned}
$$

Proof of Theorem 3.3. First we claim that $J_WK_\Theta \subset K_{\Theta'}$. To show that J_Wf belong to $K_{\Theta'}$ for every $f \in K_\Theta$, we must show that J_Wf is orthogonal to every function of the form $\Theta' g$ where $g \in H^2(E)$. This follows from the following computation. Note here we use the fact that $\Theta(e^{it})\Theta^*(e^{it}) = \Theta^*(e^{it})\Theta(e^{it}) = I$ almost everywhere on $\mathbb{T}$.

$$\begin{aligned}
\langle J_W f, \Theta' g\rangle &= \langle D_{W^*}(I-\Theta(e^{it})W^*)^{-1}f, \Theta' g\rangle\\
&= \langle f, (I-W\Theta(e^{it})^*)^{-1}D_{W^*}\Theta' g\rangle\\
&= \langle f, (I-W\Theta(e^{it})^*)^{-1}D_{W^*}[-W+D_{W^*}(I-\Theta(e^{it})W^*)^{-1}\Theta(e^{it})D_W]g\rangle\\
&= \langle f, [-(I-W\Theta^*)^{-1}D_{W^*}W+(I-W\Theta^*)^{-1}D^2_{W^*}(I-\Theta W^*)^{-1}\Theta D_W]g\rangle\\
&= \langle f, [-(I-W\Theta^*)^{-1}WD_W+(I-W\Theta^*)^{-1}D^2_{W^*}(I-\Theta W^*)^{-1}\Theta D_W]g\rangle\\
&= \langle f, (I-W\Theta^*)^{-1}[-W+D^2_{W^*}\Theta(I-W^*\Theta)^{-1}]D_W g\rangle\\
&= \langle f, (I-W\Theta^*)^{-1}[-W(I-W^*\Theta)(I-W^*\Theta)^{-1}+D^2_{W^*}\Theta(I-W^*\Theta)^{-1}]D_W g\rangle\\
&= \langle f, (I-W\Theta^*)^{-1}[-W(I-W^*\Theta)+(I-WW^*)\Theta](I-W^*\Theta)^{-1}D_W g\rangle\\
&= \langle f, (I-W\Theta^*)^{-1}[-W+WW^*\Theta+\Theta-WW^*\Theta](I-W^*\Theta)^{-1}D_W g\rangle\\
&= \langle f, (I-W\Theta(e^{it})^*)^{-1}[\Theta(e^{it})-W](I-W^*\Theta(e^{it}))^{-1}D_W g\rangle\\
&= \langle f, (I-W\Theta(e^{it})^*)^{-1}(I-W\Theta(e^{it})^*)\Theta(e^{it})(I-W^*\Theta(e^{it}))^{-1}D_W g\rangle\\
&= \langle f, \Theta(e^{it})(I-W^*\Theta(e^{it}))^{-1}D_W g\rangle\\
&= 0,
\end{aligned}$$

because the function $\Theta(e^{it})(I-W^*\Theta(e^{it}))^{-1}D_W g \in \Theta H^2(E)$. Hence it follows that $J_W K_\Theta \subset K_{\Theta'}$.

Now define the operator $J_W' : K_{\Theta'} \longrightarrow K_\Theta$ by

$$J_W' g = D_{W^*}(I+\Theta' W^*)^{-1}g, \quad \forall g \in K_{\Theta'}. \tag{3.3}$$

First we show that $J_W' K_{\Theta'} \subset K_\Theta$. For this purpose we will prove that $J_W' g$ is orthogonal to Θh for any $g \in K_{\Theta'}$ and any $h \in H^2(E)$. We have

$$\begin{aligned}
\langle J_W' g, \Theta h\rangle &= \langle D_{W^*}(I+\Theta' W^*)^{-1}f, \Theta g\rangle = \langle f, (I+W\Theta'^*)^{-1}D_{W^*}\Theta g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}D_{W^*}[W+D_{W^*}(I+\Theta')W^*)^{-1}\Theta' D_W]g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}[D_{W^*}W+D^2_{W^*}(I+\Theta' W^*)^{-1}\Theta' D_W]g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}[WD_W+D^2_{W^*}(I+\Theta' W^*)^{-1}\Theta' D_W]g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}[W+D^2_{W^*}\Theta'(I+W^*\Theta')^{-1}]D_W g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}[W(I+W^*\Theta')+D^2_{W^*}\Theta'](I+W^*\Theta')^{-1}D_W g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}[W(I+W^*\Theta')+(I-WW^*)\Theta'](I+W^*\Theta')^{-1}D_W g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}[W+WW^*\Theta'+\Theta'-WW^*\Theta'](I+W^*\Theta')^{-1}D_W g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}[\Theta'+W](I+W^*\Theta'))^{-1}D_W g\rangle\\
&= \langle f, (I+W\Theta'^*)^{-1}(I+W\Theta'^*)\Theta'(I+W^*\Theta')^{-1}D_W g\rangle\\
&= \langle f, \Theta'(I+W^*\Theta'))^{-1}D_W g\rangle\\
&= 0,
\end{aligned}$$

and so $J_W^{'} K_{\Theta'} \subset K_\Theta$.

Next we prove that $J_W^{'}$ is the inverse of J_W. If $f \in K_\Theta$, then

$$\begin{aligned}
J_W^{'} J_W f &= D_{W^*}(I+\Theta' W^*)^{-1} D_{W^*}(I-\Theta W^*)^{-1} f \\
&= D_{W^*}[I+(-W+D_{W*}(I-\Theta W^*)^{-1}\Theta D_W)W^*]^{-1} D_{W^*}(I-\Theta W^*)^{-1} f \\
&= D_{W^*}[I-WW^*+D_{W^*}(I-\Theta W^*)^{-1}\Theta W^* D_{W^*}]^{-1} D_{W^*}(I-\Theta W^*)^{-1} f \\
&= D_{W^*}[D_{W^*}^2+D_{W^*}(I-\Theta W^*)^{-1}\Theta W^* D_{W^*}]^{-1} D_{W^*}(I-\Theta W^*)^{-1} f \\
&= D_{W^*}[D_{W^*}^{-2}+D_{W^*}^{-1} W^{*-1}\Theta^{-1}(I-\Theta W^*) D_{W^*}^{-1}] D_{W^*}(I-\Theta W^*)^{-1} f \\
&= [I+W^{*-1}\Theta^{-1}(I-\Theta W^*)](I-\Theta W^*)^{-1} f \\
&= [I+(\Theta W^*)^{-1}(I-\Theta W^*)](I-\Theta W^*)^{-1} f \\
&= (I-\Theta W^*)^{-1} f+(\Theta W^*)^{-1} f \\
&= (I-\Theta W^*)^{-1} f+(\Theta W^*)^{-1} f+f-f \\
&= (I-\Theta W^*)^{-1} f-(I-\Theta W^*)^{-1} f+f=f.
\end{aligned}$$

For $g \in K_{\Theta'}$ we have

$$\begin{aligned}
J_W J_W^{'} g &= D_{W^*}(I-\Theta W^*)^{-1} D_{W^*}(I+\Theta' W^*)^{-1} g \\
&= D_{W^*}(I-\Theta W^*)^{-1} D_{W^*}[I+(-W+D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W)W^*]^{-1} g \\
&= D_{W^*}(I-\Theta W^*)^{-1} D_{W^*}[I-WW^*+D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W W^*]^{-1} g \\
&= D_{W^*}(I-\Theta W^*)^{-1} D_{W^*}[D_{W^*}^2+D_{W^*}(I-\Theta W^*)^{-1}\Theta W^* D_{W^*}]^{-1} g \\
&= D_{W^*}(I-\Theta W^*)^{-1} D_{W^*}[D_{W^*}^{-2}+D_{W^*}^{-1} W^{*-1}\Theta^{-1}(I-\Theta W^*) D_{W^*}^{-1}] g \\
&= D_{W^*}(I-\Theta W^*)^{-1}[D_{W^*}^{-1}+W^{*-1}\Theta^{-1}(I-\Theta W^*) D_{W^*}^{-1}] g \\
&= D_{W^*}(I-\Theta W^*)^{-1}[I+W^{*-1}\Theta^{-1}(I-\Theta W^*)] D_{W^*}^{-1} g \\
&= D_{W^*}(I-\Theta W^*)^{-1}(\Theta W^*)^{-1} D_{W^*}^{-1} g \\
&= D_{W^*}[(\Theta W^*)^{-1}+(I-\Theta W^*)^{-1}] D_{W^*}^{-1} g \\
&= D_{W^*}[I-I+(\Theta W^*)^{-1}+(I-\Theta W^*)^{-1}] D_{W^*}^{-1} g \\
&= D_{W^*}[I-(I-\Theta W^*)^{-1}+(I-\Theta W^*)^{-1}] D_{W^*}^{-1} g \\
&= g.
\end{aligned}$$

The above computation shows that $J_W^{'}$ is the inverse of J_W and $J_W K_\Theta = K_{\Theta'}$.

We now show that J_W is a unitary operator. By using Proposition 3.4 we obtain

$$\begin{aligned}
\langle J_W k_\lambda^\Theta x, J_W k_\mu^\Theta y\rangle &= \langle J_W k_\lambda^\Theta x, k_\mu^{\Theta'} D_{W^*}^{-1}(I-W\Theta^*(\mu))y\rangle \\
&= \langle J_W k_\lambda^\Theta(\mu)x, D_{W^*}^{-1}(I-W\Theta^*(\mu))y\rangle \\
&= \langle D_{W^*}(I-\Theta(\mu))W^*)^{-1} k_\lambda^\Theta(\mu)x, D_{W^*}^{-1}(I-W\Theta^*(\mu))y\rangle \\
&= \langle (I-\Theta(\mu)W^*) D_{W^*}^{-1} D_{W^*}(I-\Theta(\mu))W^*)^{-1} k_\lambda^\Theta(\mu)x, y\rangle \\
&= \langle k_\lambda^\Theta(\mu)x, y\rangle = \langle k_\lambda^\Theta x, k_\mu^\Theta y\rangle.
\end{aligned}$$

Therefore

$$\langle J_W f, J_W g\rangle = \langle f, g\rangle$$

for any f, g in the linear span of $k_\lambda^\Theta x$, $\lambda \in \mathbb{D}$, $x \in \mathbb{C}$. The required result follows by the density of this last set in K_Θ. $\square$

REMARK 3.5. The defect spaces of $S_{\Theta'}$ in terminology of [12] are given by

$$\begin{aligned} \mathscr{D}'_* &= \left\{\frac{1}{z}(\Theta'(z) - \Theta'(0))x : x \in \mathbb{C}^d\right\} \\ \mathscr{D}' &= \{(I - \Theta'(z)\Theta'(0)^*)x : x \in \mathbb{C}^d\}. \end{aligned} \tag{3.4}$$

COROLLARY 3.6.
(i) $f \in \mathscr{D}_*{}^\perp$ *if and only if* $J_W f \in \mathscr{D}'^\perp_*$.
(ii) $g \in \mathscr{D}'^\perp_*$ *if and only if* $J_W^* g \in \mathscr{D}_*{}^\perp$

Proof. (i) By using Proposition 3.4 we have

$$\langle J_W f, \widetilde{k_0^{\Theta'}} x\rangle = \langle f, J_W^* \widetilde{k_0^{\Theta'}} x\rangle = \langle f, \widetilde{k_0^{\Theta}} D_W y\rangle = 0.$$

(ii) Let $f \in \mathscr{D}'^\perp_*$ and $D_W y = x$ then by Proposition 3.4 we obtain

$$\langle J_W^* g, \widetilde{k_0^{\Theta}} x\rangle = \langle g, J_W \widetilde{k_0^{\Theta}} x\rangle = \langle g, \widetilde{k_0^{\Theta'}} y\rangle = 0. \quad \square$$

PROPOSITION 3.7. *Let* $f \in K_\Theta$, *we have*

$$S_{\Theta'}^* J_W f = J_W S_\Theta^* f + S_{\Theta'}^* J_W f(0).$$

Proof. Let $f \in K_\Theta$, then

$$\begin{aligned} S_{\Theta'}^* J_W f &= S_{\Theta'}^* [D_{W^*}(I - \Theta(z)W^*)^{-1} f] \\ &= \frac{1}{z}\Big(D_{W^*}(I - \Theta(z)W^*)^{-1} f(z) - D_{W^*}(I - \Theta(0)W^*)^{-1} f(0)\Big) \\ &= D_{W^*}\frac{1}{z}\Big((I - \Theta(z)W^*)^{-1} f(z) - (I - \Theta(0)W^*)^{-1} f(0)\Big) \\ &= D_{W^*}\frac{1}{z}\Big((I - \Theta(z)W^*)^{-1} f(z) - (I - \Theta(z)W^*)^{-1} f(0) \\ &\quad + (I - \Theta(z)W^*)^{-1} f(0) - (I - \Theta(0)W^*)^{-1} f(0)\Big) \\ &= D_{W^*}(I - \Theta(z)W^*)^{-1}\frac{1}{z}(f(z) - f(0)) \\ &\quad + D_{W^*}\frac{1}{z}((I - \Theta(z)W^*)^{-1} f(0) - (I - \Theta(0)W^*)^{-1} f(0)) \end{aligned}$$

$$
\begin{aligned}
&= D_{W^*}(I-\Theta(z)W^*)^{-1}\frac{1}{z}(f(z)-f(0)) \\
&\quad + \frac{1}{z}(D_{W^*}(I-\Theta(z)W^*)^{-1}f(0) - D_{W^*}(I-\Theta(0)W^*)^{-1}f(0)) \\
&= D_{W^*}(I-\Theta(z)W^*)^{-1}S_\Theta^* f + S_{\Theta'}^*(D_{W^*}(I-\Theta(z)W^*)^{-1}f(0)) \\
&= J_W S_\Theta^* f + S_{\Theta'}^* J_W f(0). \quad \square
\end{aligned}
$$

LEMMA 3.8. *$S_{\Theta'} J_W f = J_W S_\Theta f$ for $f \in \mathscr{D}_*^\perp$.*

Proof. Let $f \in \mathscr{D}^\perp$; so $f \perp k_0^\Theta x$ for any $x \in \mathbb{C}$, which by the reproducing kernel property of k_0^Θ is equivalent to $f(0) = 0$. So from Proposition 3.7 it follows that

$$S_{\Theta'}^* J_W f = J_W S_\Theta^* f \text{ for } f \in \mathscr{D}^\perp. \tag{3.5}$$

Now by (2.3), it follows that S_Θ^* is a unitary (division by z) from $\mathscr{D}^\perp$ to $\mathscr{D}_*^\perp$ (and similarly for Θ'). On the other hand, from Proposition 3.4 it follows that J_W maps (unitarily) $\mathscr{D}^\perp$ to $\mathscr{D}'^\perp$, and $\mathscr{D}_*^\perp$ to $\mathscr{D}_*'^\perp$. Using (3.5), we have the following commutative diagram of unitary operators:

$$
\begin{array}{ccc}
\mathcal{D}^\perp & \xrightarrow{S_\Theta^*} & \mathcal{D}_*^\perp \\
\big\downarrow J_W & & \big\downarrow J_W \\
\mathcal{D}'^\perp & \xrightarrow{S_{\Theta'}^*} & \mathcal{D}_*'^\perp .
\end{array}
$$

From the operators in above diagram as acting between these spaces, we have

$$S_{\Theta'}^* J_W = J_W S_\Theta^*;$$

by passing to the adjoint we get

$$J_W^* S_{\Theta'} = S_\Theta J_W^*,$$

where the two sides act from $\mathscr{D}_*'^\perp$ to $\mathscr{D}^\perp$, and then multiplying on the left and on the right with J_W,

$$S_{\Theta'} J_W = J_W S_\Theta,$$

where the two sides act from $\mathscr{D}_*^\perp$ to $\mathscr{D}'^\perp$. This completes the proof. $\square$

A characterization of matrix valued truncated Toeplitz operators is obtained (see Theorem 5.5 in [9]) by shift invariance. A bounded operator A on K_Θ is called shift invariant if

$$f, Sf \in K_\Theta \quad \textit{implies} \quad Q_A(Sf) = Q_A(f),$$

where Q_A is associated quadratic form on K_Θ defined by $Q_A(f) = \langle Af, f\rangle$. It is well known that $S_\Theta f \in K_\Theta$ if and only if $f \in \mathscr{D}_*^\perp$.

THEOREM 3.9. [9] *A bounded operator A on K_Θ is a matrix valued truncated Toeplitz operator if and only if A is shift invariant.*

The spaces of matrix valued truncated Toeplitz operators on K_Θ and $K_{\Theta'}$ are denoted respectively by $\mathscr{T}_\Theta$ and $\mathscr{T}_{\Theta'}$. The next result shows the action of the generalized Crofoot transform.

THEOREM 3.10. $\mathscr{T}_\Theta = J_W^* \mathscr{T}_{\Theta'} J_W$.

Proof. Let $A \in \mathscr{T}_{\Theta'}$, then $J_W^* A J_W \in J_W^* \mathscr{T}_{\Theta'} J_W$. We shall show that $J_W^* A J_W \in \mathscr{T}_\Theta$. Assume that $f \in \mathscr{D}_*^\perp$ then by Corollary 3.6 we have $J_W f \in \mathscr{D}_*'^\perp$. By Lemma 3.8 we obtain

$$\begin{aligned}
Q_{J_W^* A J_W}(f) &= \langle J_W^* A J_W f, f\rangle = \langle A J_W f, J_W f\rangle \\
&= \langle A S_{\Theta'} J_W f, S_{\Theta'} J_W f\rangle = \langle A J_W S_\Theta f, J_W S_\Theta f\rangle \\
&= \langle J_W^* A J_W S_\Theta f, S_\Theta f\rangle = Q_{J_W^* A J_W}(S_\Theta f).
\end{aligned}$$

It shows that $J_W^* A J_W \in \mathscr{T}_\Theta$. Therefore by Theorem 3.9 we obtain $J_W^* \mathscr{T}_{\Theta'} J_W \subset \mathscr{T}_\Theta$.

To prove the required equality we now prove the inclusion $J_W \mathscr{T}_\Theta J_W^{-1} \subset \mathscr{T}_{\Theta'}$.

Assume that $B \in \mathscr{T}_\Theta$ then we have $J_W B J_W^* \in J_W \mathscr{T}_\Theta J_W^*$. Let $f \in \mathscr{D}_*'^\perp$ then by Corollary 3.6 we get $J_W^* f \in \mathscr{D}_*^\perp$ and again by Lemma 3.8 we have

$$\begin{aligned}
Q_{J_W B J_W^*}(f) &= \langle J_W B J_W^* f, f\rangle = \langle B J_W^* f, J_W^* f\rangle \\
&= \langle B S_\Theta J_W^* f, S_\Theta J_W^* f\rangle = \langle B J_W^* S_{\Theta'} f, J_W^* S_{\Theta'} f\rangle \\
&= \langle J_W B J_W^* S_{\Theta'} f, S_{\Theta'} f\rangle = Q_{J_W B J_W^*}(S_{\Theta'} f).
\end{aligned}$$

Hence $J_W B J_W^*$ is shift invariant. Again by Theorem 3.9 we have $J_W \mathscr{T}_\Theta J_W^* \subset \mathscr{T}_{\Theta'}$ which implies that $\mathscr{T}_\Theta \subset J_W^* \mathscr{T}_{\Theta'} J_W$. The required result follows. □

4. Conjugation and Crofoot transform

A bounded linear operator T on a separable Hilbert space E is complex symmetric if there exist an orthonormal basis for E with respect to which T has self-transpose matrix representation. An equivalent definition also exist and involve conjugation. A *conjugation* on a Hilbert space E is a conjugate-linear, isometric and involutive map. We say that T is C-symmetric if $T = CT^*C$, and complex symmetric if there exist a conjugation C with respect to which T is C-symmetric (see [8]).

Let Γ be a conjugation on E and Θ is $\Gamma-$symmetric a.e on $\mathbb{T}$. Then the map $C_\Gamma : L^2(E) \longrightarrow L^2(E)$ defined by

$$C_\Gamma f = \Theta e^{-it} \Gamma f,$$

is conjugation on $L^2(E)$. The following lemma shows the relation, in this case, between conjugation and model spaces.

LEMMA 4.1. [9] *Suppose that* $\Gamma\Theta\Gamma = \Theta^*$ *a.e on* $\mathbb{T}$. *Then* $C_\Gamma K_\Theta = K_\Theta$.

Note that in the scalar case the inner function θ is always C-symmetric with respect to usual complex conjugation, which produces the standard conjugation on the model space K_θ.

Suppose that $\Gamma W^* = W\Gamma$ and $\Gamma\Theta\Gamma = \Theta^*$, then a simple calculation shows that $\Gamma\Theta^{'}\Gamma = \Theta^{*'}$, and the relation $\Gamma D_{W^*} = D_W\Gamma$ also holds.

LEMMA 4.2. *Suppose* C_Γ *is conjugation on* K_Θ *and* $C_\Gamma^{'}$ *is conjugation on* $K_{\Theta'}$. *Then generalized Crofoot transformation intertwines the conjugation on* K_Θ *with the conjugation on* $K_{\Theta'}$, *that is* $J_W C_\Gamma = C_\Gamma^{'} J_W$.

Proof. Let $f \in K_\Theta$, then we have

$$\begin{aligned}
C_\Gamma^{'} J_W f &= \Theta^{'} e^{-it}\Gamma(D_{W^*}(I-\Theta W^*)^{-1}f) \\
&= e^{-it}[-W + D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W]\Gamma(D_{W^*}(I-\Theta W^*)^{-1}f) \\
&= e^{-it}[-W\Gamma D_{W^*}(I-\Theta W^*)^{-1}f + D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W\Gamma D_{W^*}(I-\Theta W^*)^{-1}f] \\
&= e^{-it}[-WD_W\Gamma(I-\Theta W^*)^{-1}f + D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W^2\Gamma(I-\Theta W^*)^{-1}f] \\
&= e^{-it}[-D_{W^*}W\Gamma(I-\Theta W^*)^{-1}f + D_{W^*}(I-\Theta W^*)^{-1}\Theta D_W^2\Gamma(I-\Theta W^*)^{-1}f] \\
&= e^{-it}D_{W^*}[-W + (I-\Theta W^*)^{-1}\Theta D_W^2]\Gamma(I-\Theta W^*)^{-1}f \\
&= e^{-it}D_{W^*}[-(I-\Theta W^*)^{-1}(I-\Theta W^*)W + (I-\Theta W^*)^{-1}\Theta D_W^2]\Gamma(I-\Theta W^*)^{-1}f \\
&= e^{-it}D_{W^*}(I-\Theta W^*)^{-1}[-(I-\Theta W^*)W + \Theta D_W^2]\Gamma(I-\Theta W^*)^{-1}f \\
&= e^{-it}D_{W^*}(I-\Theta W^*)^{-1}[-W + \Theta W^*W + \Theta - \Theta W^*W]\Gamma(I-\Theta W^*)^{-1}f \\
&= e^{-it}D_{W^*}(I-\Theta W^*)^{-1}(\Theta - W)\Gamma(I-\Theta W^*)^{-1}f \\
&= e^{-it}D_{W^*}(I-\Theta W^*)^{-1}\Theta(I-\Theta^*W)\Gamma(I-\Theta W^*)^{-1}f,
\end{aligned}$$

since $(I-\Theta^*W)\Gamma(I-\Theta W^*)^{-1} = \Gamma$ therefore we have

$$\begin{aligned}
C_\Gamma^{'} J_W f &= e^{-it}D_{W^*}(I-\Theta W^*)^{-1}\Theta(I-\Theta^*W)\Gamma(I-\Theta W^*)^{-1}f \\
&= e^{-it}D_{W^*}(I-\Theta W^*)^{-1}\Theta\Gamma f \\
&= D_{W^*}(I-\Theta W^*)^{-1}\Theta e^{it}\Gamma f \\
&= J_W C_\Gamma f. \quad \square
\end{aligned}$$

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