Operators
and
Matrices
Volume 15, Number 1 (2021), 239–252
doi:10.7153/oam-2021-15-17

REFINED HEINZ OPERATOR INEQUALITIES AND NORM INEQUALITIES

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(*Communicated by F. Kittaneh*)

Abstract. In this article we study the Heinz and Hermite-Hadamard inequalities. We derive the whole series of refinements of these inequalities involving unitarily invariant norms, which improve some recent results, known from the literature.

We also prove that if $A,B,X \in M_n(\mathbb{C})$ such that A and B are positive definite and f is an operator monotone function on $(0,\infty)$. Then

$$|||f(A)X - Xf(B)||| \leqslant \max\{||f'(A)||, ||f'(B)||\}|||AX - XB|||.$$

Finally we obtain a series of refinements of the Heinz operator inequalities, which were proved by Kittaneh and Krnić.

1. Introduction and preliminaries

Let $M_{m,n}(\mathbb{C})$ be the space of $m \times n$ complex matrices and $M_n(\mathbb{C}) = M_{n,n}(\mathbb{C})$. Let $|||.|||$ denote any unitarily invariant norm on $M_n(\mathbb{C})$. So, $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. The Hilbert-Schmidt and trace class norm of $A = [a_{ij}] \in M_n(\mathbb{C})$ are denoted by

$$||A||_2 = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{\frac{1}{2}}, \qquad ||A||_1 = \sum_{j=1}^{n} s_j(A)$$

where $s_1(A) \geqslant s_2(A) \geqslant \ldots \geqslant s_n(A)$ are the singular values of A, which are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. For Hermitian matrices $A,B \in M_n(\mathbb{C})$, we write that $A \geqslant 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \geqslant B$ if $A - B \geqslant 0$.

The Heron means introduced by Bhatia in [2] as follows:

$$K_\nu(a,b) = (1-\nu)\sqrt{ab} + \nu\frac{a+b}{2}, \quad 0 \leqslant \nu \leqslant 1.$$

Bhatia derived the inequality

$$H_\nu(a,b) \leqslant K_{\alpha(\nu)}(a,b),$$

Mathematics subject classification (2010): 47A30, 47A63, 15A45.
Keywords and phrases: Norm inequality, operator inequality, Heinz mean.

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where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$.

The another one of means that interpolates between the geometric and the arithmetic means is the logarithmic mean:

$$L(a,b) = \int_0^1 a^\nu b^{1-\nu} d\nu.$$

Drissi in [5] showed that $\frac{\sqrt{3}-1}{2\sqrt{3}} \leqslant \nu \leqslant \frac{\sqrt{3}+1}{2\sqrt{3}}$ if and only if

$$H_\nu(a,b) \leqslant L(a,b). \tag{1.1}$$

R. Kaur and M. Singh [8] have proved that for $A,B,X \in M_n$, such that A,B are positive definite, then for any unitarily invariant norm $|||.|||$, and $\frac{1}{4} \leqslant \nu \leqslant \frac{3}{4}$ and $\alpha \in [\frac{1}{2}, \infty)$,the following inequality holds

$$\frac{1}{2}|||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \leqslant \left|\left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|. \tag{1.2}$$

They also proved the following result:

$$\begin{aligned} |||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leqslant \frac{1}{2}|||A^{\frac{2}{3}}XB^{\frac{1}{3}} + A^{\frac{1}{3}}XB^{\frac{2}{3}}||| \\ &\leqslant \frac{1}{2+t}|||AX + XB + tA^{\frac{1}{2}}XB^{\frac{1}{2}}|||, \end{aligned} \tag{1.3}$$

where $A,B,X \in M_n$, A,B are positive definite and $-2 < t \leqslant 2$.

Obviously, if $A,B,X \in M_n$, such that A,B are positive definite, then for $\frac{1}{4} \leqslant \nu \leqslant \frac{3}{4}$ and $\alpha \in [\frac{1}{2}, \infty)$, and any unitarily invariant norm $|||.|||$, the following inequalities hold

$$\begin{aligned} |||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leqslant \frac{1}{2}|||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \\ &\leqslant \left|\left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|, \end{aligned} \tag{1.4}$$

Suppose that

$$g_\circ(\nu) = \left|\left|\left|\frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2}\right|\right|\right|,$$

and

$$f_\circ(\alpha) = \left|\left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|.$$

Then, the inequalities (1.2), (1.3),(1.4), can be simply rewritten respectively as follows

$$\begin{aligned} &g_\circ(\nu) \leqslant f_\circ(\alpha), \\ &g_\circ\left(\frac{1}{2}\right) \leqslant g_\circ\left(\frac{2}{3}\right) \leqslant f_\circ\left(\frac{2}{2+t}\right), \end{aligned} \tag{1.5}$$

$$g_\circ\left(\frac{1}{2}\right) \leqslant g_\circ(\nu) \leqslant f_\circ(\alpha),$$

I. Ali, H. Yang and A. shakoor [1] gave a refinement of the inequality (1.4) as follows:

$$g_\circ(\nu) \leqslant (4r_0 - 1)g_\circ\left(\frac{1}{2}\right) + 2(1-2r_0)f_\circ(\alpha), \tag{1.6}$$

where $\frac{1}{4} \leqslant \nu \leqslant \frac{3}{4}$, $\alpha \in [\frac{1}{2}, \infty)$ and $r_0 = \min\{\nu, 1-\nu\}$.

Kittaneh [10], gave a generalization of the Heinz inequality using convexity and the Hermite-Hadamard integral inequality for $0 \leqslant \nu \leqslant 1$, as follows:

$$\begin{aligned} 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leqslant \frac{1}{|1-2\nu|}\left|\int_\nu^{1-\nu} |||A^tXB^{1-t} + A^{1-t}XB^t|||dt\right| \\ &\leqslant |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||, \end{aligned} \tag{1.7}$$

A refinement of (1.7) is given in [9]. They also proved that

$$\begin{aligned} &\left|\left|\left|A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}\right|\right|\right| \\ &\leqslant \frac{1}{|\beta-\alpha|}\left|\left|\left|\int_\alpha^\beta (A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu)d\nu\right|\right|\right| \\ &= \frac{1}{2}\left|\left|\left|A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha + A^\beta XB^{1-\beta} + A^{1-\beta}XB^\beta\right|\right|\right|. \end{aligned} \tag{1.8}$$

Heretofore the inequalities discussed above are proved in the setting of matrices. Kapil and Singh in [7], using the contractive maps proved that the relation (1.8) holds for invertible positive operators in $B(H)$. The aim of this paper is to obtain refinements of the Hermite-Hadamard inequality (1.8) in the setting of operators (see Theorem (2)). We also present a generalization of the difference version of Heinz inequality (see Theorem (1)). At the end, we study the Heinz operator inequalities, which were proved in [10] and give a series of refinements of these operator inequalities (see Theorem (4) and (5)).

2. Norm inequalities for matrices

Let $A,B,X \in M_n(\mathbb{C})$ such that A and B be positive definite and $0 \leqslant \nu \leqslant 1$. A difference version of the Heinz inequality

$$|||A^\nu XB^{1-\nu} - A^{1-\nu}XB^\nu||| \leqslant |2\nu - 1|\ |||AX - XB||| \tag{2.1}$$

was proved by Bhatia and Davis in [4].

Kapil, et.al., [6] proved that if $0 < r \leqslant 1$. Then

$$|||A^rX - XB^r||| \leqslant r\max\{||A^{r-1}||, ||B^{r-1}||\}|||AX - XB|||. \tag{2.2}$$

They also proved that if $\alpha \geqslant 1$, and $\frac{1-\alpha}{2} \leqslant \nu \leqslant \frac{1+\alpha}{2}$, then

$$\begin{aligned}\alpha|||A^{\nu}XB^{1-\nu} - A^{1-\nu}XB^{\nu}||| \\ \leqslant |2\nu-1|\max\{||A^{1-\alpha}||,||B^{1-\alpha}||\}|||A^{\alpha}X - XB^{\alpha}|||. \qquad (2.3)\end{aligned}$$

The following theorem is a generalization of (2.2).

THEOREM 1. *Let $A,B,X \in M_n(\mathbb{C})$ such that A and B be positive definite and f be an operator monotone function on $(0,\infty)$. Then*

$$|||f(A)X - Xf(B)||| \leqslant \max\{||f'(A)||,||f'(B)||\}|||AX - XB|||. \qquad (2.4)$$

Proof. It suffices to prove the required inequality in the special case which $A = B$ and A is diagonal. Then the general case follows by replacing A with $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and X with $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$. Therefore let $A = diag(\lambda_i) > 0$. Then $f(A)X - Xf(A) = Y \circ (AX - XA)$ where $Y = f^{[1]}(A)$, i.e.,

$$y_{ij} = \begin{cases} \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}, & \lambda_i \neq \lambda_j \\ f'(\lambda_i), & \lambda_i = \lambda_j. \end{cases}$$

By [3, Theorem V.3.4], $f^{[1]}(A) \geqslant 0$. Consequently

$$\begin{aligned}|||f(A)X - Xf(A)||| &= |||Y \circ (AX - XA)||| \leqslant \max y_{ii}\, |||AX - XA||| \\ &= ||f'(A)||\; |||AX - XA|||. \quad \square\end{aligned}$$

EXAMPLE 1. (i) For the function $f(t) = t^r$, $0 < r < 1$,

$$\begin{aligned}|||A^rX - XB^r||| &\leqslant r\left(\max\{\|A^{r-1}\|,\|B^{r-1}\|\}\right) \\ &= r\left(\max\{\|A^{-1}\|,\|B^{-1}\|\}\right)^{1-r} |||AX - XB|||.\end{aligned}$$

(ii) For the function $f(t) = \log t$ on $(0,\infty)$,

$$|||\log(A)X - X\log(B)||| \leqslant \left(\max\{\|A^{-1}\|,\|B^{-1}\|\}\right) |||AX - XB|||.$$

REMARK 1. Let $\alpha \geqslant 1$ and $0 \leqslant \nu \leqslant 1$. From inequality (2.4) for A^{α}, B^{α} and $f(t) = t^{\frac{1}{\alpha}}$, we get

$$|||AX - XB||| \leqslant \frac{1}{\alpha}\max\{||A^{1-\alpha}||,||B^{1-\alpha}||\}|||A^{\alpha}X - XB^{\alpha}|||. \qquad (2.5)$$

On combining (2.1), and (2.5), we obtain (2.3).

3. Norm inequalities for operators

Let $B(H)$ denote the set of all bounded linear operators on a complex Hilbert space H. An operator $A \in B(H)$ is positive, and we write $A \geqslant 0$, if $(Ax,x) \geqslant 0$ for every vector $x \in H$. If A and B are self-adjoint operators, the order relation $A \geqslant B$ means, as usual, that $A - B$ is a positive operator.

To reach inequalities for bounded self-adjoint operators on Hilbert space, we shall use the following monotonicity property for operator functions:

If $X \in B(H)$ is self adjoint with a spectrum $Sp(X)$, and f,g are continuous real valued functions on an interval containing $Sp(X)$, then

$$f(t) \geqslant g(t),\ t \in Sp(X) \Rightarrow\ f(X) \geqslant g(X). \tag{3.1}$$

For more details about this property, the reader is referred to [14].

Let L_X, R_Y denote the left and right multiplication maps on $B(H)$, respectively, that is, $L_X(T) = XT$ and $R_Y(T) = TY$. Since L_X and R_Y commute, we have

$$e^{L_X + R_Y}(T) = e^X T e^Y.$$

Let U be an invertible positive operator in $B(H)$, then there exists a self-adjoint operator $V \in B(H)$ such that $U = e^V$. Let $n \in \mathbb{N}$ and A,B be two invertible positive operators in $B(H)$. To simplify computations, we denote A and B by $e^{2^{n+1}X_1}$ and $e^{2^{n+1}Y_1}$, respectively, where X_1 and Y_1 in $B(H)$ are self-adjoint. The corresponding operator map $L_{X_1} - R_{Y_1}$ is denoted by D. With these notations, we now use the results proved in [7, 13] to derive the Hermite-Hadamard type inequalities for unitarily invariant norms.

The Hermite-Hadamard inequality and various refinements of it in the setting of operators (resp. matrices) were given in [7] (resp. [9]). The following theorem is another generalization of the Hermite-Hadamard inequality for operators.

THEOREM 2. *Let $A,B,X \in B(H)$ such that A and B be invertible positive operators and let α,β be any two real numbers and $n,m \in \mathbb{N}$. Let $\gamma(t) = (1-t)\alpha + t\beta$,*

$$E_n = \frac{1}{2^{n-1}} \sum_{i=1}^{2^{n-1}} \left(A^{\gamma(\frac{2i-1}{2^n})} X B^{1-\gamma(\frac{2i-1}{2^n})} + A^{1-\gamma(\frac{2i-1}{2^n})} X B^{\gamma(\frac{2i-1}{2^n})} \right),$$

and

$$\begin{aligned} F_m = \frac{1}{2^m} \sum_{i=1}^{2^{m-1}} \Big(& A^{\gamma(\frac{i-1}{2^{m-1}})} X B^{1-\gamma(\frac{i-1}{2^{m-1}})} + A^{1-\gamma(\frac{i-1}{2^{m-1}})} X B^{\gamma(\frac{i-1}{2^{m-1}})} \\ & + A^{\gamma(\frac{i}{2^{m-1}})} X B^{1-\gamma(\frac{i}{2^{m-1}})} + A^{1-\gamma(\frac{i}{2^{m-1}})} X B^{\gamma(\frac{i}{2^{m-1}})} \Big). \end{aligned}$$

Then

$$\begin{aligned} & \left|\left|\left| A^{\frac{\alpha+\beta}{2}} X B^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} X B^{\frac{\alpha+\beta}{2}} \right|\right|\right| = |||E_1||| \leqslant \ldots \leqslant |||E_n||| \\ & \leqslant \frac{1}{|\beta - \alpha|} \left|\left|\left| \int_\alpha^\beta (A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu) d\nu \right|\right|\right| \end{aligned}$$

$$
\begin{aligned}
&\leqslant |||F_m||| \leqslant \ldots \leqslant |||F_1||| \\
&= \frac{1}{2}\left|\left|\left|A^{\alpha}XB^{1-\alpha} + A^{1-\alpha}XB^{\alpha} + A^{\beta}XB^{1-\beta} + A^{1-\beta}XB^{\beta}\right|\right|\right|. \qquad (3.2)
\end{aligned}
$$

Proof. Put $A = e^{2^{n+1}X_1}$, $B = e^{2^{n+1}Y_1}$ and $T = A^{\frac{1}{2}}XB^{\frac{1}{2}}$, then

$$
\begin{aligned}
&A^{\gamma(\frac{2i-1}{2^n})}XB^{1-\gamma(\frac{2i-1}{2^n})} + A^{1-\gamma(\frac{2i-1}{2^n})}XB^{\gamma(\frac{2i-1}{2^n})} \\
&\qquad = 2\cosh\left(2^{n+1}\left(\gamma\Big(\frac{2i-1}{2^n}\Big) - \frac{1}{2}\right)D\right)T.
\end{aligned}
$$

Similarly, a simple calculation shows

$$
\begin{aligned}
&A^{\gamma(\frac{i-1}{2^{n-1}})}XB^{1-\gamma(\frac{i-1}{2^{n-1}})} + A^{1-\gamma(\frac{i-1}{2^{n-1}})}XB^{\gamma(\frac{i-1}{2^{n-1}})} \\
&\quad + A^{\gamma(\frac{i}{2^{n-1}})}XB^{1-\gamma(\frac{i}{2^{n-1}})} + A^{1-\gamma(\frac{i}{2^{n-1}})}XB^{\gamma(\frac{i}{2^{n-1}})} \\
&= 2\cosh\left(2^{n}\left(\gamma\Big(\frac{i-1}{2^{n-1}}\Big) - \frac{1}{2}\right)D\right)T + 2\cosh\left(2^{n}\left(\gamma\Big(\frac{i}{2^{n-1}}\Big) - \frac{1}{2}\right)D\right)T.
\end{aligned}
$$

Continuing the calculation, we have

$$
\begin{aligned}
&A^{\gamma(\frac{i-1}{2^{n-1}})}XB^{1-\gamma(\frac{i-1}{2^{n-1}})} + A^{1-\gamma(\frac{i-1}{2^{n-1}})}XB^{\gamma(\frac{i-1}{2^{n-1}})} \\
&\qquad + A^{\gamma(\frac{i}{2^{n-1}})}XB^{1-\gamma(\frac{i}{2^{n-1}})} + A^{1-\gamma(\frac{i}{2^{n-1}})}XB^{\gamma(\frac{i}{2^{n-1}})} \\
&\qquad = 4\cosh\left(2^{n-1}\left(\gamma\Big(\frac{i-1}{2^{n-1}}\Big) + \gamma\Big(\frac{i}{2^{n-1}}\Big) - 1\right)D\right) \\
&\qquad\quad \times \cosh\left(2^{n-1}\left(\gamma\Big(\frac{i-1}{2^{n-1}}\Big) - \gamma\Big(\frac{i}{2^{n-1}}\Big)\right)D\right)T \\
&\qquad = 4\cosh\left(2^{n-1}\left(\gamma\Big(\frac{i-1}{2^{n-1}}\Big) + \gamma\Big(\frac{i}{2^{n-1}}\Big) - 1\right)D\right) \\
&\qquad\quad \times \cosh((\beta-\alpha)D)T,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{2^n}{\beta-\alpha}\int_{\gamma(\frac{i-1}{2^n})}^{\gamma(\frac{i}{2^n})}(A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu})d\nu \\
&\qquad = \frac{2^n}{\beta-\alpha}\int_{\gamma(\frac{i-1}{2^n})}^{\gamma(\frac{i}{2^n})} 2\cosh\left(2^{n+1}\left(\nu - \frac{1}{2}\right)D\right)T\,d\nu \\
&\qquad = \frac{D^{-1}}{\beta-\alpha}\left[\sinh\left(2^{n+1}\left(\gamma\Big(\frac{i}{2^n}\Big) - \frac{1}{2}\right)D\right)\right. \\
&\qquad\quad \left. - \sinh\left(2^{n+1}\left(\gamma\Big(\frac{i-1}{2^n}\Big) - \frac{1}{2}\right)D\right)\right]T.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\frac{2^n}{\beta-\alpha}\int_{\gamma(\frac{i-1}{2^n})}^{\gamma(\frac{i}{2^n})}&(A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu})d\nu\\
&=\frac{2D^{-1}}{\beta-\alpha}\cosh\left(2^n\left(\gamma\Big(\frac{i-1}{2^n}\Big)+\gamma\Big(\frac{i}{2^n}\Big)-1\right)D\right)\\
&\quad\times\sinh\left(2^n\left(\gamma\Big(\frac{i}{2^n}\Big)-\gamma\Big(\frac{i-1}{2^n}\Big)\right)D\right)T\\
&=\frac{2D^{-1}}{\beta-\alpha}\cosh\left(2^n\left(\gamma\Big(\frac{i-1}{2^n}\Big)+\gamma\Big(\frac{i}{2^n}\Big)-1\right)D\right)\\
&\quad\times\sinh((\beta-\alpha)D)T.
\end{aligned}
$$

Calculus computations show that for $n\geqslant 2$, we have

$$
\begin{aligned}
E_n&=\frac{1}{2^{n-2}}\sum_{i=1}^{2^{n-1}}\cosh\left(2^{n+1}\left(\gamma\Big(\frac{2i-1}{2^n}\Big)-\frac{1}{2}\right)D\right)T\\
&=\frac{1}{2^{n-2}}\left[\sum_{i=1}^{2^{n-2}}\cosh\left(2^{n+1}\left(\gamma\Big(\frac{2i-1}{2^n}\Big)-\frac{1}{2}\right)D\right)\right.\\
&\quad\left.+\sum_{i=1+2^{n-2}}^{2^{n-1}}\cosh\left(2^{n+1}\left(\gamma\Big(\frac{2i-1}{2^n}\Big)-\frac{1}{2}\right)D\right)\right]T\\
&=\frac{1}{2^{n-2}}\sum_{i=1}^{2^{n-2}}\left[\cosh\left(2^{n+1}\left(\gamma\Big(\frac{2i-1}{2^n}\Big)-\frac{1}{2}\right)D\right)\right.\\
&\quad\left.+\cosh\left(2^{n+1}\left(\gamma\Big(1-\frac{2i-1}{2^n}\Big)-\frac{1}{2}\right)D\right)\right]T\\
&=\frac{1}{2^{n-3}}\sum_{i=1}^{2^{n-2}}\left[\cosh\left(2^n\left(\gamma\Big(\frac{2i-1}{2^n}\Big)+\gamma\Big(1-\frac{2i-1}{2^n}\Big)-1\right)D\right)\right.\\
&\quad\left.\times\cosh\left(2^n\left(\gamma\Big(\frac{2i-1}{2^n}\Big)-\gamma\Big(1-\frac{2i-1}{2^n}\Big)\right)D\right)\right]T.
\end{aligned}
$$

Using the relations $\gamma(t)+\gamma(1-t)=\alpha+\beta$ and $\gamma(t)-\gamma(1-t)=(2t-1)(\beta-\alpha)$, we obtain

$$
\begin{aligned}
E_n&=\frac{1}{2^{n-3}}\cosh\left(2^n(\alpha+\beta-1)D\right)\sum_{i=1}^{2^{n-2}}\cosh\left(2^n\Big(\frac{2i-1}{2^{n-1}}-1\Big)(\beta-\alpha)D\right)T\\
&=\frac{1}{2^{n-3}}\cosh\left(2^n(\alpha+\beta-1)D\right)\sum_{i=1}^{2^{n-2}}\cosh\left(2(2i-1)(\beta-\alpha)D\right)T\\
&=2\cosh\left(2^n(\alpha+\beta-1)D\right)\prod_{i=1}^{n-1}\cosh\left(2^{n-i}(\beta-\alpha)D\right)T.
\end{aligned}
\tag{3.3}
$$

Similarly, by simple calculations, we obtain

$$\begin{aligned}
F_{n+1} &= \frac{1}{2^{n-1}}\sum_{i=1}^{2^n}\cosh\left(2^n\left(\gamma\Big(\frac{i-1}{2^n}\Big)+\gamma\Big(\frac{i}{2^n}\Big)-1\right)D\right)\cosh((\beta-\alpha)D)T\\
&= \frac{1}{2^{n-2}}\sum_{i=1}^{2^{n-1}}\cosh\left(2^n(\alpha+\beta-1)D\right)\cosh((2i-1)(\beta-\alpha)D)\cosh((\beta-\alpha)D)T\\
&= \cosh\left(2^n(\alpha+\beta-1)D\right)\prod_{i=1}^{n-1}\cosh\left(2^{n-i}(\beta-\alpha)D\right)\left(\cosh(2(\beta-\alpha)D)+1\right)T, \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
W &:= \frac{1}{\beta-\alpha}\int_\alpha^\beta (A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu)d\nu\\
&= \frac{2D^{-1}}{\beta-\alpha}\sum_{i=1}^{2^n}\cosh\left(2^n\left(\gamma\Big(\frac{i-1}{2^n}\Big)+\gamma\Big(\frac{i}{2^n}\Big)-1\right)D\right)\sinh\left((\beta-\alpha)D\right)T\\
&= \frac{2D^{-1}}{\beta-\alpha}\cosh\left(2^n(\alpha+\beta-1)D\right)\prod_{i=1}^{n}\cosh\left(2^{n-i}(\beta-\alpha)D\right)\sinh\left((\beta-\alpha)D\right)T\\
&= \frac{D^{-1}}{2^{n-1}(\beta-\alpha)}\cosh\left(2^n(\alpha+\beta-1)D\right)\sinh\left(2^n(\beta-\alpha)D\right)T. \qquad (3.5)
\end{aligned}$$

By [13, Proposition 21], the operator map $\frac{2(\beta-\alpha)D}{\sinh(2(\beta-\alpha)D)}$ is contractive, so from equalities (3.3) and (3.5), we obtain

$$|||E_n||| \leqslant |||W|||. \qquad (3.6)$$

From equality (3.3) for E_{n-1} with $A=e^{2^{n+1}X_1}, B=e^{2^{n+1}Y_1}$, we get

$$E_{n-1} = 2\cosh\left(2^n(\alpha+\beta-1)D\right)\prod_{i=1}^{n-2}\cosh\left(2^{n-i}(\beta-\alpha)D\right)T.$$

The operator map $\frac{1}{\cosh(2(\beta-\alpha)D)}$ is contractive, so

$$|||E_{n-1}||| \leqslant |||E_n|||. \qquad (3.7)$$

By [7, Proposition 2.4], the operator map $\frac{\sinh((\beta-\alpha)D)}{(\beta-\alpha)D\cosh((\beta-\alpha)D)}$ is contractive, therefore from equalities (3.4) and (3.5), we get

$$|||W||| \leqslant |||F_{n+1}|||. \qquad (3.8)$$

From equality (3.5) for $n=2$, i.e., for $A=e^{8X_1}, B=e^{8Y_1}$, we have

$$W = \frac{D^{-1}}{2(\beta-\alpha)}\cosh\left(4(\alpha+\beta-1)D\right)\sinh\left(4(\beta-\alpha)D\right)T$$

and

$$F_2 = \cosh(4(\alpha+\beta-1)D)\Big(\cosh(4(\beta-\alpha)D)+1\Big)T.$$

In this case, we also get $|||W||| \leqslant |||F_2|||$ because the operator map $\frac{\sinh(2(\beta-\alpha)D)}{2(\beta-\alpha)D\cosh(2(\beta-\alpha)D)}$ is contractive.

From equality (3.4) for F_n with $A = e^{2^{n+1}X_1}, B = e^{2^{n+1}Y_1}$, we get

$$F_n = \cosh(2^n(\alpha+\beta-1)D)\prod_{i=1}^{n-2}\cosh\left(2^{n-i}(\beta-\alpha)D\right) \times \Big(\cosh(4(\beta-\alpha)D)+1\Big)T.$$

Therefore

$$\frac{F_{n+1}}{F_n} = \frac{\cosh(2(\beta-\alpha)D)(1+\cosh(2(\beta-\alpha)D))}{1+\cosh(4(\beta-\alpha)D)} = \frac{1}{2}\left(\frac{1}{\cosh(2(\beta-\alpha)D)}+1\right),$$

and this implies that

$$|||F_{n+1}||| \leqslant |||F_n|||. \tag{3.9}$$

From (3.6), (3.7), (3.8) and (3.9), we obtain the relation (3.2) and the proof is completed. □

THEOREM 3. *Let $A,B,X \in B(H)$ such that A and B be invertible positive operators. Let $\frac{1}{4} \leqslant \nu \leqslant \frac{3}{4}$ and $\alpha \in [\frac{1}{2},\infty)$. Then*

$$\frac{1}{2}|||A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu||| \leqslant \left|\left|\left|\int_0^1 A^tXB^{1-t}dt\right|\right|\right| \leqslant \left|\left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}+\alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|. \tag{3.10}$$

Proof. Suppose that $A = e^{2X_1}, B = e^{2Y_1}$ and $T = A^{\frac{1}{2}}XB^{\frac{1}{2}}$,then

$$\frac{1}{2}|||A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu||| = |||\cosh\big((2\nu-1)D\big)T|||,$$

and

$$\left|\left|\left|\int_0^1 A^tXB^{1-t}dt\right|\right|\right| = \left|\left|\left|\int_0^1 \exp\big((2t-1)D\big)T\,dt\right|\right|\right| = |||D^{-1}\sinh(D)T|||.$$

By [13, Proposition 21], the operator map $\frac{D\cosh\big((2\nu-1)D\big)}{\sinh(D)}$ is contractive. This proves the first inequality in (3.10). The second inequality in (3.10) has been proved in Theorem 3.9 of [7]. □

4. Improved Heinz operator inequalities

Let $A,B \in B(H)$ be two positive operators and $\nu \in [0,1]$, then the ν-weighted arithmetic mean of A and B denoted by $A\nabla_\nu B$, is defined as $A\nabla_\nu B = (1-\nu)A + \nu B$. If A is invertible, the ν-geometric mean of A and B denoted by $A\sharp_\nu B$ is defined as $A\sharp_\nu B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\nu A^{\frac{1}{2}}$. For more detail, see Kubo and Ando [12]. When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$, for brevity, respectively.

Let $A,B \in B(H)$ be two invertible positive (strictly positive) operators and $\nu \in [0,1]$. The operator version of the Heinz means are defined by

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2},$$

and the operator version of the Heron means are defined by

$$K_\nu(A,B) = (1-\nu)(A\sharp B) + \nu(A\nabla B).$$

Zhao et al. in [15] gave an inequality for the Heinz-Heron means as follows:

$$H_\nu(A,B) \leqslant K_{\alpha(\nu)}(A,B),$$

where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$.

It is easy to show that the above Heinz mean $H_\nu(\cdot,\cdot)$ interpolates between the non-weighted arithmetic mean and geometric mean, that is

$$A\sharp B \leqslant H_\nu(A,B) \leqslant A\nabla B. \tag{4.1}$$

Kittaneh and Krnić in [11] obtained the some refinements of the left and right inequalities in (4.1) for $\nu \in [0,1] - \{\frac{1}{2}\}$, as follows:

$$\begin{aligned} A\sharp B \leqslant H_{\frac{2\nu+1}{4}}(A,B) &\leqslant \frac{1}{2\nu-1}A^{\frac{1}{2}}F_\nu(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \\ &\leqslant \frac{1}{4}H_\nu(A,B) + \frac{1}{2}H_{\frac{2\nu+1}{4}}(A,B) + \frac{1}{4}A\nabla B \\ &\leqslant \frac{1}{2}H_\nu(A,B) + \frac{1}{2}A\sharp B + \leqslant H_\nu(A,B), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} H_\nu(A,B) \leqslant H_{\frac{r_0}{2}}(A,B) &\leqslant \frac{1}{2r_0}A^{\frac{1}{2}}\left[F_1(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}) + F_{r_0}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})\right]A^{\frac{1}{2}} \\ &\leqslant \frac{1}{4}H_\nu(A,B) + \frac{1}{2}H_{\frac{r_0}{2}}(A,B) + \frac{1}{4}A\nabla B \\ &\leqslant \frac{1}{2}H_\nu(A,B) + \frac{1}{2}A\nabla B \leqslant A\nabla B, \end{aligned} \tag{4.3}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and

$$F_\nu(x) = \begin{cases} \frac{x^\nu - x^{1-\nu}}{\log x}, & x > 0, x \neq 1 \\ 2\nu - 1, & x = 1. \end{cases} \tag{4.4}$$

Let f,α,β be continuous real functions on $\mathbb{R}$ and f be convex. Let $\alpha(\nu) < \beta(\nu)$ $(\nu\in\mathbb{R})$, and $\gamma_\nu(t)=(1-t)\alpha(\nu)+t\beta(\nu)$. For $n\in\mathbb{N}$, Define

$$\begin{aligned}\varphi_n(f,\nu) &= \frac{1}{2^{n-1}}\sum_{i=1}^{2^{n-1}} f\left(\left(1-\frac{2i-1}{2^n}\right)\alpha(\nu)+\frac{2i-1}{2^n}\beta(\nu)\right) \quad (\nu\in\mathbb{R})\\ &= \frac{1}{2^{n-1}}\sum_{i=1}^{2^{n-1}} f\left(\gamma_\nu\left(\frac{2i-1}{2^n}\right)\right). \end{aligned} \tag{4.5}$$

For $m\in\mathbb{N}$, we define

$$\Phi_1(f,\nu)=\frac{f(\alpha(\nu))+f(\beta(\nu))}{2},$$

and for $m\geqslant 1$

$$\begin{aligned}\Phi_{m+1}(f,\nu) &= \frac{1}{2^{m+1}}\left[f(\alpha(\nu))+f(\beta(\nu))+2\sum_{i=1}^{2^m-1} f\left(\left(1-\frac{i}{2^m}\right)\alpha(\nu)+\frac{i}{2^m}\beta(\nu)\right)\right]\\ &= \frac{1}{2^{m+1}}\left[f(\alpha(\nu))+f(\beta(\nu))+2\sum_{i=1}^{2^m-1} f\left(\gamma_\nu\left(\frac{i}{2^m}\right)\right)\right]. \end{aligned} \tag{4.6}$$

It can be easily shown that for every $n,m\in\mathbb{N}$, the sequence (φ_n), (resp.(Φ_m)) is an increasing (resp. a decreasing) sequence of continuous functions such that

$$f\left(\frac{\alpha+\beta}{2}\right)\leqslant \varphi_n(f,\nu)\leqslant \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt\leqslant \Phi_m(f,\nu)\leqslant \frac{f(\alpha)+f(\beta)}{2} \tag{4.7}$$

and

$$\lim_{n\to\infty}\varphi_n(f,\nu)=\lim_{m\to\infty}\Phi_m(f,\nu)=\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt. \tag{4.8}$$

Now, we consider the function $f_x:[0,1]\to\mathbb{R}$, $x>0$, by

$$f_x(t)=\frac{x^t+x^{1-t}}{2}, \tag{4.9}$$

and $0\leqslant\alpha(\nu)<\beta(\nu)\leqslant 1$. The functions $\varphi_n(f_x,\nu)$ and $\Phi_n(f_x,\nu)$ are continuous functions of x. If $A,B\in B(H)$ are two invertible positive operators, using the functional calculus at $x=A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ for $\varphi_n(f_x,\nu)$, we have

$$\varphi_n(f_{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}},\nu)=\frac{1}{2^{n-1}}\sum_{i=1}^{2^{n-1}}\frac{(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\gamma_\nu\left(\frac{2i-1}{2^n}\right)}+(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{1-\gamma_\nu\left(\frac{2i-1}{2^n}\right)}}{2}. \tag{4.10}$$

Multiplying (4.10) by $A^{\frac{1}{2}}$ on the left and right sides, we get

$$A^{\frac{1}{2}}\varphi_n(f_{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}},\nu)A^{\frac{1}{2}}=\frac{1}{2^{n-1}}\sum_{i=1}^{2^{n-1}}H_{\gamma_\nu\left(\frac{2i-1}{2^n}\right)}(A,B). \tag{4.11}$$

We denote it by $\varphi_n(\alpha,\beta;A,B)$. Similarly,

$$\begin{aligned}\Phi_{m+1}(\alpha,\beta;A,B) &:= A^{\frac{1}{2}}\Phi_{m+1}(f_x,\nu)A^{\frac{1}{2}} \\ &= \frac{1}{2^{m+1}}\left[H_{\alpha(\nu)}(A,B)+H_{\beta(\nu)}(A,B)+2\sum_{i=1}^{2^m-1}H_{\gamma_\nu(\frac{i}{2^m})}(A,B)\right].\end{aligned} \tag{4.12}$$

In the following Theorem we give a series of refinements of (4.2).

THEOREM 4. *Let $n,m\in\mathbb{N}$ and $n>1, m>2$. If $A,B\in B(H)$ are two invertible positive operators, then the series of inequalities holds*

$$\begin{aligned}A\sharp B &\leqslant H_{\frac{2\nu+1}{4}}(A,B)=\varphi_1\left(\nu,\frac{1}{2};A,B\right)\leqslant\varphi_n\left(\nu,\frac{1}{2};A,B\right) \\ &\leqslant \frac{1}{2\nu-1}A^{\frac{1}{2}}F_\nu(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}\leqslant\Phi_m\left(\nu,\frac{1}{2};A,B\right) \\ &\leqslant \Phi_2\left(\nu,\frac{1}{2};A,B\right)=\frac{1}{4}H_\nu(A,B)+\frac{1}{2}H_{\frac{2\nu+1}{4}}(A,B)+\frac{1}{4}A\sharp B \\ &\leqslant \frac{1}{2}H_\nu(A,B)+\frac{1}{2}A\sharp B+\leqslant H_\nu(A,B),\end{aligned} \tag{4.13}$$

for all $\nu\in[0,1]-\{\frac{1}{2}\}$, where F_ν is the function given in (4.4).

Proof. Let $0\leqslant\nu<\frac{1}{2}$. Applying inequality (4.7) to the function f_x and $\alpha(\nu)=\nu, \beta(\nu)=\frac{1}{2}$, we get

$$\begin{aligned}f_x\left(\frac{2\nu+1}{4}\right)\leqslant\varphi_n(f_x,\nu)\leqslant\frac{2}{1-2\nu}\int_\nu^{\frac{1}{2}}f(t)dt \\ \leqslant\Phi_m(f_x,\nu)\leqslant\frac{f_x(\nu)+f_x(\frac{1}{2})}{2}.\end{aligned} \tag{4.14}$$

Clearly, $\varphi_n(\alpha,\beta;A,B)=\varphi_n(\beta,\alpha;A,B)$ and $\Phi_m(\alpha,\beta;A,B)=\Phi_m(\beta,\alpha;A,B)$ since $H_{1-\nu}(A,B)=H_\nu(A,B)$. Therefore (4.14) also holds for $\frac{1}{2}<\nu\leqslant 1$ because $F_{1-\nu}(x)=-F_\nu(x)$.

Utilizing of the monotonicity property (3.1), the relation (4.14) holds when x is replaced with the positive operator $A^{\frac{-1}{2}}BA^{\frac{1}{2}}$. Finally, multiplying both sides of such obtained series of inequalities by $A^{\frac{1}{2}}$ and applying (4.11) and (4.12), we deduced the inequalities (4.13). □

In the following Theorem we give a series of refinements of (4.3).

THEOREM 5. *Let* $1 \leqslant n,m \in \mathbb{N}$ *and* $\nu \in [0,1]-\{\frac{1}{2}\}$. *If* $A,B \in B(H)$ *are two invertible positive operators, then the series of inequalities holds*

$$\begin{aligned} H_\nu(A,B) &\leqslant H_{\frac{r_0}{2}}(A,B) \leqslant \varphi_n(0,r_0;A,B) \\ &\leqslant \frac{1}{2r_0}A^{\frac{1}{2}}\left[F_1(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})+F_{r_0}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})\right]A^{\frac{1}{2}} \\ &\leqslant \Phi_m(0,r_0;A,B) \\ &\leqslant \frac{1}{4}H_\nu(A,B)+\frac{1}{2}H_{\frac{r_0}{2}}(A,B)+\frac{1}{4}A\nabla B \\ &\leqslant \frac{1}{2}H_\nu(A,B)+\frac{1}{2}A\nabla B \leqslant A\nabla B, \end{aligned} \tag{4.15}$$

where $r_0=\min\{\nu,1-\nu\}$ *and* F_ν *is the function given in* (4.4).

Proof. By the symmetry of the Heinz means and the fact that $F_{1-\nu}=-F_\nu$, it is sufficient that, we prove (4.15) for $0 \leqslant \nu < \frac{1}{2}$. Applying inequality (4.7) to the function f_x and $\alpha(\nu)=0, \beta(\nu)=r_0=\min\{\nu,1-\nu\}=\nu$, we get

$$\begin{aligned} f_x\left(\frac{\nu}{2}\right) \leqslant \varphi_n(f_x,\nu) &\leqslant \frac{1}{\nu}\int_0^\nu f(t)dt \\ &\leqslant \Phi_m(f_x,\nu) \leqslant \frac{f_x(0)+f_x(\nu)}{2}. \end{aligned} \tag{4.16}$$

By the same argument used in the proof of Theorem 4, we obtain the inequalities (4.15). □

REFERENCES

[1] I. ALI, H. YANG AND A. SHAKOOR, *Refinements of the Heron and Heinz means inequalities for matrices*, J. Math. Inequal. 1 (2014), 107–112.

[2] R. BHATIA, *Interpolating the arithmetic mean inequality and its operator version*, Linear Algebra Appl. **413** (2006), 355–363.

[3] R. BHATIA, *Matrix analysis*, Springer-Verlag, New York, 1997.

[4] R. BHATIA, C. DAVIS, *A Cauchy-Schwarz inequality for operators with applications*, Linear Algebra Appl. 223/224 (1995) 119–129.

[5] D. DRISSI, *Sharp inequalities for some operator means*, J. Matrix Anal. Appl. **28** (2006), 822–828.

[6] Y. KAPIL, C. CONDE, M. S. MOSLEHIAN, M. SINGH, M. SABABHEH, *Norm Inequalities Related to the Heron and Heinz Means*, Mediterr. J. Math. (2017), 14:213.

[7] Y. KAPIL, M. SINGH, *Contractive maps on operator ideals and norm inequalities*, Linear Algebra Appl. **459** (2014), 475–492.

[8] R. KAUR, M. SINGH, *Complete interpolation of matrix version of Heron and Heinz means*, J. Math. inequal., **16**, 1 (2013), 93–99.

[9] R. KAUR, M. SAL MOSLEHIAN, M. SINGH, C. CONDE, *Further refinements of the Heinz inequality*, Linear Algebra and its Applications 447 (2014) 26–37.

[10] F. KITTANEH, *On the convexity of the Heinz means*, Integr. Equ. Oper. Theory **68** (2010), 519–527.

[11] F. KITTANEH AND M. KRNIĆ, *Refined Heinz operator inequalities*, Linear and Multilinear Algebra, **61** (8) (2013), 1148–1157.

[12] F. KUBO, T. ANDO, *Means of positive linear operators*, Math. Ann. 246 (1980) 205–224.

[13] G. LAROTONDA, *Norm inequalities in operator ideals*, J. Funct. Anal. 255 (2008) 3208–3228.

[14] J. E. PEČARIĆ, T. FURUTA, J. MIĆIĆ HOT, Y. SEO, *Mond-Pečarić Method in operator inequalities*, Element, Zagreb, 2005.

[15] J. G. ZHAO, J. L. WU, *Operator inequalities involving improved Young and its reverse inequalities*, J. Math. Anal. Appl. **421**, 2 (2015), 1779–1789.

(Received January 25, 2020)

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